Elementary Probability
2nd Edition

Now available in a fully revised and updated new edition, this well-established textbook provides a straightforward introduction to the theory of probability. The presentation is entertaining without any sacrifice of rigour; important notions are covered with the clarity that the subject demands.

Topics covered include conditional probability, independence, discrete and continuous random variables, basic combinatorics, generating functions and limit theorems, and an introduction to Markov chains. This edition includes an elementary approach to martingales and the theory of Brownian motion, which supply the cornerstones for many topics in modern financial mathematics such as option and derivative pricing. The text is accessible to undergraduate students, and provides numerous worked examples and exercises to help build the important skills necessary for problem solving.

‘[T]he author succeeds in combining the utmost succinctness with clarity and genuine readability. . . . This textbook can be recommended unreservedly.’

Internationale Mathematische Nachrichten

‘[T]his book is a superb resource of theory and application, which should be on every lecturer’s shelves, and those of many students. You may never need to buy another book on probability.’
Keith Hirst, The Mathematical Gazette

‘Excellent! A vast number of well-chosen worked examples and exercises guide the reader through the basic theory of probability at the elementary level . . . an excellent text which I am sure will give a lot of pleasure to students and teachers alike.’

International Statistics Institute

‘[W]ould make a fine addition to an undergraduate library. A student with a solid background in calculus, linear algebra, and set theory will find many useful tools of elementary probability here.’

Phil Gilbert, The Mathematics Teacher

‘Stirzaker does an excellent job of developing problem-solving skills in an introductory probability text. Numerous examples and practice exercises are provided that only serve to enhance a student’s problem-solving abilities . . . Highly recommended.’

D.J. Gougeon, Choice

‘The book would make an excellent text for the properly prepared class, a solid instructor’s reference for both probability applications and problems, as well as a fine work for purposes of self-study.’

J. Philip Smith, School Science and Mathematics
Elementary Probability
2nd Edition

by

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# Contents

Preface to the Second Edition  
0 Introduction  
0.1 Chance  
0.2 Models  
0.3 Symmetry  
0.4 The Long Run  
0.5 Pay-Offs  
0.6 Introspection  
0.7 FAQs  
0.8 History  
Appendix: Review of Elementary Mathematical Prerequisites  
1 Probability  
1.1 Notation and Experiments  
1.2 Events  
1.3 The Addition Rules for Probability  
1.4 Properties of Probability  
1.5 Sequences of Events  
1.6 Remarks  
1.7 Review and Checklist for Chapter 1  
1.8 Example: Dice  
1.9 Example: Urn  
1.10 Example: Cups and Saucers  
1.11 Example: Sixes  
1.12 Example: Family Planning  
1.13 Example: Craps  
1.14 Example: Murphy’s Law  
Problems  
2 Conditional Probability and Independence  
2.1 Conditional Probability  
2.2 Independence  
2.3 Recurrence and Difference Equations  
2.4 Remarks
### Contents

