Symmetrization in Analysis

Symmetrization is a rich area of mathematical analysis whose history reaches back to antiquity. This book presents many aspects of the theory, including symmetric decreasing rearrangement and circular and Steiner symmetrization in Euclidean spaces, spheres, and hyperbolic spaces. Many energies, frequencies, capacities, eigenvalues, perimeters, and function norms are shown to either decrease or increase under symmetrization.

The book begins by focusing on Euclidean space, building up from two-point polarization with respect to hyperplanes. Background material in geometric measure theory and analysis is carefully developed, yielding self-contained proofs of all the major theorems. This leads to the analysis of functions defined on spheres and hyperbolic spaces, and then to convolutions, multiple integrals, and hypercontractivity of the Poisson semigroup. The author's star function, which preserves subharmonicity, is developed with applications to semilinear partial differential equations. The book concludes with a thorough self-contained account of the star function's role in complex analysis, covering value distribution theory, conformal mapping, and the hyperbolic metric.

Albert Baernstein II was a professor in the Department of Mathematics at Washington University in St. Louis until his death in 2014. He gained international renown for innovative solutions to extremal problems in complex and harmonic analysis. His invention of the "star function" method in the 1970s prompted an invitation to the International Congress of Mathematicians held in Helsinki in 1978, and during the 1980s and 1990s he substantially extended the breadth and applications of this method.

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Symmetrization in Analysis

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Notation

$|A|$, the operator norm of a matrix A, §7.8
$|A|$, the linear measure of a subset A of $\mathbb{R}$ or $\mathbb{T}$, §11.1
α_n, the volume of the unit n-ball, §1.4
AL and SAL classes, §2.1
AL_0 and SAL_0 classes, §2.2
$A(R_1, R_2)$, open spherical shell, §9.2
β_{n-1}, the surface measure of the unit $(n-1)$-sphere, §4.5
$\mathcal{B}$, the Borel σ-algebra, §1.3
$\mathcal{B}_c$, the set of Borel sets contained in some compact set, §9.2
$\mathbb{B}^n(r)$, the open n-ball of radius r centered at the origin, §1.4
$\mathbb{B}^n(a, r)$, the open n-ball of radius r centered at a, §1.5
B_t, Brownian motion, §5.7
BV, the space of functions of bounded variation, §4.2
$\widehat{\mathbb{C}}$, extended complex numbers $\mathbb{C} \cup \{\infty\}$, §11.4
C_c, continuous functions with compact support, §2.2
$c(E)$, the center of mass of E, §2.6
C^γ, Hölder spaces, §4.8
$\operatorname{Cap} K$, the Newtonian capacity of K, §5.5
$\operatorname{Cap}_p K$, the variational p-capacity of K, §5.6
$C_\alpha K$, the Riesz α-capacity of K, §5.6
$\mathbb{D}$, open unit disk, §11.1
$\mathbb{D}(r)$, open disk of radius r, §11.1
$\operatorname{diam} E$, the diameter of E, §1.7
$d(x, E)$, the distance from point x to set E, §2.4
$\partial^* E$, the reduced boundary of E, §4.3
$\partial_i f$, the partial derivatives of f, §3.4
$\partial_v f$, the derivative of f in direction v, §3.1
$D^\alpha f$, multiindex notation for derivatives, §4.8

Δ, the Laplace operator, §5.1
$d(x, y)$, spherical distance in §7.1; hyperbolic distance in §7.6
Δ_s, spherical Laplacian, §7.2
$\Delta^\star$ and related elliptic operators, §9.6, 9.7, 9.8, 9.10, 9.11
$\Delta^{\star}$, a variant of the $\Delta^\star$ operator, §10.6
∇_s, spherical gradient, §7.2
∇_h, hyperbolic gradient, §7.6
$E^\#$, rearrangement of a set E, §1.6 and later. The particular type of rearrangement (e.g., symmetric decreasing, Steiner, spherical, cap) depends on the context.
$E(\delta)$, the δ-collar of E, §4.4
$E(-\delta)$, the δ-core of E, §4.4
E_x, expected value with respect to the Brownian motion starting at x, §5.7
$\tilde{E}$, the Gauss symmetrization of set E, §7.7
f^*, the decreasing rearrangement of f, §1.2
$f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$, positive and negative parts of f, §1.3
$f^\#$, rearrangement of a function f, §1.6 and later. The particular type of rearrangement (e.g., symmetric decreasing, Steiner, spherical, cap) depends on the context.
$f^\star$, the $\star$-function, for various types of rearrangement, §9.1, 9.2, 9.6, 9.7, 9.8, 9.10, 9.11
$f^{\star}$, a variant of the $\star$-function, §11.1
f_H, the polarization of f with respect to H, §1.7
$\tilde{f}$, the Gauss symmetrization of function f, §7.7
$f_\#\mu$, the pushforward of μ by f, §1.3
$G(x, y, \Omega)$, Green's function, §9.4
$G(x)$, Green's function of unit ball with pole at 0, §9.4
$G(\mathbb{R}^n)$, the conformal group of $\mathbb{R}^n$, §7.6
γ_n, the Gauss measure, §7.7
$\mathbb{H}^n$, the n-dimensional hyperbolic space, §1.6
$\mathbb{H}$, the upper halfplane, §11.1
$\mathcal{H}(\mathbb{R}^n)$, the set of affine hyperplanes in $\mathbb{R}^n$, §1.7
$\mathcal{H}(\mathbb{S}^n)$, the intersections of linear hyperplanes in $\mathbb{R}^{n+1}$ with $\mathbb{S}^n$, §7.1
H^+ and H^-, two halfspaces determined by affine hyperplane H, §1.7
$\mathcal{H}^s$, Hausdorff measure, §4.1
$\mathcal{H}(f)$, the Riesz energy of f, §8.7
$\mathcal{H}^{k,n}$, spherical harmonics on $\mathbb{S}^n$ of degree k, §8.8
J_f, the Jacobian determinant of f, §4.1
$J(f, g, h)$, triple convolution evaluated at zero, §8.1. More general version in §8.5.

