NUMBERS, LANGUAGE, AND THE HUMAN MIND

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CHAPTER I

Numbers and objects

A striking feature of numbers is their enormous flexibility. A quality like colour, for instance, can only be conceived for visual objects, so that we have the notion of a red flower, but not the notion of a red thought. In contrast to that, there seem to be no restrictions on the objects numbers can apply to. In 1690 John Locke put it this way, in his 'Essay Concerning Human Understanding':

> number applies itself to men, angels, actions, thoughts; everything that either doth exist, or can be imagined. (Locke 1690: Book II, ch. XVI, § 1)

This refers to our usage of numbers as in 'four men' or 'four angels', where we identify a cardinality. This number assignment works for any objects, imagined or existent, no matter what qualities they might have otherwise; the only criterion here is that the objects must be distinct in order to be counted. In a seminal work on numbers from the nineteenth century, the mathematician and logician Gottlob Frege took this as an indication for the intimate relationship between numbers and thought, a relationship that will be a recurring topic throughout this book:

> The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the existent, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought?[1] (Frege 1884: § 14)

And this is only one respect in which numbers are flexible. Not only can we assign them to objects of all kinds, we can also assign them to objects in ways that are so diverse that, on first sight, they seem not to be related at all. Of these number assignments, the cardinality assignment that is based on counting is probably the first that comes to mind when thinking of

[1] This is quoted from the translation provided by John Austin (1950). One might want to replace 'numerable' by the more specific term 'countable' here, since this translation would be closer to Frege's term 'zählbar' in the German original.

numbers and objects, but it is by no means the only way we can assign numbers to objects.[2]

The same number, say 3, can be used to give the cardinality of pens on my desk ('three pens'); to indicate, together with a unit of measurement, the amount of wine needed for a dinner with friends ('three litres of wine') or the temperature of the mineral water in my glass ('water of 3 °C'); it can tell us the rank of a runner in a Marathon race ('the third runner'); or identify the bus that goes to the opera ('bus #3' / 'the #3 bus'). The following example from a paper on the acquisition of number concepts by Karen Fuson and James Hall illustrates, in one sentence, the various ways in which we employ numbers in our daily lives:

> Despite a *seventy-eight yard run* by *number thirty-four* the Bears lost by *two touchdowns* and dropped into *sixth place*. (Fuson and Hall 1983: 49)

What is it that makes numbers so flexible, allowing them to occur naturally in so many different contexts? How are their different usages related to each other? To answer these questions, let us have a closer look at the different ways numbers apply to objects. In a first approach, let us distinguish three kinds of number assignments: cardinal, ordinal, and nominal assignments. To give you an idea of what I mean by this classification, I give a brief characterisation for each of these number assignments in the following paragraphs (we will analyse them in more detail later in this chapter).

We encounter cardinal number assignments in contexts like 'three pens', 'three litres of wine' and 'three degrees Celsius', where the number indicates how many. In our examples the number indicates how many pens there are, how many litres of wine, and how many degrees Celsius. In the first case, 'three' identifies the cardinality of a set of objects: it tells us how many elements the set of pens has. In the second case, 'three' quantifies over litres, and by so doing identifies, say, the amount of wine needed for dinner. In 'three degrees Celsius', 'three' quantifies over degrees of temperature.

Ordinal number assignments are illustrated by our Marathon example above, 'the third runner'. Unlike in cardinal assignments, the number does not apply to a set, but to an individual element of a set; more precisely, to an individual element of a sequence. For instance, in 'the third runner', 3 is not the number of the entire set of Marathon participants:

[2] Although I speak of 'numbers *and* objects' here, you should be aware of the fact that numbers themselves might also be among the objects, that is, for instance, among the things that are counted.

it indicates the rank of one particular person within the sequence of runners.

Nominal number assignments are ones like 'bus #3'. This is probably not what you would call a typical number context (Fuson and Hall (1983) go so far as to call them '*non-numerical* number contexts', which sounds a bit like a contradiction in terms). However, nominal number assignments are actually quite common in our daily lives; we encounter them in the numbering of football players, in subway and bus systems, and also in telephone numbers and in the numbers on an ID card, to name just a few examples. What these cases have in common is the fact that the numbers identify objects within a set: in nominal assignments, numbers are used like proper names. So rather than thinking of names like 'Mike' or 'Lucy' for buses, we just assign them numbers when we want to identify them, in the same way that we assign numbers to the members of a football team, employ them as telephone numbers, or use ID numbers as a means to identify students within a university.

Figure 1 shows a photograph I took on Fehmarn, an island in the Baltic Sea. As you can see, the lamb in this picture was given the number '289'. This is an instance of a nominal number assignment that distinguishes sheep and might for instance help the farmer to keep track of which lamb belongs to which mother sheep, or which sheep are on which part of the dyke.

So when we investigate the different ways we apply numbers to objects, we must include in our analysis nominal number assignments as well as cardinal and ordinal usages of numbers. A theory that allows us to discuss the different usages under a unified notion of number assignments is the Representational Theory of Measurement, a theory that has been developed within the fields of philosophy and psychology.[3] The Representational Theory of Measurement is concerned with the features that make a number assignment significant; it aims to establish the criteria that make sure that the number we assign to an object does in fact tell us something about the property we want to assess (this property might be, for instance, cardinality, or volume, or the rank in a sequence).

