Cambridge University Press
0521831660 - Functional Analysis for Probability and Stochastic Processes: An Introduction
A. Bobrowski
Excerpt
More information

1

Preliminaries, notations and conventions

Finite measures and various classes of functions, including random variables, are examples of elements of natural Banach spaces and these spaces are central objects of functional analysis. Before studying Banach spaces in Chapter 2, we need to introduce/recall here the basic topological, measure-theoretic and probabilistic notions, and examples that will be used throughout the book. Seen from a different perspective, Chapter 1 is a big "tool-box" for the material to be covered later.

1.1 Elements of topology

1.1.1 *Basics of topology* We assume that the reader is familiar with basic notions of topology. To set notation and refresh our memory, let us recall that a pair $(S, \mathcal{U})$ where S is a set and $\mathcal{U}$ is a collection of subsets of S is said to be a **topological space** if the empty set and S belong to $\mathcal{U}$, and unions and finite intersections of elements of $\mathcal{U}$ belong to $\mathcal{U}$. The family $\mathcal{U}$ is then said to be the **topology** in S, and its members are called **open sets**. Their complements are said to be **closed**. Sometimes, when $\mathcal{U}$ is clear from the context, we say that the set S itself is a topological space. Note that all statements concerning open sets may be translated into statements concerning closed sets. For example, we may equivalently define a topological space to be a pair $(S, \mathcal{C})$ where $\mathcal{C}$ is a collection of sets such that the empty set and S belong to $\mathcal{C}$, and intersections and finite unions of elements of $\mathcal{C}$ belong to $\mathcal{C}$.

An open set containing a point $s \in S$ is said to be a **neighborhood** of s. A topological space $(S, \mathcal{U})$ is said to be **Hausdorff** if for all $p_1, p_2 \in S$, there exists $A_1, A_2 \in \mathcal{U}$ such that $p_i \in A_i, i = 1, 2$ and $A_1 \cap A_2 = \emptyset$. Unless otherwise stated, we assume that all topological spaces considered in this book are Hausdorff.

The **closure**, $cl(A)$, of a set $A \subset S$ is defined to be the smallest closed set that contains A. In other words, $cl(A)$ is the intersection of all closed sets that contain A. In particular, $A \subset cl(A)$. A is said to be **dense** in S iff $cl(A) = S$.

A family $\mathcal{V}$ is said to be a **base** of topology $\mathcal{U}$ if every element of $\mathcal{U}$ is a union of elements of $\mathcal{V}$. A family $\mathcal{V}$ is said to be a **subbase** of $\mathcal{U}$ if the family of finite intersections of elements of $\mathcal{V}$ is a base of $\mathcal{U}$.

If $(S,\mathcal{U})$ and $(S',\mathcal{U}')$ are two topological spaces, then a map $f : S \to S'$ is said to be **continuous** if for any open set A' in $\mathcal{U}'$ its inverse image $f^{-1}(A')$ is open in S.

Let S be a set and let $(S',\mathcal{U}')$ be a topological space, and let $\{f_t, t \in \mathbb{T}\}$ be a family of maps from S to S' (here $\mathbb{T}$ is an abstract indexing set). Note that we may introduce a topology in S such that all maps f_t are continuous, a trivial example being the topology consisting of all subsets of S. Moreover, an elementary argument shows that intersections of finite or infinite numbers of topologies in S is a topology. Thus, there exists the smallest topology (in the sense of inclusion) under which the f_t are continuous. This topology is said to be **generated** by the family $\{f_t, t \in \mathbb{T}\}$.

1.1.2 **Exercise** Prove that the family $\mathcal{V}$ composed of sets of the form $f_t^{-1}(A'), t \in \mathbb{T}, A' \in \mathcal{U}'$ is a subbase of the topology generated by $f_t, t \in \mathbb{T}$.

