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978-0-521-83054-6 - Non-linear Modeling and Analysis of Solids and Structures

Steen Krenk

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# Non-linear Modeling and Analysis of Solids and Structures

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CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press

The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521830546](http://www.cambridge.org/9780521830546)

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First published 2009

Printed in the United Kingdom at the University Press, Cambridge

*A catalog record for this publication is available from the British Library*

ISBN 978-0-521-83054-6 hardback

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## Preface

The aim of this book is to take the reader on a concentrated tour of some of the central issues of non-linear modeling and analysis of structures and solids. Traditionally, the non-linear theories of solids have been treated in books on continuum mechanics, while the questions of analysis have formed the focus of books on finite element techniques. The idea of the present book is to place the emphasis on modeling with a view to its numerical implementation right from the outset. Two guiding principles have determined the main style of the book: the story should be told in the form of concentrated chapters, each giving the central ideas of a specific aspect such as ‘finite rotations’ or ‘elasto-plastic solids’, and the reader should have the possibility of getting a feel for the numerical implementation by access and use of simple high-level implementations of the basic algorithms. A text based on these principles cannot provide exhaustive coverage, but aims at giving an interesting introduction to the basic ideas, which can then be studied elsewhere in greater detail as needed. It is hoped that the combination of a concise theoretical presentation in plain language supported by specific algorithms will make the text of interest to graduate students as well as professionals.

The book contains nine chapters: a brief introductory chapter setting the scene by use of elementary arguments, four chapters on structures, two chapters on non-linear deformation and material behavior of solids, and finally two chapters on numerical techniques for non-linear quasi-static and dynamic analysis. The theory is combined with demonstrations and exercises using a small MATLAB toolbox FEMFILES providing routines for creation and assembly of element matrices and permitting the solution of non-linear finite element problems in a fairly simple script file format. The toolbox FEMFILES is available from the author via the internet. Exercises that require the use of a high-level program like FEMFILES are marked \*.



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The text started as a draft manuscript prepared for a short introductory course on non-linear aspects of the finite element method at Aalborg University in the fall of 1992. A visit to Lund Institute of Technology sponsored by NorFA provided an opportunity to include additional material on the numerical aspects. The text was later extended with material on finite rotations, co-rotating formulation of elements, potential theory of plasticity theory and plasticity models for geotechnical materials, and conservation algorithms for numerical integration of dynamic problems. Several parts of this work have been sponsored by the Danish Technical Research Council. The work on bringing it all together was initiated during a visiting appointment as Melchor Professor at the University of Notre Dame, Indiana, in the fall of 2001. The final stage has been combined with courses at Helsinki University of Technology 2004, and at Aalborg University and Lund Institute of Technology 2005.