I will use the machinery of this theory for a somewhat different purpose, putting it into service for our investigation of numbers and objects. In particular, I am going to employ the Representational Theory of Measurement

[3] Cf. Stevens (1946); Suppes and Zinnes (1963); Krantz et al. (1971); Roberts (1979); Narens (1985). The Representational Theory has its roots in the philosophical works of Fechner (1858); Helmholtz (1887); Mach (1896); Hölder (1901); and Russell (1903).

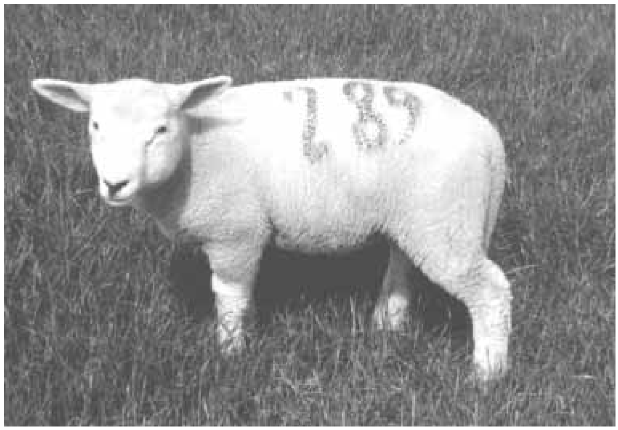

Figure 1 Nominal number assignment: a numbered lamb on Fehmarn

as a generalised theory of number assignments that can give us a handle on the different ways we assign numbers to objects and allows us to find out which properties of numbers we make use of in each case. As a result, we will be in a position to identify the crucial properties that numbers need to have.

In order to take into account all meaningful assignments of numbers to objects, the Representational Theory takes a very broad view of 'measurement'. Within this framework, any assignment of numbers to objects is regarded as an instance of measurement, as long as certain relations between the numbers represent relations between the objects. So, for example, we can regard the correlation between the pile of books in Figure 2 and the numbers from 1 to 6 as a kind of measurement, because the numbers express a relation that holds between the books, namely the relation 'lies further up'. In our example, books receive higher or lower numbers depending on the higher or lower position they occupy within the pile: if a book lies further to the top than another book, it receives a higher number than that book; and vice versa, if a book receives a higher number than another book, you know that it must lie further up than that book.[4] This is depicted in Figure 2: the bottom book has been assigned the number 1, the next book has been numbered '2', and so on, with the top book receiving the highest number in our set, namely 6. Hence, the property we 'measure' in Figure 2 is the position of a book in relation to other books in the pile, and this

[4] Hence, this number assignment is not an instance of counting, where the books could receive numbers in any order, that is, their relative position would not play a role. We will discuss counting as a subroutine in number assignments below.

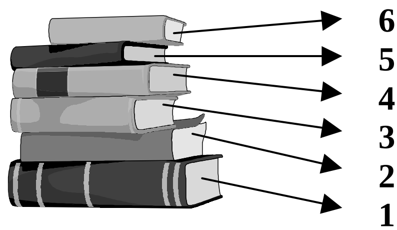

Figure 2 Numbering of books as a form of 'measurement'

property is identified by the '>' relation that holds between the numbers we assigned to the books. (Note that '>' is just the ordering relation for our number sequence here, hence '$2 > 1$' can be understood as '2 comes after 1 in the number sequence' and does not necessarily relate to a quantitative view of numbers, as the reading 'greater than' might suggest. For the time being we leave it open what status numbers have and how their ordering might work. We will tackle this question in the following two chapters.)

Calling this kind of book numbering an instance of 'measurement' may strike you as a bit odd, since this is not what we normally mean when we talk about measuring objects. Our use of the term 'measurement' is normally restricted to cases like the three litres of wine from our dinner-example above, that is, cases where the number assignment tells us something about properties like volume (as in the dinner-example), weight, length, temperature, and so on.

However, while being at odds with our pre-theoretical terminology, it is exactly this generalisation that makes the Representational Theory so powerful for our investigation of number contexts. By expanding the notion of 'measurement' to include all meaningful assignments of numbers to objects, the Representational Theory can capture the whole range of number contexts within one unified framework.

To avoid unnecessary terminological confusion, though, I will talk of *measurement*^{RTM} when I use 'measurement' as a technical term within the Representational Theory of Measurement, and where possible I will use the more intuitive term '(meaningful) number assignment'. Without superscript, 'measurement' will refer to those number assignments where we measure properties like weight, length, or temperature (in accordance with our pre-theoretical terminology).

Applying the Representational Theory to our analysis of the relationship between numbers and objects does not only allow us to determine what is common to all number contexts. As I am going to show in this chapter, it will also enable us to identify the characteristic features of the different kinds of number assignments, and to relate them to each other within a unified framework. Together this will give us a clear idea how numbers work and, in doing so, will give us the key to see what properties are crucial for our concept of numbers and what it is that makes them so powerful.

Let me now introduce the basic elements of the Representational Theory. As mentioned in the introduction, here I will concentrate on the essential theoretical concepts that are relevant for our investigation of numbers and the purpose they serve within the model, while the technical definitions are spelled out in the appendices. You can find the definitions relevant to the present chapter in Appendix 1.