J-operator for definite integration over balls, spherical caps, and so on, depending on the type of rearrangement, §9.6, 9.7, 9.10, 9.11
$\mathcal{K}(\theta)$, open spherical cap on $\mathbb{S}^n$ centered at e_1, §7.1
$K(x, y, t)$ the Dirichlet heat kernel, §8.6
$K_\lambda(x) = |x|^{-\lambda}$, the Riesz kernel, §8.7
$\mathcal{L}$, the Lebesgue measure on $\mathbb{R}$, §1.2
$\mathcal{L}^n$, the Lebesgue measure on $\mathbb{R}^n$, §1.4
λ_f, distribution function, §1.1
$\lambda(t+)$ and $\lambda(t-)$, one-sided limits, §1.1
Lip, the Lipschitz class, §3.1
$\lambda_1(\Omega)$, the principal Dirichlet eigenvalue of Ω, §5.3
Lcap K, the logarithmic capacity of K, §5.6
$\mathcal{M}^s$, $\mathcal{M}^s_*$, Minkowski content, §4.4
Mod Ω, the conformal modulus of Ω, §5.6 (extended to dimensions $n > 2$ in §7.8)
$M(K)$, the space of finite measures on K, §9.2
$M_{\text{loc}}(X)$, the space of locally finite measures on X, §9.2
$\mu^\#$, rearrangement of a measure, for various types of rearrangement, §§9.5–9.9, 9.11
$\mu^\star$, star operation applied to a measure, for various types of rearrangement, §9.6, 9.7, 9.8, 9.10, 9.11
ν_n, normalized spherical measure, §8.8
$O(n)$, orthogonal group, §7.1
$\omega(t,f)$, the modulus of continuity of f, §1.7
p^*, the Sobolev conjugate exponent of p, §4.6
$P(E)$, the perimeter of set E, §4.3
$\mathbb{R}^+ = [0, \infty)$, the set of nonnegative real numbers
R_G and R_T, the Grötszsch and Teichmüller rings, §7.8
ρ_H, reflection in hyperplane H, §1.7
$\mathbb{S}^n$, the n-dimensional unit sphere, §1.6
$\mathbb{S}(r)$, the circle of radius r centered at the origin, §11.1
σ_{n-1}, the restriction of $\mathcal{H}^{n-1}$ to $\mathbb{S}^{n-1}$, §4.5
$\star$-function, see entries above for $f^\star$ and $\mu^\star$
$\star$-function, see entry above for $f^{\star}$
$T(\Omega)$, the torsional rigidity of Ω, §5.7
$T^{\star}$, the $\star$-function of the Nevanlinna characteristic T, §11.2
$\tau(E)$, the canonical measure on hyperbolic space, §7.6
$\text{Tr}(t, \Omega)$, the trace of the heat kernel, §8.6
u_K, the equilibrium potential of K, §5.5
$V(f)$, the total variation of f, §4.2

$W^{1,p}$, $W_0^{1,p}$, Sobolev spaces, §3.4
$\mathcal{W}(\Omega)$, the space of functions whose weak Laplacian is a measure, §9.2
$(X, \mathcal{M}, \mu)$, measure space, §1.1
χ_A, characteristic function of set A, §1.1
Y_k, a spherical harmonic of degree k, §8.8

Foreword

At the 1973 Symposium at Canterbury on Complex Analysis there were two stars. One was the star function and the other was its inventor, Al Baernstein.

Suppose $u(z)$ is subharmonic in the annulus $\{re^{i\theta} : r_1 < r < r_2\}$, and define

$$u^{\star}(re^{i\theta}) = \sup_E \int_E u(re^{it})\,dt,$$

where the supremum is taken over all sets E of measure 2θ in $[0, 2\pi]$. Then $u^{\star}$ is the star function of u.

Theorem I *With the above hypotheses, $u^{\star}$ is subharmonic in the semiannulus $\{re^{i\theta} : r_1 < r < r_2, 0 < \theta < \pi\}$.*

(See Corollary 9.10 and the Chapter 9 Notes, with $u^{\star} = u^{\star}/r$.)

