

Contents

<i>Preface</i>	<i>page xi</i>
<i>Notation and conventions</i>	<i>xvi</i>
1 Fundamentals	1
1.1 Vectors, dual vectors, and tensors	2
1.2 Covariant differentiation	4
1.3 Geodesics	6
1.4 Lie differentiation	8
1.5 Killing vectors	10
1.6 Local flatness	11
1.7 Metric determinant	12
1.8 Levi-Civita tensor	13
1.9 Curvature	15
1.10 Geodesic deviation	16
1.11 Fermi normal coordinates	18
1.11.1 Geometric construction	19
1.11.2 Coordinate transformation	20
1.11.3 Deviation vectors	21
1.11.4 Metric on γ	22
1.11.5 First derivatives of the metric on γ	22
1.11.6 Second derivatives of the metric on γ	22
1.11.7 Riemann tensor in Fermi normal coordinates	24
1.12 Bibliographical notes	24
1.13 Problems	25
2 Geodesic congruences	28
2.1 Energy conditions	29
2.1.1 Introduction and summary	29
2.1.2 Weak energy condition	30
2.1.3 Null energy condition	31

vi	<i>Contents</i>	
	2.1.4 Strong energy condition	31
	2.1.5 Dominant energy condition	32
	2.1.6 Violations of the energy conditions	32
2.2	Kinematics of a deformable medium	33
	2.2.1 Two-dimensional medium	33
	2.2.2 Expansion	34
	2.2.3 Shear	34
	2.2.4 Rotation	35
	2.2.5 General case	35
	2.2.6 Three-dimensional medium	35
2.3	Congruence of timelike geodesics	36
	2.3.1 Transverse metric	37
	2.3.2 Kinematics	37
	2.3.3 Frobenius' theorem	38
	2.3.4 Raychaudhuri's equation	40
	2.3.5 Focusing theorem	40
	2.3.6 Example	41
	2.3.7 Another example	42
	2.3.8 Interpretation of θ	43
2.4	Congruence of null geodesics	45
	2.4.1 Transverse metric	46
	2.4.2 Kinematics	47
	2.4.3 Frobenius' theorem	48
	2.4.4 Raychaudhuri's equation	50
	2.4.5 Focusing theorem	50
	2.4.6 Example	51
	2.4.7 Another example	52
	2.4.8 Interpretation of θ	52
2.5	Bibliographical notes	54
2.6	Problems	54
3	Hypersurfaces	59
3.1	Description of hypersurfaces	60
	3.1.1 Defining equations	60
	3.1.2 Normal vector	60
	3.1.3 Induced metric	62
	3.1.4 Light cone in flat spacetime	63
3.2	Integration on hypersurfaces	64
	3.2.1 Surface element (non-null case)	64
	3.2.2 Surface element (null case)	65
	3.2.3 Element of two-surface	67

	<i>Contents</i>	vii
3.3	Gauss–Stokes theorem	69
3.3.1	First version	69
3.3.2	Conservation	71
3.3.3	Second version	72
3.4	Differentiation of tangent vector fields	73
3.4.1	Tangent tensor fields	73
3.4.2	Intrinsic covariant derivative	73
3.4.3	Extrinsic curvature	75
3.5	Gauss–Codazzi equations	76
3.5.1	General form	76
3.5.2	Contracted form	78
3.5.3	Ricci scalar	78
3.6	Initial-value problem	79
3.6.1	Constraints	79
3.6.2	Cosmological initial values	80
3.6.3	Moment of time symmetry	81
3.6.4	Stationary and static spacetimes	82
3.6.5	Spherical space, moment of time symmetry	82
3.6.6	Spherical space, empty and flat	82
3.6.7	Conformally-flat space	84
3.7	Junction conditions and thin shells	84
3.7.1	Notation and assumptions	85
3.7.2	First junction condition	86
3.7.3	Riemann tensor	87
3.7.4	Surface stress-energy tensor	87
3.7.5	Second junction condition	88
3.7.6	Summary	89
3.8	Oppenheimer–Snyder collapse	90
3.9	Thin-shell collapse	93
3.10	Slowly rotating shell	94
3.11	Null shells	98
3.11.1	Geometry	98
3.11.2	Surface stress-energy tensor	100
3.11.3	Intrinsic formulation	102
3.11.4	Summary	104
3.11.5	Parameterization of the null generators	104
3.11.6	Imploding spherical shell	107
3.11.7	Accreting black hole	109
3.11.8	Cosmological phase transition	112
3.12	Bibliographical notes	114