2.5 Review and Checklist for Chapter 2  
Worked examples and exercises 65

2.6 Example: Sudden Death 65

2.7 Example: Polya's Urn 66

2.8 Example: Complacency 67

2.9 Example: Dogfight 68

2.10 Example: Smears 69

2.11 Example: Gambler's Ruin 70

2.12 Example: Accidents and Insurance 72

2.13 Example: Protocols 73

2.14 Example: Eddington's Controversy 75

Problems 76

3 Counting 83

3.1 First Principles 83

3.2 Permutations: Ordered Selection 84

3.3 Combinations: Unordered Selection 86

3.4 Inclusion–Exclusion 87

3.5 Recurrence Relations 88

3.6 Generating Functions 90

3.7 Techniques 93

3.8 Review and Checklist for Chapter 3 95

Worked examples and exercises 97

3.9 Example: Railway Trains 97

3.10 Example: Genoese Lottery 98

3.11 Example: Ringing Birds 99

3.12 Example: Lottery 101

3.13 Example: The Ménages Problem 101

3.14 Example: Identity 102

3.15 Example: Runs 103

3.16 Example: Fish 105

3.17 Example: Colouring 106

3.18 Example: Matching (Rencontres) 107

Problems 108

4 Random Variables: Distribution and Expectation 114

4.1 Random Variables 114

4.2 Distributions 115

4.3 Expectation 120

4.4 Conditional Distributions 127

4.5 Sequences of Distributions 130

4.6 Inequalities 131

4.7 Review and Checklist for Chapter 4 134

Worked examples and exercises 137

4.8 Example: Royal Oak Lottery 137

4.9 Example: Misprints 138

4.10 Example: Dog Bites: Poisson Distribution 139
## Contents

4.11 Example: Guesswork 141  
4.12 Example: Gamblers Ruined Again 142  
4.13 Example: Postmen 143  
4.14 Example: Acme Gadgets 144  
4.15 Example: Roulette and the Martingale 145  
4.16 Example: Searching 146  
4.17 Example: Duelling 147  
4.18 Binomial Distribution: The Long Run 149  
4.19 Example: Uncertainty and Entropy 150  
   Problems 151  

5 Random Vectors: Independence and Dependence 158  
5.1 Joint Distributions 158  
5.2 Independence 162  
5.3 Expectation 165  
5.4 Sums and Products of Random Variables: Inequalities 172  
5.5 Dependence: Conditional Expectation 177  
5.6 Simple Random Walk 183  
5.7 Martingales 190  
5.8 The Law of Averages 196  
5.9 Convergence 199  
5.10 Review and Checklist for Chapter 5 203  
   Worked examples and exercises 206  
5.11 Example: Golf 206  
5.12 Example: Joint Lives 208  
5.13 Example: Tournament 209  
5.14 Example: Congregations 210  
5.15 Example: Propagation 211  
5.16 Example: Information and Entropy 212  
5.17 Example: Cooperation 214  
5.18 Example: Strange But True 215  
5.19 Example: Capture–Recapture 216  
5.20 Example: Visits of a Random Walk 218  
5.21 Example: Ordering 219  
5.22 Example: More Martingales 220  
5.23 Example: Simple Random Walk Martingales 221  
5.24 Example: You Can’t Beat the Odds 222  
5.25 Example: Matching Martingales 223  
5.26 Example: Three-Handed Gambler’s Ruin 224  
   Problems 226  

6 Generating Functions and Their Applications 232  
6.1 Introduction 232  
6.2 Moments and the Probability Generating Function 236  
6.3 Sums of Independent Random Variables 239  
6.4 Moment Generating Functions 245  
6.5 Joint Generating Functions 247
## Contents

6.6 Sequences 251
6.7 Regeneration 254
6.8 Random Walks 259
6.9 Review and Checklist for Chapter 6 263
   Appendix: Calculus 265
   Worked examples and exercises 268
6.10 Example: Gambler’s Ruin and First Passages 268
6.11 Example: “Fair” Pairs of Dice 269
6.12 Example: Branching Process 271
6.13 Example: Geometric Branching 272
6.14 Example: Waring’s Theorem: Occupancy Problems 274
6.15 Example: Bernoulli Patterns and Runs 275
6.16 Example: Waiting for Unusual Light Bulbs 277
6.17 Example: Martingales for Branching 278
6.18 Example: Wald’s Identity 279
6.19 Example: Total Population in Branching Problems 280