The Representational Theory of Measurement is based on three principal notions: *measurement^{RTM}*, the *scales* underlying a number assignment, and the features that make a numerical statement *meaningful*. Let us have a look at these in turn.

'Measurement$^{\mathrm{RTM}}$' is defined as a mapping between empirical objects (in the above example, the books) and numbers. As mentioned above, we want to express certain relations between our objects by this mapping. This is determined by two requirements. The first requirement is that the objects and the numbers form *relational structures*, that is, sets of elements that stand in specific relationships to each other. For instance, in the book example we regarded the books not as unrelated individual objects, but as elements of a particular pile. The relational structure is here constituted by the relation 'lies further up'. The relation between the numbers that we focused on in our number assignment was '>'. All other relations that might hold between the objects (for example, the size of the books) or between the numbers (for example, odd numbers versus even numbers) are ignored for the purposes of this measurement. The two relational structures are distinguished as *numerical relational structure* (the relational structure constituted by the numbers) and *empirical relational structure* (the one established by the objects, here: the pile of books).

The second requirement for measurement$^{\mathrm{RTM}}$ is that the mapping underlying our number assignment be *homomorphic*. This means that it should translate the property we want to measure in the objects into a property of our numbers. A mapping from a relational structure A (for example, the pile of books) into a relational structure B (the numbers) is homomorphic when it not only correlates the elements of A and B, but also preserves the relations between them. In our example we did not just randomly line up books and numbers, but we linked them in a way such that the relation 'lies further up' from the empirical relational structure (the pile of books) was associated with the '>' relation in our numerical relational structure (the numbers). So if I tell you that one book received the number 1 and another book got the number 3, you know that the second book lies further up than the first one, because $3 > 1$.

Had I assigned numbers to books without taking into account the relation 'lies further up', it would not help you at all to know which numbers two books received if you wanted to find out which one lies further up in the pile. For instance, it might easily turn out that the lowest book had been given the 3, while a higher book had the 1. The mapping was not homomorphic with respect to the two relations. However, this does not necessarily mean that the mapping was random and did not preserve any relations at all. I might not have bothered about which book lies further up in the pile, but my numbering might have focused on another property of the books, for instance their age. In this case, the book that got the 1 might not lie further up in the pile than the one that got the 3, but it would be newer. Hence, the mapping would indeed have been homomorphic, it would have preserved a relation between the books, although a different one: I regarded the books as elements of a different relational structure, namely one that is based on the relation 'is newer than' (rather than the relation 'lies further up than').

Another possibility would be that I focused on the same empirical property as before, namely the position of books in the pile, but employed a different relation between the numbers, for instance the relation 'lesser than'. In this case, you would know that the book with the number 1 lies further up in the pile than the book with the number 3, because $1 < 3$. Again, the mapping would be homomorphic, but this time with respect to a different numerical relational structure.

The interesting aspect for our investigation of numbers and objects is now that from this analysis, it follows that number assignments are essentially links between relations: for the purpose of number assignment it is not so much the correlation between individual objects and individual numbers that counts, but the association between relations that hold between the empirical objects and relations that hold between the numbers. For instance, in our initial number assignment for the book pile, the links between the books and the numbers were grounded in the association of two relations, 'lies further up' and '>', but we could also associate other relations, for instance, the relations 'is newer than' and '>', or the relations 'lies further up' and '<', and as a result we might get different links between individual

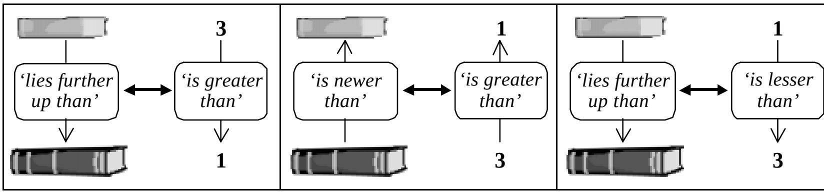

Figure 3 Measurement as an association of relations

books and individual numbers. Figure 3 gives an illustration for the different associations we discussed.

In our example from Figure 2 (illustrated by the graphic on the left), there is no reason why one should assign the number 3 to the grey book on the top if one looks at the book and the number as individuals in their own standing, and there is nothing in the number 1 itself that makes it particularly prone to be assigned to the black book on the bottom. We are associating relational structures here: it is because of their relations with other numbers and with other books, respectively, that 3 is linked up with the grey book and 1 with the black book. To put it plainly: we link up 3 with the grey book and 1 with the black book because 3 is greater than 1 and the grey book is higher in the pile than the black one – and not because 3 has anything to do with the grey book or 1 with the black book if we look at them as individuals outside their respective systems (namely, the number sequence and the pile of books). The links are dependent on the systems. Accordingly, I will call such a linking an instance of 'system-dependent linking', or in short: *dependent linking*.

So when assigning numbers to objects, we are not interested in numbers as individuals in their own right; it is the relations between them that we want. Accordingly, when analysing the different kinds of number assignments, I will focus on the *relations* between numbers that are relevant in each case, that is, I will focus on the numerical relations that reflect, in each case, the properties we want to assess in the objects.

The homomorphism that establishes the number assignment identifies its underlying *scale*: given a certain empirical property, the scale tells us which relation between the numbers is relevant in the assignment, that is, which numerical relational structure the mapping employs. Accordingly, in number assignments that are based on the same type of scale, the empirical property is reflected by the same relation between the numbers. This means that the number assignments can be transformed into each other. For instance in the measurement of weight, we can transform a number assignment like ‘The pumpkin weighs 3 kg’ into ‘The pumpkin weighs 6.6138 lb’, because both are based on the same type of scale.