1.1.3 *Compact sets* A subset K of a topological space $(S,\mathcal{U})$ is said to be **compact** if every open cover of K contains a finite subcover. This means that if $\mathcal{V}$ is a collection of open sets such that $K \subset \bigcup_{B\in\mathcal{V}} B$, then there exists a finite collection of sets $B_1, \dots, B_n \in \mathcal{V}$ such that $K \subset \bigcup_{1=1}^{n} B_i$. If S is compact itself, we say that the space $(S,\mathcal{U})$ is compact (the reader may have noticed that this notion depends as much on S as it does on $\mathcal{U}$). Equivalently, S is compact if, for any family $C_t, t \in \mathbb{T}$ of closed subsets of S such that $\bigcap_{t\in\mathbb{T}} C_t = \emptyset$, there exists a finite collection $C_{t_1}, \dots, C_{t_n}$ of its members such that $\bigcap_{i=1}^{n} C_{t_i} = \emptyset$. A set K is said to be **relatively compact** iff its closure is compact. A topological space $(S,\mathcal{U})$ is said to be **locally compact** if for every point $p \in S$ there exist an open set A and a compact set K, such that $s \in A \subset K$. The **Bolzano–Weierstrass Theorem** says that a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded. In particular, $\mathbb{R}^n$ is locally compact.

1.1.4 *Metric spaces* Let $\mathbb{X}$ be an abstract space. A map $d : \mathbb{X}\times\mathbb{X} \to \mathbb{R}^+$ is said to be a **metric** iff for all $x, y, z \in \mathbb{X}$

(a) $d(x, y) = d(y, x)$,
(b) $d(x, y) \le d(x, z) + d(z, y)$,
(c) $d(x, y) = 0$ iff $x = y$.

A sequence x_n of elements of $\mathbb{X}$ is said to **converge** to $x \in \mathbb{X}$ if $\lim_{n\to\infty} d(x_n, x) = 0$. We call x the **limit** of the sequence $(x_n)_{n\ge1}$ and write $\lim_{n\to\infty} x_n = x$. A sequence is said to be **convergent** if it converges to some x. Otherwise it is said to be **divergent**.

An **open ball** $B(x, r)$ with radius r and center x is defined as the set of all $y \in \mathbb{X}$ such that $d(x, y) < r$. A closed ball with radius r and center x is defined similarly as the set of y such $d(x, y) \le r$. A natural way to make a metric space into a topological space is to take all open balls as the base of the topology in $\mathbb{X}$. It turns out that under this definition a subset A of a metric space is closed iff it contains the limits of sequences with elements in A. Moreover, A is compact iff every sequence of its elements contains a converging subsequence and its limit belongs to the set A. (If S is a topological space, this last condition is necessary but not sufficient for A to be compact.)

A function $f : \mathbb{X} \to \mathbb{Y}$ that maps a metric space $\mathbb{X}$ into a normed space $\mathbb{Y}$ is continuous at $x \in \mathbb{X}$ if for any sequence x_n converging to x, $\lim_{n\to\infty} f(x_n)$ exists and equals $f(x)$ (x_n converges in $\mathbb{X}$, $f(x_n)$ converges in $\mathbb{Y}$). f is called continuous if it is continuous at every $x \in \mathbb{X}$ (this definition agrees with the definition of continuity given in 1.1.1).

1.2 Measure theory

1.2.1 *Measure spaces and measurable functions* Although we assume that the reader is familiar with the rudiments of measure theory as presented, for example, in [103], let us recall the basic notions. A family $\mathcal{F}$ of subsets of an abstract set Ω is said to be a **σ-algebra** if it contains Ω and complements and countable unions of its elements. The pair $(\Omega, \mathcal{F})$ is then said to be a **measurable space**. A family $\mathcal{F}$ is said to be an **algebra** or a **field** if it contains Ω, complements and finite unions of its elements.

A function μ that maps a family $\mathcal{F}$ of subsets of Ω into $\mathbb{R}^+$ such that

$$\mu\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.1}$$

for all pairwise-disjoint elements $A_n, n \in \mathbb{N}$ of $\mathcal{F}$ such that the union $\bigcup_{n\in\mathbb{N}} A_n$ belongs to $\mathcal{F}$ is called a **measure**. In most cases $\mathcal{F}$ is a σ-algebra but there are important situations where it is not, see e.g. 1.2.8 below. If $\mathcal{F}$ is a σ-algebra, the triple $(\Omega, \mathcal{F}, \mu)$ is called a **measure space**.

Property (1.1) is termed **countable additivity**. If $\mathcal{F}$ is an algebra and $\mu(S) < \infty$, (1.1) is equivalent to

$$\lim_{n\to\infty} \mu(A_n) = 0 \quad \text{whenever } A_n \in \mathcal{F}, A_n \supset A_{n+1}, \bigcap_{n=1}^{\infty} A_n = \emptyset. \tag{1.2}$$

The reader should prove it.

