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To

Brigitta, Karen, and Kenneth Gasper

and

Babu, Raja, and to the memory of Parul S. Rahman

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Foreword

My education was not much different from that of most mathematicians of my generation. It included courses on modern algebra, real and complex variables, both point set and algebraic topology, some number theory and projective geometry, and some specialized courses such as one on Riemann surfaces. In none of these courses was a hypergeometric function mentioned, and I am not even sure if the gamma function was mentioned after an advanced calculus course. The only time Bessel functions were mentioned was in an undergraduate course on differential equations, and the only thing done with them was to find a power series solution for the general Bessel equation. It is small wonder that with a similar education almost all mathematicians think of special functions as a dead subject which might have been interesting once. They have no idea why anyone would care about it now.

Fortunately there was one part of my education which was different. As a junior in college I read Widder's book *The Laplace Transform* and the manuscript of its very important sequel, Hirschman and Widder's *The Convolution Transform*. Then as a senior, I. I. Hirschman gave me a copy of a preprint of his on a multiplier theorem for Legendre series and suggested I extend it to ultraspherical series. This forced me to become acquainted with two other very important books, Gabor Szegő's great book *Orthogonal Polynomials*, and the second volume of *Higher Transcendental Functions*, the monument to Harry Bateman which was written by Arthur Erdélyi and his co-workers W. Magnus, F. Oberhettinger and F. G. Tricomi.

From this I began to realize that the many formulas that had been found, usually in the 18th or 19th century, but once in a while in the early 20th century, were useful, and started to learn about their structure. However, I had written my Ph.D. thesis and worked for three more years before I learned that not every fact about special functions I would need had already been found, and it was a couple of more years before I learned that it was essential to understand hypergeometric functions. Like others, I had been put off by all the parameters. If there were so many parameters that it was necessary to put subscripts on them, then there has to be a better way to solve a problem than this. That was my initial reaction to generalized hypergeometric functions, and a very common reaction to judge from the many conversations I have had on these functions in the last twenty years. After learning a little more about hypergeometric functions, I was very surprised to realize that they had occurred regularly in first year calculus. The reason for the subscripts on the parameters is that not all interesting polynomials are of degree one or two. For a generalized hypergeometric function has a series representation

$$\sum_{n=0}^{\infty} c_n \tag{1}$$

with c_{n+1}/c_n a rational function of n. These contain almost all the examples of infinite series introduced in calculus where the ratio test works easily. The ratio c_{n+1}/c_n can be factored, and it is usually written as

$$\frac{c_{n+1}}{c_n} = \frac{(n+a_1)\cdots(n+a_p)x}{(n+b_1)\cdots(n+b_q)(n+1)}. \tag{2}$$

Introduce the shifted factorial

$$\begin{aligned} &(a)_0 = 1, \\ &(a)_n = a(a+1)\cdots(a+n-1), \quad n = 1,2,\ldots\ . \end{aligned} \tag{3}$$

Then if $c_0 = 1$, equation (2) can be solved for c_n as

$$c_n = \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!} \tag{4}$$

and

$${}_pF_q\left[\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix};x\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!} \tag{5}$$

is the usual notation.

The first important result for a ${}_pF_q$ with $p>2,\ q>1$ is probably Pfaff's sum

$${}_3F_2\left[\begin{matrix} -n,\ a,\ b \\ c,\ a+b+1-c-n \end{matrix};1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \quad n=0,1,\ldots\ . \tag{6}$$

This result from 1797, see Pfaff [1797], contains as a limit when $n\to\infty$, another important result usually attributed to Gauss [1813],

$${}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re}\ (c-a-b)>0. \tag{7}$$

The next instance is a very important result of Clausen [1828]:

$$\left\{{}_2F_1\left[\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix};x\right]\right\}^2 = {}_3F_2\left[\begin{matrix} 2a,\ 2b,\ a+b \\ a+b+\frac{1}{2},\ 2a+2b \end{matrix};x\right]. \tag{8}$$

Some of the interest in Clausen's formula is that it changes the square of a class of ${}_2F_1$'s to a ${}_3F_2$. In this direction it is also interesting because it was probably the first instance of anyone finding a differential equation satisfied by $[y(x)]^2,\ y(x)z(x)$ and $[z(x)]^2$ when $y(x)$ and $z(x)$ satisfy