In this example, the ‘>’ relation between numbers reflects the property ‘weight’ (more precisely: it reflects the relation ‘weighs more’ between the objects), so if you tell me your measurement yielded ‘3 kg’ for one pumpkin, and ‘2 kg’ for another pumpkin, I know that the first one weighs more than the second one, and in just the same way I know that a 6 lb pumpkin weighs more than one of 4 lb. In both cases, the ‘>’ relation between the numbers is associated with the relation ‘weighs more than’ in a particular way: in both cases, the numbers tell us that the first pumpkin weighs 1½ times as much as the second pumpkin. We can always transform one number assignment into another one, as long as the association between the relevant numerical and empirical relations stays intact, because it is this association of relations that establishes our number assignment. The transformations that satisfy this requirement are the *admissible transformations* for a scale.

A numerical statement can now be defined as *meaningful* if and only if its truth-value is constant under admissible scale transformations (cf. Suppes and Zinnes 1963: 66). This means that if a numerical statement is true (or false), it should still be true (or false, respectively) when we translate it into another numerical statement, as long as this transformation is one that is allowed for the type of scale that underlies our number assignment. To put it plainly, if you tell me that ‘The pumpkin weighs 3 kg’ is true, but ‘The pumpkin weighs 6.6138 lb’ turns out to be false, something is wrong with your measurement. In this case, the numbers you apply to your objects seem not to reflect the property you intended to measure (namely, weight).

This is now a good point to see where we have got so far in our investigation of numbers and objects. Using the Representational Theory of Measurement, we have outlined a unified view of number assignments as mappings from empirical objects to numbers, and we have spelled out the characteristic features of this mapping: the empirical objects enter the number assignment with respect to a particular property, the property we want to assess. This property is then associated with a relation that holds between the numbers, and it is this association that determines the correlation between numbers and objects and makes the number assignment meaningful. According to this analysis, the numbers and objects are not correlated as individuals, but as elements of two systems, they are correlated in a way we described as ‘dependent linking’.

Having thus spelled out the framework for our investigation, we are now in a position to examine the different ways in which we assign numbers to objects, as different instances of the same general scheme of 'measurement^{RTM}'. Above, I identified three different types of number assignments: cardinal, ordinal, and nominal assignments. Let us have a look at each of the three types in turn. In each case, we will ask which properties of numbers we make use of, that is, which properties numbers need to have in order to represent properties of empirical objects in a meaningful way. In our discussion I will concentrate on the natural numbers starting from 1, that is on the positive integers, ignoring, for the time being, 0, negative numbers, rational numbers, and so forth.

CARDINAL NUMBER ASSIGNMENTS:[5] '3 PENS', '3 KG', '3 °C'

In cardinal number assignments, the mapping from empirical objects to numbers takes advantage of the numerical relation '$>$' (or '$<$', respectively). As mentioned above, we can distinguish two subclasses of cardinal number assignments: those like '3 pens', on the one hand, and those like 'a 3 kg pumpkin' or 'bathwater of 3 °C' on the other hand. In the first case, the number identifies the cardinality of a set, that is, '3' tells us how many elements the set has, in this case: how many pens there are on my desk. The second class of cardinal assignments is the kind we mean when talking about 'measurement' in the familiar, pre-theoretical usage of the word. In these cases, '3' does not give us the cardinality of the measured objects (the pumpkin or the bathwater in our examples), but instead the cardinality of certain units of measurement (kg or °C). With the help of these units, we can use numbers to measure properties like weight or temperature. In 'a 3 kg pumpkin', '3' tells us how many kilograms, and '3 kg' specifies the pumpkin's weight. In 'bathwater of 3 °C', '3' tells us how many degrees Celsius, and '3 °C' specifies the temperature of the bathwater. As I am going to argue later in this section, we can regard these measurements as a special case of cardinal number assignments. (Admittedly bathwater of 3 °C is not a very pleasant image. If this causes you cold shivers, just think of a spa where you might want to dip into a 3 °C bath tub after a really hot sauna . . .)

Cardinality assignments for sets of objects: '3 pens'

In a number assignment like '3 pens', we apply a number to a set of objects; for instance the number 3 to the set of pens on my desk. This number tells us the cardinality of that set; it tells us how many pens there are

[5] Recall that with this terminology I refer to a classification of number assignments, that is, by 'cardinal number assignments' I do not mean the assignment of 'cardinal numbers', but number assignments that are cardinal in nature.