7 Continuous Random Variables 287
   7.1 Density and Distribution 287
   7.2 Functions of Random Variables 297
   7.3 Simulation of Random Variables 301
   7.4 Expectation 302
   7.5 Moment Generating Functions 306
   7.6 Conditional Distributions 310
   7.7 Ageing and Survival 312
   7.8 Stochastic Ordering 314
   7.9 Random Points 315
   7.10 Review and Checklist for Chapter 7 318
   Worked examples and exercises 321
   7.11 Example: Using a Uniform Random Variable 321
   7.12 Example: Normal Distribution 323
   7.13 Example: Bertrand’s Paradox 324
   7.14 Example: Stock Control 326
   7.15 Example: Obtaining Your Visa 327
   7.16 Example: Pirates 329
   7.17 Example: Failure Rates 330
   7.18 Example: Triangles 330
   7.19 Example: Stirling’s Formula 332
   Problems 334

8 Jointly Continuous Random Variables 337
   8.1 Joint Density and Distribution 337
   8.2 Change of Variables 342
   8.3 Independence 344
   8.4 Sums, Products, and Quotients 348
   8.5 Expectation 351
   8.6 Conditional Density and Expectation 355
8.7 Transformations: Order Statistics 361
8.8 The Poisson Process: Martingales 364
8.9 Two Limit Theorems 368
8.10 Review and Checklist for Chapter 8 371
   Worked examples and exercises 375
8.11 Example: Bivariate Normal Density 375
8.12 Example: Partitions 376
8.13 Example: Buffon’s Needle 377
8.14 Example: Targets 379
8.15 Example: Gamma Densities 380
8.16 Example: Simulation – The Rejection Method 381
8.17 Example: The Inspection Paradox 382
8.18 Example: von Neumann’s Exponential Variable 383
8.19 Example: Maximum from Minima 385
8.20 Example: Binormal and Trinormal 387
8.21 Example: Central Limit Theorem 388
8.22 Example: Poisson Martingales 389
8.23 Example: Uniform on the Unit Cube 390
8.24 Example: Characteristic Functions 390
   Problems 391
9 Markov Chains 396
9.1 The Markov Property 396
9.2 Transition Probabilities 400
9.3 First Passage Times 406
9.4 Stationary Distributions 412
9.5 The Long Run 418
9.6 Markov Chains with Continuous Parameter 425
9.7 Forward Equations: Poisson and Birth Processes 428
9.8 Forward Equations: Equilibrium 431
9.9 The Wiener Process and Diffusions 436
9.10 Review and Checklist for Chapter 9 449
   Worked examples and exercises 451
9.11 Example: Crossing a Cube 451
9.12 Example: Reversible Chains 453
9.13 Example: Diffusion Models 454
9.14 Example: The Renewal Chains 456
9.15 Example: Persistence 457
9.16 Example: First Passages and Bernoulli Patterns 459
9.17 Example: Poisson Processes 461
9.18 Example: Decay 462
9.19 Example: Disasters 463
9.20 Example: The General Birth Process 465
9.21 Example: The Birth–Death Process 466
9.22 Example: Wiener Process with Drift 468
9.23 Example: Markov Chain Martingales 469
9.24 Example: Wiener Process Exiting a Strip 470
<table>
<thead>
<tr>
<th>Contents</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.25 Example: Arcsine Law for Zeros</td>
<td>471</td>
</tr>
<tr>
<td>9.26 Example: Option Pricing: Black–Scholes Formula</td>
<td>472</td>
</tr>
<tr>
<td>Problems</td>
<td>473</td>
</tr>
<tr>
<td>Appendix: Solutions and Hints for Selected Exercises and Problems</td>
<td>478</td>
</tr>
<tr>
<td>Further Reading</td>
<td>514</td>
</tr>
<tr>
<td>Index of Notation</td>
<td>515</td>
</tr>
<tr>
<td>Index</td>
<td>517</td>
</tr>
</tbody>
</table>
Preface to the Second Edition

The calculus of probabilities, in an appropriate form, should interest equally the mathematician, the experimentalist, and the statesman. . . . It is under its influence that lotteries and other disgraceful traps cunningly laid for greed and ignorance have finally disappeared.