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{9}$$

This problem was considered for (9) by Appell, see Watson [1952], but the essence of his general argument occurs in Clausen's paper. This is a common phenomenon, which is usually not mentioned when the general method is introduced to students, so they do not learn how often general methods come from specific problems or examples. See D. and G. Chudnovsky [1988] for an instance of the use of Clausen's formula, where a result for a ${}_2F_1$ is carried to a ${}_3F_2$ and from that to a very interesting set of expansions of π^{-1}. Those identities were first discovered by Ramanujan. Here is Ramanujan's most impressive example:

$$\frac{9801}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty}[1103 + 26390n]\frac{(1/4)_n(1/2)_n(3/4)_n}{(1)_n(1)_n n!}\frac{1}{(99)^{4n}}. \tag{10}$$

There is another important reason why Clausen's formula is important. It leads to a large class of ${}_3F_2$'s that are nonnegative for the power series variable between -1 and 1. The most famous use of this is in the final step of de Branges' solution of the Bieberbach conjecture, see de Branges [1985]. The integral of the ${}_2F_1$ or Jacobi polynomial he had is a ${}_3F_2$, and its positivity is an easy consequence of Clausen's formula, as Gasper had observed ten years earlier. There are other important results which follow from the positivity in Clausen's identity.

Once Kummer [1836] wrote his long and important paper on ${}_2F_1$'s and ${}_1F_1$'s, this material became well-known. It has been reworked by others. Riemann redid the ${}_2F_1$ using his idea that the singularities of a function go a long way toward determining the function. He showed that if the differential equation (9) has regular singularities at three points, and every other point in the extended complex plane is an ordinary point, then the equation is equivalent to the hypergeometric equation

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0, \tag{11}$$

which has regular singular points at $x = 0, 1, \infty$. Riemann's work was very influential, so much so that much of the mathematical community that considered hypergeometric functions studied them almost exclusively from the point of view of differential equations. This is clear in Klein's book [1933], and in the work on multiple hypergeometric functions that starts with Appell in 1880 and is summarized in Appell and Kampé de Fériet [1926].

The integral representations associated with the differential equation point of view are similar to Euler's integral representation. This is

$${}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};x\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 (1-xt)^{-a}t^{b-1}(1-t)^{c-b-1}dt, \tag{12}$$

$|x| < 1$, Re $c >$ Re $b > 0$, and includes related integrals with different contours. The differential equation point of view is very powerful where it works, but it

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does not work well for $p \geq 3$ or $q \geq 2$ as Kummer discovered. Thus there is a need to develop other methods to study hypergeometric functions.

In the late 19th and early 20th century a different type of integral representation was introduced. These two different types of integrals are best represented by Euler's beta integral

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \text{Re } (a,b) > 0 \tag{13}$$

and Barnes' beta integral

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)\ dt \qquad (14)\\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \quad \text{Re } (a,b,c,d) > 0. \end{aligned}$$

There is no direct connection with differential equations for integrals like (14), so it stands a better chance to work for larger values of p and q.

While Euler, Gauss, and Riemann and many other great mathematicians wrote important and influential papers on hypergeometric functions, the development of basic hypergeometric functions was much slower. Euler and Gauss did important work on basic hypergeometric functions, but most of Gauss' work was unpublished until after his death and Euler's work was more influential on the development of number theory and elliptic functions.

Basic hypergeometric series are series $\sum c_n$ with c_{n+1}/c_n a rational function of q^n for a fixed parameter q, which is usually taken to satisfy $|q| < 1$, but at other times is a power of a prime. In this Foreword $|q| < 1$ will be assumed.

Euler summed three basic hypergeometric series. The one which had the largest impact was

$$\sum_{-\infty}^{\infty}(-1)^n q^{(3n^2-n)/2} = (q;q)_\infty, \tag{15}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n). \tag{16}$$

If

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty \tag{17}$$

then Euler also showed that

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty}\frac{x^n}{(q;q)_n}, \quad |x| < 1, \tag{18}$$

and

$$(x;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n}. \tag{19}$$

Eventually all of these were contained in the q-binomial theorem

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n, \quad |x| < 1. \tag{20}$$

While (18) is clearly the special case $a = 0$, and (19) follows easily on replacing x by xa^{-1} and letting $a \to \infty$, it is not so clear how to obtain (15) from (20). The easiest way was discovered by Cauchy and many others. Take $a = q^{-2N}$, shift n by N, rescale and let $N \to \infty$. The result is called the triple product, and can be written as

$$(x;q)_\infty (qx^{-1};q)_\infty (q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n. \tag{21}$$

Then $q \to q^3$ and $x = q$ gives Euler's formula (15).