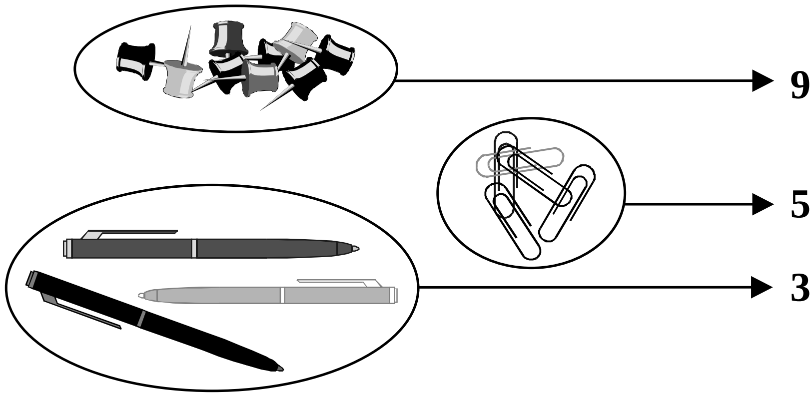

Figure 4 'Measurement^{RTM}' of cardinality

on my desk. This is what distinguishes these from other kinds of number assignments: when we use numbers to represent cardinalities, it is crucial that the empirical objects in our assignment are not individuals, but sets. The empirical relation that we focus on in our number assignment is 'has more elements than', and it is linked up with the numerical relation '>'.

Figure 4 illustrates this kind of number assignment. The sets in my example are different sets of utensils from my desk: a set of plastic pins, a set of paperclips, and the by now well-known set of pens. What makes this number assignment meaningful? How can numbers indicate cardinalities of sets? 3, 5, and 9 in Figure 4 occur as single objects. So where does the cardinality come from; what property of numbers do we make use of when we 'measure^{RTM}' cardinality in our empirical objects? To answer this question, let us have an explicit demonstration of the procedure that leads to a cardinal number assignment. The task is to tell the cardinality of the set in Figure 5, or, to put it a bit clumsily in measurement terms: to map the empirical object 'set of stars' in Figure 5 onto a number that indicates its cardinality.

The most straightforward way to determine the correct number is to assign a number to every star, starting with 1, then 2, and so on, until all the stars have a number; in short: we count the stars. Then we use the last number from the counting procedure – which will be 20 if the count is correct – in order to indicate the cardinality of the whole set of stars: 20 stars.

This gives us an insight into what lies behind our cardinal number assignment: first, we map the *elements* of the empirical object (that is, the

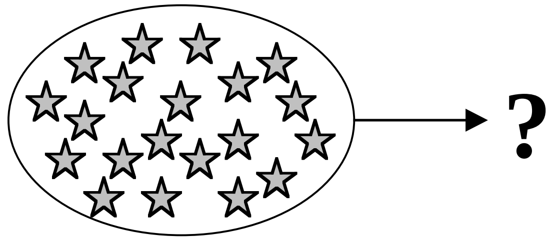

Figure 5 How many stars?

elements of our set) onto numbers, we 'count' the set; in our example: we mapped the elements of our empirical object 'set of stars' onto the numbers from 1 to 20. On the basis of this 'auxiliary number assignment', we then map the whole set onto one number, namely onto the last number we used in counting.

What makes this assignment meaningful is the fact that the counted set has as many elements as the initial sequence of numbers we used in counting: there are as many stars as there are numbers from 1 to 20. This is because when counting the stars, we match each star with exactly one number. We establish a one-to-one-mapping between stars and numbers that guarantees that the set of stars and the set of numbers we used in counting have the same cardinality.

So the number we assigned to the set of stars, 20, is something like a placeholder for a whole set of numbers, the numbers from 1 to 20. The cardinality of this *number set* represents the cardinality of our empirical object, the set of stars. In a nutshell, in cardinal number assignments we map a set onto a number *n* such that the set of numbers less than or equal to *n* – the number sequence from 1 to *n* – has as many elements as that set.

Accordingly, when employing counting as a verification procedure for our cardinal number assignment, we do not use numbers in random order. For instance, in Figure 5, you presumably did not count "5, 7, 1, 12, 24, 8, 3", and so on, and then suddenly came up with 20 as your last number. Much more likely, you started with 1, then 2, and so on, applying numbers to stars sequentially, with each number being followed by its successor in the number line. This is crucial to make sure the last number, 20, is the endpoint of a particular subsequence of the set N of natural numbers, namely the sequence <1, 2, 3, . . . , 20>. This sequence consists of the number 20 and all predecessors of 20 within the number line.

In Figure 6, this procedure is illustrated for our set of pens: we establish an ordered one-to-one mapping between the pens and the numbers from 1 to 3, in sequential order (column A). On this basis, we assign the number 3

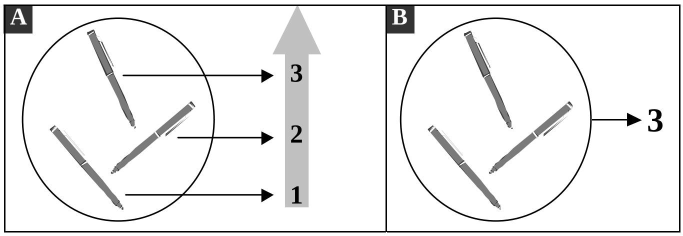

Figure 6 Procedure underlying cardinal number assignments ('3 pens')

to the entire set of pens (column B): 3 functions as a place-holder for the set consisting of the numbers 1, 2, and 3 – a set that has as many elements as the set of pens. By so doing, 3 can indicate the cardinality of the set of pens.

Let me sum up our analysis of cardinal number assignments. In cardinal number assignments, the empirical objects are sets, and the property we want to assess is their numerical quantity: their cardinality as identified by a number. When we apply a number n to a set s, we want to specify how many elements s has. The numerical statement is meaningful if and only if the number sequence from 1 to n has the same cardinality as the set s, that is, if there is a one-to-one-mapping between the numbers up to n and the elements of s. This one-to-one-mapping can be established via counting.