The smallest σ-algebra containing a given class $\mathcal{F}$ of subsets of a set is denoted $\sigma(\mathcal{F})$. If Ω is a topological space, then $\mathcal{B}(\Omega)$ denotes the smallest σ-algebra containing open sets, called the **Borel σ-algebra**. A measure μ on a measurable space $(\Omega, \mathcal{F})$ is said to be **finite** (or **bounded**) if $\mu(\Omega) < \infty$. It is said to be **σ-finite** if there exist measurable subsets Ω_n, $n \in \mathbb{N}$, of Ω such that $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup_{n\in\mathbb{N}} \Omega_n$.

A measure space $(\Omega, \mathcal{F}, \mu)$ is said to be **complete** if for any set $A \subset \Omega$ and any measurable B conditions $A \subset B$ and $\mu(B) = 0$ imply that A is measurable (and $\mu(A) = 0$, too). When Ω and $\mathcal{F}$ are clear from the context, we often say that the measure μ itself is complete. In Exercise 1.2.10 we provide a procedure that may be used to construct a complete measure from an arbitrary measure. Exercises 1.2.4 and 1.2.5 prove that properties of complete measure spaces are different from those of measure spaces that are not complete.

A map f from a measurable space $(\Omega, \mathcal{F})$ to a measurable space $(\Omega', \mathcal{F}')$ is said to be $\mathcal{F}$ **measurable**, or just **measurable** iff for any set $A \in \mathcal{F}'$ the inverse image $f^{-1}(A)$ belongs to $\mathcal{F}$. If, additionally, all inverse images of measurable sets belong to a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, then we say that f is $\mathcal{G}$ **measurable**, or more precisely $\mathcal{G}/\mathcal{F}'$ **measurable**. If f is a measurable function from $(\Omega, \mathcal{F})$ to $(\Omega', \mathcal{F}')$ then

$$\sigma_f = \{A \in \mathcal{F} | A = f^{-1}(B) \text{ where } B \in \mathcal{F}'\}$$

is a sub-σ-algebra of $\mathcal{F}$. σ_f is called the σ-algebra **generated by** f. Of course, f is $\mathcal{G}$ measurable if $\sigma_f \subset \mathcal{G}$.

The σ-algebra of Lebesgue measurable subsets of a measurable subset $A \subset \mathbb{R}^n$ is denoted $\mathcal{M}_n(A)$ or $\mathcal{M}(A)$ if n is clear from the context, and the Lebesgue measure in this space is denoted leb_n, or simply leb. A standard result says that $\mathcal{M} := \mathcal{M}(\mathbb{R}^n)$ is the smallest complete σ-algebra containing $\mathcal{B}(\mathbb{R}^n)$. In considering the measures on $\mathbb{R}^n$ we will always assume that they are defined on the σ-algebra of Lebesgue measurable sets, or Borel sets. The interval $[0, 1)$ with the family of its Lebesgue subsets and the Lebesgue measure restricted to these subsets is often referred to as **the standard probability space**. An **n-dimensional random vector** (or simply n-vector) is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. A **complex-valued random variable** is simply a two dimensional random vector; we tend to use the former name if we want to consider complex products of two-dimensional random vectors. Recall that any random n-vector $\underline{X}$ is of the form $\underline{X} = (X_1, ..., X_n)$ where X_i are random variables $X_i : \Omega \to \mathbb{R}$.

1.2.2 **Exercise** Let A be an open set in $\mathbb{R}^n$. Show that A is union of all balls contained in A with rational radii and centers in points with rational coordinates. Conclude that $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by open (resp. closed) intervals. The same result is true for intervals of the form $(a, b]$ and $[a, b)$. Formulate and prove an analog in $\mathbb{R}^n$.

1.2.3 **Exercise** Suppose that Ω and Ω' are topological spaces. If a map $f : \Omega \to \Omega'$ is continuous, then f is measurable with respect to Borel σ-fields in Ω and Ω'. More generally, suppose that f maps a measurable space $(\Omega, \mathcal{F})$ into a measurable space $(\Omega, \mathcal{F}')$, and that $\mathcal{G}'$ is a class of measurable subsets of Ω' such $\sigma(\mathcal{G}') = \mathcal{F}'$. If inverse images of elements of $\mathcal{G}'$ are measurable, then f is measurable.