Gauss used a basic hypergeometric series identity in his first proof of the determination of the sign of the Gauss sum, and Jacobi used some to determine the number of ways an integer can be written as the sum of two, four, six and eight squares. However, this particular aspect of Gauss' work on Gauss sums was not very influential, as his hypergeometric series work had been, and Jacobi's work appeared in his work on elliptic functions, so its hypergeometric character was lost in the great interest in the elliptic function work. Thus neither of these led to a serious treatment of basic hypergeometric series. The result that seems to have been the crucial one was a continued fraction of Eisenstein. This along with the one hundredth anniversary of Euler's first work on continued fractions seem to have been the motivating forces behind Heine's introduction of a basic hypergeometric extension of ${}_2F_1(a, b; c; x)$. He considered

$${}_2\phi_1 \left[\begin{matrix} q^a, \ q^b \\ q^c \end{matrix} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(q^a;q)_n (q^b;q)_n}{(q^c;q)_n (q;q)_n} x^n, \quad |x| < 1. \tag{22}$$

Observe that

$$\lim_{q \to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n,$$

so

$$\lim_{q \to 1} {}_2\phi_1 \left[\begin{matrix} q^a, \ q^b \\ q^c \end{matrix} ; q, x \right] = {}_2F_1 \left[\begin{matrix} a, \ b \\ c \end{matrix} ; x \right].$$

Heine followed the pattern of Gauss' published paper on hypergeometric series, and so obtained contiguous relations and from them continued fraction

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expansions. He also obtained some series transformations, and the sum

$${}_2\phi_1\left[\begin{matrix} q^a, \; q^b \\ q^c \end{matrix}; q, q^{c-a-b}\right] = \frac{(q^{c-a};q)_\infty (q^{c-b};q)_\infty}{(q^c;q)_\infty (q^{c-a-b};q)_\infty}, \quad |q^{c-a-b}| < 1. \tag{23}$$

This sum becomes (7) when $q \to 1$.

As often happens to path breaking work, this work of Heine was to a large extent ignored. When writing the second edition of *Kugelfunctionen* (Heine [1878]) Heine decided to include some of his work on basic hypergeometric series. This material was printed in smaller type, and it is clear that Heine included it because he thought it was important, and he wanted to call attention to it, rather than because he thought it was directly related to spherical harmonics, the subject of his book. Surprisingly, his inclusion of this material led to some later work, which showed there was a very close connection between Heine's work on basic hypergeometric series and spherical harmonics. The person Heine influenced was L. J. Rogers, who is still best known as the first discoverer of the Rogers–Ramanujan identities. Rogers tried to understand this aspect of Heine's work, and one transformation in particular. Thomae [1879] had observed this transformation of Heine could be written as an extension of Euler's integral representation (12), but Rogers was unaware of this explanation, and so discovered a second reason. He was able to modify the transformation so it became the permutation symmetry in a new series. While doing this he introduced a new set of polynomials which we now call the continuous q-Hermite polynomials. In a very important set of papers which were unjustly neglected for decades, Rogers discovered a more general set of polynomials and found some remarkable identities they satisfy, see Rogers [1893a,b, 1894, 1895]. For example, he found the linearization coefficients of these polynomials which we now call the continuous q-ultraspherical polynomials. These polynomials contain many of the spherical harmonics Heine studied. Contained within this product identity is the special case of the square of one of these polynomials as a double series. As Gasper and Rahman have observed, one of these series can be summed, and the resulting identity is an extension of Clausen's sum in the terminating case. Earlier, others had found a different extension of Clausen's identity to basic hypergeometric series, but the resulting identity was not satisfactory. The identity had the product of two functions, the same functions but one evaluated at x and the other at qx, and so was not a square. Thus the nonnegativity that is so useful in Clausen's formula was not true for the corresponding basic hypergeometric series. Rogers' result for his polynomials led directly to the better result which contains the appropriate nonnegativity. From this example and many others, one sees that orthogonal polynomials provide an alternative approach to the study of hypergeometric and basic hypergeometric functions. Both this approach and that of differential equations are most useful for small values of the degrees of the numerator and denominator polynomials in the ratio c_{n+1}/c_n, but orthogonal polynomials work for a larger class of series, and are much more useful for basic hypergeometric series. However, neither of these approaches is powerful enough to encompass all aspects of these functions. Direct series manipulations are surprisingly useful, when done by a master, or when a computer algebra system is used as an aid. Gasper and Rahman are both experts at symbolic calculations, and I regularly marvel at some of the formulas they have found. As quantum groups become better known, and as Baxter's work spreads to other parts of mathematics as it has started to do, there will be many people trying to learn how to deal with basic hypergeometric series. This book is where I would start.