Cardinal number assignments like the ones described here – that is, those that are about cardinality and not about properties like weight or temperature – are based upon absolute scales. These scales do not allow any transformations but the identity transformation, hence a transformation that does not change the number assignment at all: it yields for a numerical statement like 'There are 3 pens on my desk' the very same statement ('There are 3 pens on my desk'). This is because in these cardinal number assignments, we unambiguously refer to the sequence of numbers starting with 1, and set them in a one-to-one correlation with the elements of the empirical set. Accordingly, there is always only one number that applies to a given set.

For our investigation of numbers we now want to know what it is in particular that qualifies numbers for this kind of assignment: what enables numbers to represent cardinalities the way they do? What is the system on which the dependent linking is based in this case? As we have seen, when indicating a cardinality, a number n points to the set of numbers less than or equal to n. The elements of this set form a sequence from 1 to n, they constitute an initial sequence of the natural numbers N. This is possible because of the sequential order of numbers, as established by the '$<$'-relation: every number has a fixed position within N, hence, for a particular number *n*, we can always specify the set of numbers up to *n*, the sequence from 1 to *n*. This sequence is unique for every number, and it has always a unique cardinality.

Take any two natural numbers, say 3 and 5, and you can always identify the respective sets of predecessors (including the numbers themselves) that are relevant in cardinal number assignments: the set with the elements '1, 2, 3' in the case of 3, and '1, 2, 3, 4, 5' in the case of 5. These sets are different for any two different numbers, and they never have the same cardinality, because two different numbers will always occupy different positions within the number line, so they will always have a different set of predecessors.

In our example, the set we identified for 5 has more elements than that for 3, and it has either more or less elements than the sequence from 1 to *n* for any other number *n*. This might sound trivial, but it is part and parcel of our use of numbers in cardinal number assignments. If 3 and 5 were not elements of a sequence – and what is more, of the same sequence – in these assignments, we would have no guarantee that they indicate different cardinalities, or any cardinalities at all. Figure 7 depicts the way 3 and 5 relate to sets of unique cardinalities – namely to distinct initial sequences of N – due to their position in the number sequence.

The sequential position within N is hence the crucial numerical feature in cardinal number assignments. It is the sequential order of N that enables us to represent cardinalities with numbers the way we do. What is more, it is the only feature N needs to have for this task. As our discussion has shown, there are no other requirements on numbers in cardinal number assignments; their sequential position within N was the only feature we referred to when using numbers to identify the cardinalities of our empirical objects. Being an element of a sequence is hence an essential property of numbers.

We can define a sequence as a particular set that is ordered by a relation R (for instance, '>') with the following properties: R is antireflexive

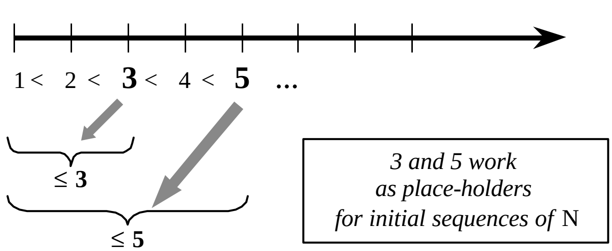

Figure 7 Sequential order as the basis for cardinal number assignments

(no number is greater than itself), asymmetric (if a number x is greater than another number y, then y cannot be greater than x), and transitive (if a number x is greater than another number y, and y is greater than a third number z, then x is also greater than z), and the order introduced by R must be total (for any two different numbers x and y, one is greater than the other). If we want to use this sequence in counting, we must make sure that there is a unique path to each of its elements. This is the case if each of its elements has only finitely many predecessors, starting with an initial element, '1'. I will call a sequence that fulfils these conditions a 'progression' (cf. also the definition given in Appendix 1).

Cardinal number assignments including measures: 'a 3 kg pumpkin', 'water of 3 °C'

The second kind of cardinal number assignments is the measurement we meet in examples like 'a 3 kg pumpkin' or 'bathwater of 3 °C'. This is the kind of number assignment we refer to when using the term 'measurement' in the familiar sense. Unlike in number assignments such as '3 pens', the empirical objects in measurement – the pumpkin and the bathwater in our examples – are not treated as sets. Of course, they can be sets anyway, as in '3 kg of apples' (where we measure the weight of a set of apples), but this is not what the number assignment takes into account; there is no set-specific feature of the empirical objects (like cardinality) that the number assignment refers to. My three pens weigh together 20 g, and this is about as much as the floppy disc lying next to them on my desk, but to find that out, I totally ignore the fact that the pens form a set of three, whereas the floppy disc is a single object – for the purpose of measurement, it comes out just the same.

So whereas in cardinality assessments it is imperative for the number assignments that the empirical objects be sets, we ignore this feature in measurement; sets and non-sets are treated alike. In fact, cardinality assessments are the only kind of number assignments where the numbers apply to sets *as sets*, and only to sets. This is because only for sets can we identify the number of elements. Only sets have a cardinality, whereas the properties we identify in other kinds of number assignments apply to individual objects as well as sets, so 'set-ness' is not a feature we need to focus on; it is not part of our empirical relational structure in the number assignment.