1.2.4 **Exercise** Suppose that $(\Omega, \mathcal{F}, \mu)$ is a measure space, and f maps Ω into $\mathbb{R}$. Equip $\mathbb{R}$ with the σ-algebra of Borel sets and prove that f is measurable iff sets of the form $\{\omega | f(\omega) \le t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$. (Equivalently: sets of the form $\{\omega | f(\omega) < t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$.) Prove by example that a similar statement is not necessarily true if Borel sets are replaced by Lebesgue measurable sets.

1.2.5 **Exercise** Let $(\Omega, \mathcal{F}, \mu)$ be a *complete* measure space, and f be a map $f : \Omega \to \mathbb{R}$. Equip $\mathbb{R}$ with the algebra of Lebesgue measurable sets and prove that f is measurable iff sets of the form $\{\omega | f(\omega) \le t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$. (Equivalently: sets of the form $\{\omega | f(\omega) < t\}$, $t \in \mathbb{R}$ belong to $\mathcal{F}$.)

1.2.6 **Exercise** Let $(S, \mathcal{U})$ be a topological space and let S' be its subset. We can introduce a natural topology in S', termed **induced**

topology, to be the family of sets $U' = U \cap S'$ where U is open in S. Show that

$$\mathcal{B}(S') = \{B \subset S' | B = A \cap S', A \in \mathcal{B}(S)\}. \tag{1.3}$$

1.2.7 *Monotone class theorem* A class $\mathcal{G}$ of subsets of a set Ω is termed a **π-system** if the intersection of any two of its elements belongs to the class. It is termed a **λ-system** if (a) Ω belongs to the class, (b) $A, B \in \mathcal{G}$ and $A \subset B$ implies $B \setminus A \in \mathcal{G}$ and (c) if $A_1, A_2, ... \in \mathcal{G}$, and $A_1 \subset A_2 \subset \ldots$ then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{G}$. The reader may prove that a λ-system that is at the same time a π-system is also a σ-algebra. In 1.4.3 we exhibit a natural example of a λ-system that is not a σ-algebra. The **Monotone Class Theorem** or **π–λ theorem**, due to W. Sierpiński, says that if $\mathcal{G}$ is a π-system and $\mathcal{F}$ is a λ-system and $\mathcal{G} \subset \mathcal{F}$, then $\sigma(\mathcal{G}) \subset \mathcal{F}$. As a corollary we obtain the uniqueness of extension of a measure defined on a π-system. To be more specific, if $(\Omega, \mathcal{F})$ is a measure space, and $\mathcal{G}$ is a π-system such that $\sigma(\mathcal{G}) = \mathcal{F}$, and if μ and μ' are two finite measures on $(\Omega, \mathcal{F})$ such that $\mu(A) = \mu'(A)$ for all $A \in \mathcal{G}$, then the same relation holds for $A \in \mathcal{F}$. See [5].

1.2.8 *Existence of an extension of a measure* A standard construction involving the so-called outer measure shows the existence of an extension of a measure defined on a field. To be more specific, if μ is a finite measure on a field $\mathcal{F}$, then there exists a measure $\tilde{\mu}$ on $\sigma(\mathcal{F})$ such that $\tilde{\mu}(A) = \mu(A)$ for $A \in \mathcal{F}$, see [5]. It is customary and convenient to omit the "~" and denote both the original measure and its extension by μ. This method allows us in particular to prove existence of the Lebesgue measure [5, 106].

1.2.9 *Two important properties of the Lebesgue measure* An important property of the Lebesgue measure is that it is **regular**, which means that for any Lebesgue measurable set A and $\epsilon > 0$ there exists an open set $G \supset A$ and a compact set $K \subset A$ such that $leb(G \setminus K) < \epsilon$. Also, the Lebesgue measure is **translation invariant**, i.e. $leb\, A = leb\, A_t$ for any Lebesgue measurable set A and $t \in \mathbb{R}$, where

$$A_t = \{s \in \mathbb{R}; s - t \in A\}. \tag{1.4}$$

1.2.10 **Exercise** Let $(\Omega, \mathcal{F})$ be a measure space and μ be a measure, not necessarily complete. Let $\mathcal{F}_0$ be the class of subsets B of Ω such that there exists a $C \in \mathcal{F}$ such that $\mu(C) = 0$ and $B \subset C$. Let $\mathcal{F}_\mu = \sigma(\mathcal{F} \cup \mathcal{F}_0)$. Show that there exists a unique extension of μ to $\mathcal{F}_\mu$, and $(\Omega, \mathcal{F}_\mu, \mu)$ is a complete measure space. Give an example of two Borel measures μ and ν such that $\mathcal{F}_\mu \neq \mathcal{F}_\nu$.

