Non-linear dynamics and statistical theories for basic geophysical flows
Non-linear dynamics and statistical theories for basic geophysical flows

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Preface

This book is an introduction to the fascinating and important interplay between non-linear dynamics and statistical theories for geophysical flows. The book is designed for a multi-disciplinary audience ranging from beginning graduate students to senior researchers in applied mathematics as well as theoretically inclined graduate students and researchers in atmosphere/ocean science. The approach in this book emphasizes the serendipity between physical phenomena and modern applied mathematics, including rigorous mathematical analysis, qualitative models, and numerical simulations. The book includes more conventional topics for non-linear dynamics applied to geophysical flows, such as long time selective decay, the effect of large-scale forcing, non-linear stability and fluid flow on the sphere, as well as emerging contemporary research topics involving applications of chaotic dynamics, equilibrium statistical mechanics, and information theory. The various competing approaches for equilibrium statistical theories for geophysical flows are compared and contrasted systematically from the viewpoint of modern applied mathematics, including an application for predicting the Great Red Spot of Jupiter in a fashion consistent with the observational record. Novel applications of information theory are utilized to simplify, unify, and compare the equilibrium statistical theories and also to quantify aspects of predictability in non-linear dynamical systems with many degrees of freedom. No previous background in geophysical flows, probability theory, information theory, or equilibrium statistical mechanics is needed to read the text. These topics and related background concepts are all introduced and developed through elementary examples and discussion throughout the text as they arise. The book is also of wider interest to applied mathematicians and other scientists to illustrate how ideas from statistical physics can be applied in novel ways to inhomogeneous large-scale complex non-linear systems.

The material in the book is based on lectures of the first author given at the Courant Institute in 1995, 1997, 2001, and 2004. The first author thanks Professor
Preface

Pedro Embid as well as his former Ph.D. students Professor Pete Kramer and Seuyung Shim for their help with early versions of Chapters 1, 2, 3, 4, and 6 of the present book. Joint research work with Professors Richard Kleeman and Bruce Turkington as well as Majdas former Courant post docs, Professors Marcus Grote, Ilya Timofeyev, Rafail Abramov, and Mark DeBattista have been incorporated into the book; their explicit and implicit contributions are acknowledged warmly. The authors acknowledge generous support of the National Science Foundation and the Office of Naval Research during the development of this book, including partial salary support for Xiaoming Wangs visit to Courant in the spring semester of 2001.