In measurement, we are concerned with empirical properties like weight, length, or temperature: dimensional properties other than cardinality. As for all cardinal number assignments, we associate the empirical property with the numerical relation '>' in our mapping. Let us distinguish two kinds of measurement with respect to the property we relate to: measurement of extensive properties like weight and length, and measurement of non-extensive properties like temperature.

An extensive property is one that depends on the amount of the measured objects. This is the case because an extensive property changes with the size of our empirical sample; it increases when we add more objects with the same property. Take weight: when you have a basket with apples, and you add more apples, it becomes heavier (its weight increases), because the property 'weight' is extensive. It is the same with volume: pour more wine into a glass, and the volume of wine in that glass increases. And two of my pens, if arranged end to end in a line, are together longer than one pen, because 'length' is an extensive property, too.

In contrast to this, a non-extensive property like temperature does not increase if we add 'more of the same'. When you add more wine of the same temperature to the wine in a glass, the wine becomes more, but it does not become warmer: the volume of the wine – an extensive property – increases, but its temperature does not. The tea in my mug might become hotter when I add fresh tea from the thermos to it, but that is because when we mix fluids their temperature equals out, not because it adds up.

Direct measurement: 'a 3 kg pumpkin'

Let us first have a look at measurements as expressed in 'a 3 kg pumpkin'. I call this 'direct measurement', because – as we will see below – in this instance of measurement we have a direct link between cardinality and our units of measurement. In this kind of number assignment, we want the numbers to tell us something about an extensive property of the empirical objects. In particular, we want the relation '>' between the numbers to reflect differences in this extensive property between our empirical objects.

Figure 8 illustrates this kind of cardinal number assignment. In the example, we measure the weight of three empirical objects: a watermelon, a pumpkin, and a squash. Higher or lower numbers are applied with respect to higher or lower weight: the watermelon as the heaviest object receives the highest number, 5; the squash is the lightest vegetable with 1, and the pumpkin is in the middle and gets the number 3. This does not quite look like a weight measurement as we know it. What is missing in this example is the specification of measurement units. When asking for the weight of a pumpkin in a grocery store, you would probably not be satisfied with an answer like '3'. What you expect the sales person to tell you is rather something like '3 kg' or '3 pounds'. Expressions like 'kg' or 'pound'

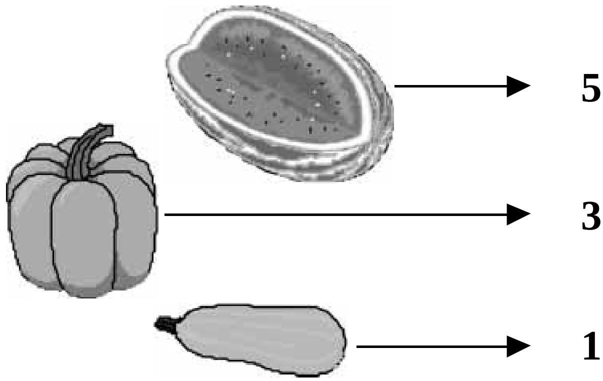

Figure 8 Direct measurement as a cardinal number assignment: measurement of weight

identify different units for the measurement of weight; they introduce supporting measure items that we employ for our number assignment. In the following discussion, I am going to argue that measure items relate extensive properties like weight to cardinality and thus enable us to draw on the cardinal aspect of numbers in the course of measurement.

Before we discuss the use of measure items in detail, let me first illustrate this with an example, to give you an idea of the procedure that underlies this kind of number assignment. Figure 9 spells out the use of measure items for the pumpkin from our example. To measure the pumpkin's weight, I employed objects of 1 kg as measure items. I then found a set of these items that weigh together as much as the pumpkin. Counting the elements of this set yielded the number I assigned to the pumpkin. Hence I identified the weight of the pumpkin via the cardinality of a set of measure items: three kilograms. Let us have a look at what lies behind the usage of measure items illustrated in Figure 9. How does it work? And why do we employ measure items in the first place? The problem we face in measurement is that we want the '>' relation between numbers to indicate 'more' ('weighs more than', 'is longer than' etc.) – measurement being a form of cardinal number assignment – but, since our empirical objects are not sets, we cannot employ the counting procedure for them that worked so well in our first kind of cardinal number assignments (i.e., assignments as in '3 pens') – it is of no use to try and count a pumpkin in order to find out its weight.

However, when measuring extensive properties like weight, there is always a way to relate the property we want to measure to some cardinality. Remember our apple example from page 23 above: when we add more apples to a basket of apples, it becomes heavier. So the total weight of the

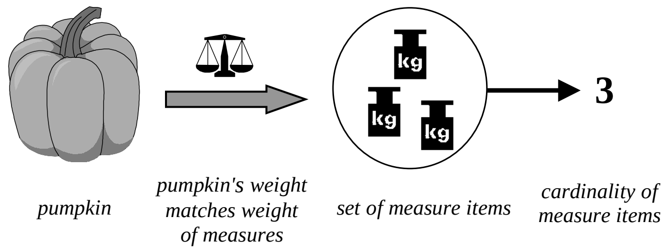

Figure 9 Procedure underlying direct measurement: weight ('a 3 kg pumpkin')

apples increases with their cardinality – the more apples, the heavier the basket. If we find a way to employ this relation between cardinality and weight for our measurement, we can base our measurement on numerical quantification after all. This is what the seventeenth- to eighteenth-century philosopher Gottfried Wilhelm Leibniz called the 'recourse from continuous quantity to discrete quantity', when he discussed the measurement of size ('grandeur'), another example of measurement:

> we cannot distinctly recognise sizes without having recourse to whole numbers . . . , and so, where distinct knowledge of size is sought, we must leave continuous quantity and have recourse to discrete quantity. (Leibniz 1703/05: § 4)

If we use standardised measure items now instead of apples, we have an elegant way to realise this recourse, that is, to measure an extensive property like weight via cardinality. Two requirements are important for the objects that are to fulfil the task of measure items:

(1) The measure items must possess the empirical property that we want to measure, and they must be identical with respect to this property.