1.2.11 *Integral* Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. The integral $\int f \, \mathrm{d}\mu$ of a **simple measurable function** f, i.e. of a function of the form $f = \sum_{i=1}^n c_i 1_{A_i}$ where n is an integer, c_i are real constants, A_i belong to $\mathcal{F}$, and $\mu(A_i) < \infty$, is defined as $\int f \, \mathrm{d}\mu = \sum_{i=1}^n c_i \mu(A_i)$. We check that this definition of the integral does not depend on the choice of representation of a simple function. The integral of a non-negative measurable function f is defined as the supremum over integrals of non-negative simple measurable functions f_s such that $f_s \le f$ (μ a.e.). This last statement means that $f_s(\omega) \le f(\omega)$ for all $\omega \in \Omega$ outside of a measurable set of μ-measure zero. If this integral is finite, we say that f is **integrable**.

Note that in our definition we may include functions f such that $f(\omega) = \infty$ on a measurable set of ωs. We say that such functions have their values in an extended non-negative half-line. An obvious necessary requirement for such a function to be integrable is that the set where it equals infinity has measure zero (we agree as it is customary in measure theory that $0 \cdot \infty = 0$).

If a measurable function f has the property that both $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are integrable then we say that f is **absolutely integrable** and put $\int f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu$. The reader may check that for a simple function this definition of the integral agrees with the one given initially. The integral of a complex-valued map f is defined as the integral of its real part plus $\imath$ (the imaginary unit) times the integral of its imaginary part, whenever these integrals exist. For any integrable function f and measurable set A the integral $\int_A f \, \mathrm{d}\mu$ is defined as $\int 1_A f \, \mathrm{d}\mu$.

This definition implies the following elementary estimate which proves useful in practice:

$$\left| \int_A f \, \mathrm{d}\mu \right| \le \int_A |f| \, \mathrm{d}\mu. \tag{1.5}$$

Moreover, for any integrable functions f and g and any α and β in $\mathbb{R}$, we have

$$\int (\alpha f + \beta g) \, \mathrm{d}\mu = \alpha \int f \, \mathrm{d}\mu + \beta \int g \, \mathrm{d}\mu.$$

In integrating functions defined on $(\mathbb{R}^n, \mathcal{M}_n(\mathbb{R}^n), leb_n)$ it is customary to write $ds_1 \dots ds_n$ instead of $d\,leb_n(\underline{s})$ where $\underline{s} = (s_1, \dots, s_n)$. In one dimension, we write ds instead of $d leb(s)$.

There are two important results concerning limits of integrals defined this way that we will use often. The first one is called **Fatou's Lemma** and the second **Lebesgue Dominated Convergence Theorem**. The former says that for a sequence of measurable functions f_n with values in the extended non-negative half-line $\limsup_{n\to\infty} \int f_n \,d\mu \geq \int \limsup_{n\to\infty} f_n \,d\mu$, and the latter says that if f_n is a sequence of measurable functions and there exists an integrable function f such that $|f_n| \leq f$ (μ a.e.), then $\lim_{n\to\infty} \int f_n \,d\mu = \int g \,d\mu$, provided f_n tends to g pointwise, except perhaps on a set of measure zero. Observe that condition $|f_n| \leq f$ implies that f_n and g are absolutely integrable; the other part of the Lebesgue Dominated Convergence Theorem says that $\int |f_n - g| \,d\mu$ tends to zero, as $n \to \infty$. The reader may remember that both above results may be derived from the **Monotone Convergence Theorem**, which says that if f_n is a sequence of measurable functions with values in the extended non-negative half-line, and $f_{n+1}(\omega) \geq f_n(\omega)$ for all ω except maybe on a set of measure zero, then $\int_A f_n \,d\mu$ tends to $\int_A \lim_{n\to\infty} f_n(\omega) \,d\mu$ regardless of whether the last integral is finite or infinite. Here A is the set where $\lim_{n\to\infty} f_n(\omega)$ exists, and by assumption it is a complement of a set of measure zero.