François Arago, Eulogy on Laplace, 1827

Lastly, one of the principal uses to which this Doctrine of Chances may be applied, is the discovering of some truths, which cannot fail of pleasing the mind, by their generality and simplicity; the admirable connexion of its consequences will increase the pleasure of the discovery; and the seeming paradoxes wherewith it abounds, will afford very great matter of surprize and entertainment to the inquisitive.

Abraham de Moivre, The Doctrine of Chances, 1756

This book provides an introduction to elementary probability and some of its simple applications. In particular, a principal purpose of the book is to help the student to solve problems. Probability is now being taught to an ever wider audience, not all of whom can be assumed to have a high level of problem-solving skills and mathematical background. It is also characteristic of probability that, even at an elementary level, few problems are entirely routine. Successful problem solving requires flexibility and imagination on the part of the student. Commonly, these skills are developed by observation of examples and practice at exercises, both of which this text aims to supply.

With these targets in mind, in each chapter of the book, the theoretical exposition is accompanied by a large number of examples and is followed by worked examples incorporating a cluster of exercises. The examples and exercises have been chosen to illustrate the subject, to help the student solve the kind of problems typical of examinations, and for their entertainment value. (Besides its practical importance, probability is without doubt one of the most entertaining branches of mathematics.) Each chapter concludes with problems: solutions to many of these appear in an appendix, together with the solutions to most of the exercises.

The ordering and numbering of material in this second edition has for the most part been preserved from the first. However, numerous alterations and additions have been included to make the basic material more accessible and the book more useful for self-study. In
Preface to the Second Edition

particularly, there is an entirely new introductory chapter that discusses our informal and intuitive ideas about probability, and explains how (and why) these should be incorporated into the theoretical framework of the rest of the book. Also, all later chapters now include a section entitled, “Review and checklist,” to aid the reader in navigation around the subject, especially new ideas and notation.

Furthermore, a new section of the book provides a first introduction to the elementary properties of martingales, which have come to occupy a central position in modern probability. Another new section provides an elementary introduction to Brownian motion, diffusion, and the Wiener process, which has underpinned much classical financial mathematics, such as the Black–Scholes formula for pricing options. Optional stopping and its applications are introduced in the context of these important stochastic models, together with several associated new worked examples and exercises.

The basic structure of the book remains unchanged; there are three main parts, each comprising three chapters.

The first part introduces the basic ideas of probability, conditional probability, and independence. It is assumed that the reader has some knowledge of elementary set theory. (We adopt the now conventional formal definition of probability. This is not because of high principles, but merely because the alternative intuitive approach seems to lead more students into errors.) The second part introduces discrete random variables, probability mass functions, and expectation. It is assumed that the reader can do simple things with functions and series. The third part considers continuous random variables, and for this a knowledge of the simpler techniques of calculus is desirable.

In addition, there are chapters on combinatorial methods in probability, the use of probability (and other) generating functions, and the basic theory of Markov processes in discrete and continuous time. These sections can be omitted at a first reading, if so desired.

In general, the material is presented in a conventional order, which roughly corresponds to increasing levels of knowledge and dexterity on the part of the reader. Those who start with a sufficient level of basic skills have more freedom to choose the order in which they read the book. For example, you may want to read Chapters 4 and 7 together (and then Chapters 5 and 8 together), regarding discrete and continuous random variables as two varieties of the same species (which they are). Also, much of Chapter 9 could be read immediately after Chapter 5, if you prefer.

In particular, the book is structured so that the first two parts are suitable to accompany the probability component of a typical course in discrete mathematics; a knowledge of calculus is not assumed until the final part of the book. This layout entails some repetition of similar ideas in different contexts, and this should help to reinforce the reader’s knowledge of the less elementary concepts and techniques.

The ends of examples, proofs, and definitions are indicated by the symbols ●, ■, and ▲, respectively.

Finally, you should note that the book contains a random number of errors. I entreat readers to inform me of all those they find.

D.S.
Oxford, January 2003