(2) The property must be additive, that is, it must be a property that can be associated with cardinality. More precisely, there must be a physical operation of concatenation for the objects that has the effect of addition if we represent it numerically.

The first requirement means, for instance, that for the measurement of weight each measure item must have the same weight. This might sound circular at first – after all, was not weight what we set out to measure eventually? So how are we supposed to know whether our measure items have the same weight beforehand? This is possible because, in order to know whether two objects have *the same weight*, we need not know *which weight* they have. We can compare the weight of two metal blocks that we might want to use as measure items and determine whether they have the same weight or not, for instance by using a balance, without measuring their weight in terms of numbers and units of measurements. And, similarly, we can arrange two objects parallel to each other, starting at the same point, in order to determine whether they have the same length. This is a general phenomenon: we can always compare two objects with respect to an empirical property without employing a number assignment.

The second requirement, namely that the relevant property of our measure items must be additive, is met by extensive properties. As illustrated in our examples above, the values of these properties for several objects add up. For instance, the weight of our set of apples increased and decreased along with its cardinality. In this case, to 'concatenate' means 'put together': if we put together two apples, their total weight adds up. Similarly, the total length of several pens, when arranged end to end in a straight line ('concatenation' of length), increases and decreases along with their cardinality: if we lay two pens end to end, their total length adds up.

If these requirements are met, the cardinality of a set of measure items is linked up with the property we want to measure (more precisely: the two properties are monotonically covarying). For instance, when we measure weight as illustrated in Figure 9 above, our measure items could be metal blocks of 1 kg each. Since each of these blocks has the same weight, the cardinality of the set of blocks indicates their total weight – if we know how many blocks there are, we know how heavy they are together.

Using this set of measure items, we can then measure the weight of an empirical object like our pumpkin. All we have to do is make sure that the total weight of our set of measure items matches that of the pumpkin. And again, in order to do this, we do not have to employ numbers yet, but can make a simple comparison, for example with the help of a balance as sketched in Figure 9. The cardinality of the set of metal blocks then indicates the weight of the pumpkin: the cardinality tells us the total weight of the blocks, and, since the blocks together weigh as much as the pumpkin, it also tells us the pumpkin's weight. This way, measure items mediate between cardinality and an extensive property like weight, and it is this correlation that makes the number assignment meaningful.

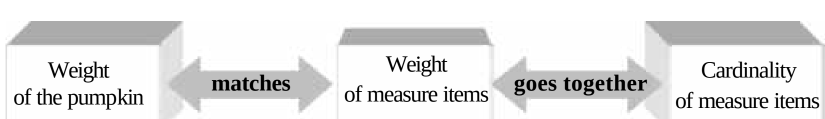

Figure 10 Measure items as agents between an extensive property (weight) and cardinality

Standardised units like 'kg' or 'metre' point to measure items of a specified weight or length, respectively; so when using these units in our number assignments, we do not have to identify the particular objects one might use in the course of measurement (particular metal blocks etc.), but can relate to standardised units. We will come back to the use of different measure items in our discussion of measure concepts in chapter 6. What is important for our present investigation is that the introduction of measure items enables us to base direct measurement *entirely* on cardinality assignments, as far as numbers are concerned. We do not make use of any properties of numbers here that we did not use in simple cardinality assignments (as in '3 pens') anyway. All we need to ask from our numbers is that they form a progression, so we can use them to identify cardinalities. The additional features of measurement come in with measure items, but they do not put any further requirements on numbers.

Let me summarise the analysis of direct measurement that I have proposed here. In direct measurement, the empirical objects are not sets; they do not have any elements that we can count for the number assignment. So what we do in this case is we find something that *is* a set and quantify it instead. This set is a set of measure items, that is, its cardinality is connected with the extensive property we want to measure items in our empirical objects. The cardinality of this set of measure items can therefore identify our extensive property. In a nutshell, direct measurement boils down to a cardinality assignment for measure items. Accordingly, nothing is required for numbers over and above the condition that they are elements of a progression[6] – the numerical feature we need for cardinality assignments.

Because there are always different kinds of measure items one can use to measure the same property (that is, measure items of a different – standardised – weight, length etc.), this kind of number assignment is not based on absolute scales (like our first kind of cardinal number assignments), but on rational scales. This means that you can always multiply the numbers you used in our measurement with a positive real number – provided you are using the same number in each measurement procedure – without changing the truth (or falsity) of your numerical statement. So for rational scales, the numbers themselves are not kept invariant (as is the case for absolute scales), only the ratios between them are: in a number assignment based on a rational scale, we determine uniquely the ratios between the values we assign to our empirical objects.

[6] That is, they are elements of a sequence each of whose elements has only finitely many precursors (see the definition of a progression on p. 23 above).