Note that these theorems are true also when, instead of a sequence of functions, we have a family of functions indexed, say, by real numbers and consider a limit at infinity or at some point of the real line.

1.2.12 **Exercise** Let (a, b) be an interval and let, for τ in this interval, $x(\tau, \omega)$ be a given integrable function on a measure space $(\Omega, \mathcal{F}, \mu)$. Suppose furthermore that for almost all $\omega \in \Omega$, $\tau \to x(\tau, \omega)$ is continuously differentiable and there exists an integrable function y such that $\sup_{\tau\in(a,b)} |x'(\tau, \omega)| \leq y(\omega)$. Prove that $z(\tau) = \int_\Omega x(\tau, \omega)\, \mu(d\omega)$ is differentiable and that $z'(\tau) = \int_\Omega x'(\tau, \omega)\, \mu(d\omega)$.

1.2.13 *Product measures* Let $(\Omega, \mathcal{F}, \mu)$ and $(\Omega', \mathcal{F}', \mu')$ be two σ-finite measure spaces. In the Cartesian product $\Omega \times \Omega'$ consider the **rectangles**, i.e. the sets of the form $A \times A'$ where $A \in \mathcal{F}$ and $A' \in \mathcal{F}'$, and the function $\mu \otimes \mu'(A \times A') = \mu(A)\mu'(A')$. Certainly, rectangles form a π-system, say $\mathcal{R}$, and it may be proved that $\mu \otimes \mu'$ is a measure on $\mathcal{R}$ and that there exists an extension of $\mu \otimes \mu'$ to a measure on $\sigma(\mathcal{R})$, which is necessarily unique. This extension is called the **product measure** of μ and μ'. The assumption that μ and μ' are σ-finite is crucial for the existence of $\mu \otimes \mu'$. Moreover, $\mu \otimes \mu'$ is σ-finite, and it is finite if μ and μ' are. The **Tonelli Theorem** says that if a function $f : \Omega \times \Omega' \to \mathbb{R}$ is $\sigma(\mathcal{R})$ measurable, then for all $\omega \in \Omega$ the function $f_\omega : \Omega' \to \mathbb{R}, f_\omega(\omega') = f(\omega, \omega')$ is $\mathcal{F}'$ measurable and the function $f^{\omega'} : \Omega \to \mathbb{R}, f^{\omega'}(\omega) = f(\omega, \omega')$ is $\mathcal{F}$ measurable. Furthermore, the **Fubini Theorem** says that for a $\sigma(\mathcal{R})$ measurable function $f : \Omega \times \Omega' \to \mathbb{R}^+$,

$$\int_{\Omega \times \Omega'} f \, \mathrm{d}(\mu \otimes \mu') = \int_\Omega \Big[\int_{\Omega'} f_\omega(\omega') \, \mu(d\omega')\Big] \, \mu(d\omega) = \int_{\Omega'} \Big[\int_\Omega f^{\omega'}(\omega) \, \mu(d\omega)\Big] \, \mu(d\omega'),$$

finite or infinite; measurability of the integrands is a part of the theorem. Moreover, this relation holds whenever f is absolutely integrable.

1.2.14 *Absolute continuity* Let μ and ν be two measures on a measure space $(\Omega, \mathcal{F})$; we say that μ is **absolutely continuous** (with respect to ν) if there exists a non-negative (not necessarily integrable) function f such that $\mu(A) = \int_A f \, \mathrm{d}\nu$ for all $A \in \mathcal{F}$. In such a case f is called the **density** of μ (with respect to ν). Observe that f is integrable (with respect to ν) iff μ is finite, i.e. iff $\mu(\Omega) < \infty$. When it exists, the density is unique up to a set of ν-measure zero.

1.2.15 *Change of variables formula* Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space and f is a measurable map from $(\Omega, \mathcal{F})$ to another measurable space $(\Omega', \mathcal{F}')$. Consider the set function μ_f on $\mathcal{F}'$ defined by $\mu_f(A) = \mu(f^{-1}(A)) = \mu(f \in A)$. We check that μ_f is a measure in $(\Omega', \mathcal{F}')$. It is called the **transport of the measure** μ **via** f or a measure **induced** on $(\Omega', \mathcal{F}')$ by μ and f. In particular, if μ is a probability measure, and $\Omega' = (\mathbb{R}^n, \mathcal{M}_n(\mathbb{R}^n))$, μ_f is called the **distribution** of f.