For many years people have asked me what is the best book on special functions. My response was George Gasper's copy of Bailey's book, which was heavily annotated with useful results and remarks. Now others can share the information contained in these margins, and many other very useful results.

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Preface

The study of basic hypergeometric series (also called q-hypergeometric series or q-series) essentially started in 1748 when Euler considered the infinite product $(q;q)_\infty^{-1} = \prod_{k=0}^{\infty} (1 - q^{k+1})^{-1}$ as a generating function for $p(n)$, the number of partitions of a positive integer n into positive integers (see §8.10). But it was not until about a hundred years later that the subject acquired an independent status when Heine converted a simple observation that $\lim_{q\to 1}[(1-q^a)/(1-q)] = a$ into a systematic theory of ${}_2\phi_1$ basic hypergeometric series parallel to the theory of Gauss' ${}_2F_1$ hypergeometric series. Heine's transformation formulas for ${}_2\phi_1$ series and his q-analogue of Gauss' ${}_2F_1(1)$ summation formula are derived in Chapter 1, along with a q-analogue of the binomial theorem, Jacobi's triple product identity, and some formulas for q-analogues of the exponential, gamma and beta functions.

Apart from some important work by J. Thomae and L. J. Rogers the subject remained somewhat dormant during the latter part of the nineteenth century until F. H. Jackson embarked on a lifelong program of developing the theory of basic hypergeometric series in a systematic manner, studying q-differentiation and q-integration and deriving q-analogues of the hypergeometric summation and transformation formulas that were discovered by A. C. Dixon, J. Dougall, L. Saalschütz, F. J. W. Whipple, and others. His work is so pervasive that it is impossible to cover all of his contributions in a single volume of this size, but we have tried to include many of his important formulas in the first three chapters. In particular, a derivation of his summation formula for an ${}_8\phi_7$ series is given in §2.6. During the 1930's and 1940's many important results on hypergeometric and basic hypergeometric series were derived by W. N. Bailey. Some mathematicians consider Bailey's greatest work to be the Bailey transform (an equivalent form of which is covered in Chapter 2), but equally significant are his nonterminating extensions of Jackson's ${}_8\phi_7$ summation formula and of Watson's transformation formula connecting very-well-poised ${}_8\phi_7$ series with balanced ${}_4\phi_3$ series. Much of the material on summation, transformation and expansion formulas for basic hypergeometric series in Chapter 2 is due to Bailey.

D. B. Sears, L. Carlitz, W. Hahn, and L. J. Slater were among the prominent contributors during the 1950's. Sears derived several transformation formulas for ${}_3\phi_2$ series, balanced ${}_4\phi_3$ series, and very-well-poised ${}_{n+1}\phi_n$ series. Simple proofs of some of his ${}_3\phi_2$ transformation formulas are given in Chapter 3. Three of his very-well-poised transformation formulas are derived in Chapter 4, where we follow G. N. Watson and Slater to develop the theory of basic hypergeometric series from a contour integral point of view, an idea first introduced by Barnes in 1907.

Chapter 5 is devoted to bilateral basic hypergeometric series, where the most fundamental formula is Ramanujan's ${}_1\psi_1$ summation formula. Substantial contributions were also made by Bailey, M. Jackson, Slater and others, whose works form the basis of this chapter.

During the 1960's R. P. Agarwal and Slater each published a book partially devoted to the theory of basic hypergeometric series, and G. E. Andrews initiated his work in number theory, where he showed how useful the summation and transformation formulas for basic hypergeometric series are in the theory of partitions. Andrews gave simpler proofs of many old results, wrote review articles pointing out many important applications and, during the mid 1970's, started a period of very fruitful collaboration with R. Askey. Thanks to these two mathematicians, basic hypergeometric series is an active field of research today. Since Askey's primary area of interest is orthogonal polynomials, q-series suddenly provided him and his co-workers with a very rich environment for deriving q-extensions of beta integrals and of the classical orthogonal polynomials of Jacobi, Gegenbauer, Legendre, Laguerre and Hermite. Askey and his students and collaborators who include W. A. Al-Salam, M. E. H. Ismail, T. H. Koornwinder, W. G. Morris, D. Stanton, and J. A. Wilson have produced a substantial amount of interesting work over the past fifteen years. This flurry of activity has been so infectious that many researchers found themselves hopelessly trapped by this alluring "q-disease", as it is affectionately called.