viii	<i>Contents</i>	
3.13	Problems	114
4	Lagrangian and Hamiltonian formulations of general relativity	118
4.1	Lagrangian formulation	119
4.1.1	Mechanics	119
4.1.2	Field theory	120
4.1.3	General relativity	121
4.1.4	Variation of the Hilbert term	122
4.1.5	Variation of the boundary term	124
4.1.6	Variation of the matter action	125
4.1.7	Nondynamical term	126
4.1.8	Bianchi identities	127
4.2	Hamiltonian formulation	128
4.2.1	Mechanics	128
4.2.2	$3 + 1$ decomposition	129
4.2.3	Field theory	131
4.2.4	Foliation of the boundary	134
4.2.5	Gravitational action	136
4.2.6	Gravitational Hamiltonian	139
4.2.7	Variation of the Hamiltonian	141
4.2.8	Hamilton's equations	145
4.2.9	Value of the Hamiltonian for solutions	146
4.3	Mass and angular momentum	146
4.3.1	Hamiltonian definitions	146
4.3.2	Mass and angular momentum for stationary, axially symmetric spacetimes	148
4.3.3	Komar formulae	149
4.3.4	Bondi–Sachs mass	151
4.3.5	Distinction between ADM and Bondi–Sachs masses: Vaidya spacetime	152
4.3.6	Transfer of mass and angular momentum	155
4.4	Bibliographical notes	156
4.5	Problems	157
5	Black holes	163
5.1	Schwarzschild black hole	163
5.1.1	Birkhoff's theorem	163
5.1.2	Kruskal coordinates	164
5.1.3	Eddington–Finkelstein coordinates	167
5.1.4	Painlevé–Gullstrand coordinates	168
5.1.5	Penrose–Carter diagram	168
5.1.6	Event horizon	170

	<i>Contents</i>	ix
5.1.7	Apparent horizon	171
5.1.8	Distinction between event and apparent horizons: Vaidya spacetime	172
5.1.9	Killing horizon	175
5.1.10	Bifurcation two-sphere	176
5.2	Reissner–Nordström black hole	176
5.2.1	Derivation of the Reissner–Nordström solution	176
5.2.2	Kruskal coordinates	178
5.2.3	Radial observers in Reissner–Nordström spacetime	181
5.2.4	Surface gravity	184
5.3	Kerr black hole	187
5.3.1	The Kerr metric	187
5.3.2	Dragging of inertial frames: ZAMOs	188
5.3.3	Static limit: static observers	188
5.3.4	Event horizon: stationary observers	189
5.3.5	The Penrose process	191
5.3.6	Principal null congruences	192
5.3.7	Kerr–Schild coordinates	194
5.3.8	The nature of the singularity	195
5.3.9	Maximal extension of the Kerr spacetime	196
5.3.10	Surface gravity	198
5.3.11	Bifurcation two-sphere	199
5.3.12	Smarr’s formula	200
5.3.13	Variation law	201
5.4	General properties of black holes	201
5.4.1	General black holes	202
5.4.2	Stationary black holes	204
5.4.3	Stationary black holes in vacuum	205
5.5	The laws of black-hole mechanics	206
5.5.1	Preliminaries	207
5.5.2	Zeroth law	208
5.5.3	Generalized Smarr formula	209
5.5.4	First law	211
5.5.5	Second law	212
5.5.6	Third law	213
5.5.7	Black-hole thermodynamics	214
5.6	Bibliographical notes	215
5.7	Problems	216
	<i>References</i>	224
	<i>Index</i>	229