Note that a measurable function x defined on Ω' is integrable with respect to μ_f iff $x \circ f$ is integrable with respect to μ and

$$\int_{\Omega'} x \, \mathrm{d}\mu_f = \int_\Omega x \circ f \, \mathrm{d}\mu. \tag{1.6}$$

To prove this relation, termed the **change of variables formula**, we check it first for simple functions, and then use approximations to show the general case. A particular case is that where a measure, say ν, is already defined on $(\Omega', \mathcal{F}')$, and μ_f is absolutely continuous with respect to ν. If ϕ is the density of μ_f with respect to ν, then the change of variables formula reads:

$$\int_\Omega x \circ f \,\mathrm{d}\mu = \int_{\Omega'} x \,\mathrm{d}\mu_f = \int_{\Omega'} x\phi \,\mathrm{d}\nu.$$

Of particular interest is the case when $\Omega' = \mathbb{R}^n$ and $\nu = leb_n$.

If $\mu = \mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ and $\Omega' = \mathbb{R}$, we usually denote measurable maps by the capital letter X. We say that X has a first moment iff X is integrable, and then write $E\,X \equiv \int X \,\mathrm{d}\mathbb{P}$. $E\,X$ is called the **first moment** or **expected value** of X. The Hölder inequality (see 1.5.8 below) shows that if X^2 has a first moment then X also has a first moment (but the opposite statement is in general not true). $E\,X^2$ is called the (non-central) **second moment** of X. If $E\,X^2$ is finite, we also define the central second moment or **variance** of X as $D^2\,X = \sigma_X^2 = E\,(X - E\,X)^2$. The reader will check that σ_X^2 equals $E\,X^2 - (E\,X)^2$.

If the distribution of a random variable X has a density ϕ with respect to Lebesgue measure, than $E\,X$ exists iff $f(\xi) = \xi\phi(\xi)$ is absolutely integrable and then $E\,X = \int_{-\infty}^{\infty} \xi\phi(\xi)\,\mathrm{d}\xi$.

1.2.16 *Convolution of two finite measures* Let μ and ν be two finite measures on $\mathbb{R}$. Consider the product measure $\mu \otimes \nu$ on $\mathbb{R} \times \mathbb{R}$, and a measurable map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(\varsigma, \tau) = \varsigma + \tau$. The **convolution** $\mu * \nu$ of μ with ν is defined as the transport of $\mu \otimes \nu$ via f. Thus, $\mu * \nu$ is a bounded measure on $\mathbb{R}$ and, by the change of variables formula,

$$\int x \,\mathrm{d}(\mu * \nu) = \int\int x(\varsigma + \tau)\,\mu(\mathrm{d}\varsigma)\nu(\mathrm{d}\tau). \tag{1.7}$$

We have $\mu * \nu(\mathbb{R}) = \mu \otimes \nu(\mathbb{R} \times \mathbb{R}) = \mu(\mathbb{R})\nu(\mathbb{R})$. In particular, the convolution of two probability measures on $\mathbb{R}$ is a probability measure. Observe also that $\mu * \nu = \nu * \mu$, and that $(\mu * \mu') * \mu'' = \mu * (\mu' * \mu'')$ for all bounded measures μ, μ' and μ''.

1.2.17 *Convolution of two integrable functions* For two Lebesgue integrable functions ϕ and ψ on $\mathbb{R}$ their convolution $\phi * \psi$ is defined by $\varphi(\xi) = \int_{-\infty}^{\infty} \phi(\xi - \varsigma)\psi(\varsigma)\,\mathrm{d}\varsigma$. The reader will use the Fubini–Tonelli Theorem to check that $\phi * \psi$ is well-defined for almost all $\xi \in \mathbb{R}$.

1.2.18 **Exercise** Suppose that μ and ν are two finite measures on $\mathbb{R}$, absolutely continuous with respect to Lebesgue measure. Let ϕ and ψ be the densities of μ and ν, respectively. Show that $\mu * \nu$ is absolutely continuous with respect to Lebesgue measure and has a density $\varphi = \phi * \psi$.