Our primary motivation for writing this book was to present in one modest volume the significant results of the past two hundred years so that they are readily available to students and researchers, to give a brief introduction to the applications to orthogonal polynomials that were discovered during the current renaissance period of basic hypergeometric series, and to point out important applications to other fields. Most of the material is elementary enough so that persons with a good background in analysis should be able to use this book as a textbook and a reference book. In order to assist the reader in developing a deeper understanding of the formulas and proof techniques and to include additional formulas, we have given a broad range of exercises at the end of each chapter. Additional information is provided in the Notes following the Exercises, particularly in relation to the results and relevant applications contained in the papers and books listed in the References. Although the References may have a bulky appearance, it is just an introduction to the vast literature available. Appendices I, II, and III are for quick reference, so that it is not necessary to page through the book in order to find the most frequently needed identities, summation formulas, and transformation formulas. It can be rather tedious to apply the summation and transformation formulas to the derivation of other formulas. But now that several symbolic computer algebraic systems are available, persons having access to such a system can let it do some of the symbolic manipulations, such as computing the form of Bailey's ${}_{10}\phi_9$ transformation formula when its parameters are replaced by products of other parameters.

Due to space limitations, we were unable to be as comprehensive in our coverage of basic hypergeometric series and their applications as we would have liked. In particular, we could not include a systematic treatment of basic hypergeometric series in two or more variables, covering F. H. Jackson's work on basic Appell series and the works of R. A. Gustafson and S. C. Milne on $U(n)$ multiple series generalizations of basic hypergeometric series referred to in the References. But we do highlight Askey and Wilson's fundamental work on their beautiful q-analogue of the classical beta integral in Chapter 6 and develop its connection with very-well-poised ${}_8\phi_7$ series. Chapter 7 is devoted to applications to orthogonal polynomials, mostly developed by Askey and his collaborators. We conclude the book with some further applications in Chapter 8, where we present part of our work on product and linearization formulas, Poisson kernels, and nonnegativity, and we also manage to point out some elementary facts about applications to the theory of partitions and the representations of integers as sums of squares of integers. The interested reader is referred to the books and papers of Andrews and N. J. Fine for additional applications to partition theory, and recent references are pointed out for applications to affine root systems (Macdonald identities), association schemes, combinatorics, difference equations, Lie algebras and groups, physics (such as representations of quantum groups and R. J. Baxter's work on the hard hexagon model of phase transitions in statistical mechanics), statistics, etc.

We use the common numbering system of letting (k.m.n) refer to the n-th numbered display in Section m of Chapter k, and letting (I.n), (II.n), and (III.n) refer to the n-th numbered display in Appendices I, II, and III, respectively. To refer to the papers and books in the References, we place the year of publication in square brackets immediately after the author's name. Thus Bailey [1935] refers to Bailey's 1935 book. Suffixes a, b, ... are used after the years to distinguish different papers by an author that appeared in the same year. Papers that have not yet been published are referred to with the year 2004, even though they might be published later due to the backlogs of journals. Since there are three Agarwals, two Chiharas and three Jacksons listed in the References, to minimize the use of initials we drop the initials of the author whose works are referred to most often. Hence Agarwal, Chihara, and Jackson refer to R. P. Agarwal, T. S. Chihara, and F. H. Jackson, respectively.

We would like to thank the publisher for their cooperation and patience during the preparation of this book. Thanks are also due to R. Askey, W. A. Al-Salam, R. P. Boas, T. S. Chihara, B. Gasper, R. Holt, M. E. H. Ismail, T. Koornwinder, and B. Nassrallah for pointing out typos and suggesting improvements in earlier versions of the book. We also wish to express our sincere thanks and appreciation to our TEXtypist, Diane Berezowski, who suffered through many revisions of the book but never lost her patience or sense of humor.

Preface to the second edition

In 1990 it was beyond our wildest imagination that we would be working on a second edition of this book thirteen years later. In this day and age of rapid growth in almost every area of mathematics, in general, and in Orthogonal Polynomials and Special Functions, in particular, it would not be surprising if the book became obsolete by now and gathered dust on the bookshelves. All we hoped for is a second printing. Even that was only a dream since the main competitor of authors and publishers these days are not other books, but the ubiquitous copying machine. But here we are: bringing out a second edition with full support of our publisher.

The main source of inspiration, of course, has been the readers and users of this book. The response has been absolutely fantastic right from the first weeks the book appeared in print. The kind of warm reception we enjoyed far exceeded our expectations. Years later many of the leading researchers in the field kept asking us if an updated version would soon be forthcoming. Yes, indeed, an updated and expanded version was becoming necessary during the latter part of the 1990's in view of all the explosive growth that the subject was experiencing in many different areas of applications of basic hypergeometric series (also called q-series). However, the most important and significant impetus came from an unexpected source — Statistical Mechanics. In trying to find elliptic (doubly periodic meromorphic) solutions of the so-called Yang-Baxter equation arising out of an eight-vertex model in Statistical Mechanics the researchers found that the solutions are, in fact, a form of hypergeometric series $\sum a_n z^n$, where a_{n+1}/a_n is an elliptic function of n, with n regarded as a complex variable. In a span of only five years the study of elliptic hypergeometric series and integrals has become almost a separate area of research on its own, whose leading researchers include, in alphabetical order, R. J. Baxter, E. Date, J. F. van Diejen, G. Felder, P. J. Forrester, I. B. Frenkel, M. Jimbo, Y. Kajihara, K. Kajiwara, H. T. Koelink, A. Kuniba, T. Masuda, T. Miwa, Y. van Norden, M. Noumi, Y. Ohta, M. Okado, E. Rains, H. Rosengren, S. N. M. Ruijsenaars, M. Schlosser, V. P. Spiridonov, L. Stevens, V. G. Turaev, A. Varchenko, S. O. Warnaar, Y. Yamada, and A. S. Zhedanov.

Even though we had not had any research experience in this exciting new field, it became quite clear to us that a new edition could be justified only if we included a chapter on elliptic hypergeometric series (and modular and theta hypergeometric series), written in an expository manner so that it would be more accessible to non-experts and be consistent with the rest of the book. Chapter 11 is entirely devoted to that topic. Regrettably, we had to be ruthlessly selective about choosing one particular approach from many possible approaches, all of which are interesting and illuminating on their own. We were guided by the need for brevity and clarity at the same time, as well as consistency of notations.

In addition to Chapter 11 we added Chapter 9 on generating and bilinear generating functions in view of their central importance in the study of orthogonal polynomials, as well as Chapter 10, covering briefly the huge topic of multivariable q-series, restricted mostly to F. H. Jackson's q-analogues of the four Appell functions F_1, F_2, F_3, F_4, and some of their more recent extensions. In these chapters we have attempted to describe the basic methods and results, but left many important formulas as exercises. Some parts of Chapters 1–8 were updated by the addition of textual material and a number of exercises, which were added at the ends of the sections and exercises in order to retain the same numbering of the equations and exercises as in the first edition.

We have corrected a number of minor typos in the first edition, some discovered by ourselves, but most kindly pointed out to us by researchers in the field, whose contributions are gratefully acknowledged. A list of errata, updates of the references, etc., for the first edition and its translation into Russian by N. M. Atakishiyev and S. K. Suslov may be downloaded at arxiv.org/abs/math.CA/9705224 or at www.math.northwestern.edu/~george/preprints/bhserrata, which is usually the most up-to-date. Analogous to the first edition, papers that have not been published by November of 2003 are referred to with the year 2003, even though they might be published later.

The number of people to whom we would like to express our thanks and gratitude is just too large to acknowledge individually. However, we must mention the few whose help has been absolutely vital for the preparation of this edition. They are S. O. Warnaar (who proof-read our files with very detailed comments and suggestions for improvement, not just for Chapter 11, which is his specialty, but also the material in the other chapters), H. Rosengren (whose e-mails gave us the first clue as to how we should present the topic of Chapter 11), M. Schlosser (who led us in the right directions for Chapter 11), S. K. Suslov (who sent a long list of errata, additional references, and suggestions for new exercises), S. C. Milne (who suggested some improvements in Chapters 5, 7, 8, and 11), V. P. Spiridonov (who suggested some improvements in Chapter 11), and M. E. H. Ismail (whose comments have been very helpful). Thanks are also due to R. Askey for his support and useful comments. Finally, we need to mention the name of a behind-the-scene helper, Brigitta Gasper, who spent many hours proofreading the manuscript. The mention of our ever gracious TEXtypist, Diane Berezowski, is certainly a pleasure. She did not have to do the second edition, but she said she enjoys the work and wanted to be a part of it. Where do you find a more committed friend? We owe her immensely.