It is amazing how many consequences have been deduced by Baernstein and others from this innocent-looking theorem, and one purpose of this volume is to provide a coherent account of what have been the most important. There are generalizations to higher dimensional Euclidean space but some of the most interesting results occur when $u(z) = \log|f(z)|$ and f is analytic or meromorphic. These are described in Chapter 11 of this book.

In a short foreword it is impossible to do justice to the full portfolio of results in this comprehensive book on symmetrization. So I would like to concentrate on two results in Chapter 11, which were first announced at the Canterbury conference.

1. Let $\mathcal{S}$ be the class of functions

$$f(z) = z + a_2 z^2 + \cdots$$

analytic and univalent in the unit disk $|z| < 1$. If Φ is a convex increasing function we have

$$\int_0^{2\pi} \Phi\big(\log|f(re^{i\theta})|\big)\,d\theta \le \int_0^{2\pi} \Phi\big(\log|k(re^{i\theta})|\big)\,d\theta,$$

where $k(z) = z/(1-z)^2$ is the Koebe function, with strict inequality unless f equals the Koebe function or one of its rotates $(e^{-it}k(ze^{it}))$.

As a corollary, we have for $0 < p < \infty$ and $0 < r < 1$ that

$$M_p(r,f) \le M_p(r,k),$$

where

$$M_p(r,f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\right)^{1/p}.$$

2. Baernstein originally introduced the star function in order to prove A. Edrei's spread conjecture. This is more complicated to formulate and needs a few definitions. Suppose that $u(z)$ is δ-subharmonic in the plane, that is, $u(z) = u_1(z) - u_2(z)$ where u_1 and u_2 are subharmonic. We define $u^+(z) = \max(u(z), 0)$, and

$$m(r) = m(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{it})\,dt$$

for $r > 0$. Let

$$n(r) = \frac{1}{2\pi}\int_{\{|z|\le r\}} \Delta u_2\,dxdy$$

and

$$N(r) = N(r,u) = \int_0^r t^{-1}\,(n(t) - n(0))\;dt + n(0)\log r.$$

Then

$$T(r) = T(r,u) = m(r,u) + N(r,u)$$

is the Nevanlinna characteristic of u (the designation is in honor of Rolf Nevanlinna, the founder of this theory). The deficiency of u is defined by

$$\delta(u) = \liminf_{r\to\infty} \frac{m(r,u)}{T(r,u)}$$

(so that $0 \le \delta(u) \le 1$), and the lower order of u is

$$\mu = \liminf_{r\to\infty} \frac{\log T(r,u)}{\log r}.$$

Then Baernstein's spread theorem states:

Theorem II *If* $\delta(u) > 0$ *then for every positive* η *there is a positive* ε *and a sequence of radii* r_n *and sets* E_n *of measure at least* $2\beta - \eta$ *in* $[0, 2\pi]$ *so that* $r_n \to \infty$ *and*

$$u(r_n e^{i\theta}) > \varepsilon T(r_n) \text{ on } E_n,$$

where

$$\beta = \min\left(\pi, (2/\mu)\sin^{-1}\sqrt{\delta/2}\right).$$

See Proposition 11.7, which handles the case of $u = \log|f|$ with f meromorphic and is phrased in terms of a more general growth measure than μ.

This value of β is sharp. When f is meromorphic in the plane and $u = \log|f|$, we deduce Edrei's spread conjecture on the size of the set where $|f| > 1$; in fact on this set $\log|f|$ is comparable to $T(r)$. Applying the result to $u = -\log|f|$ we obtain sharp bounds for the sum of the deficiencies of f. These were known for $\mu \leq 1/2$ and conjectured by Edrei when $1/2 < \mu < 1$. Very little is known when $\mu > 1$ other than for certain isolated values, as discussed in Note 3 at the end of Chapter 11.

—Walter Hayman

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Preface

Albert Baernstein passed away June 10, 2014, a great loss to mathematics and to his many friends, colleagues, and students. As recounted in Walter Hayman's foreword, Al's early discovery of the star function, along with its immediate applications to classical complex analysis, gave him international prominence, and provided the foundation for a long and productive career in mathematical analysis. An obituary, including a mathematical sketch and bibliography, has appeared in the *Notices of the American Mathematical Society* (Drasin, 2015).

Al worked on this symmetrization book for many years. Toward the end of his life, when it became clear he might not finish the work, a group of friends and former students committed to completing the project in the manner he envisaged. Al gave his blessing, and shared his files. He had planned eleven chapters. Eight and a half were essentially complete, and he left a rough outline of his goals for the remainder.

Richard Laugesen[1] and David Drasin were the general coordinators, aided by Juan Manfredi. In the early stages, Leonid Kovalev provided an essential service by editing Al's original LaTeX files to label the results, create an index file, put references into .bib format, and make the notation list. Juan Manfredi wrote most of the Introduction, and he and Almut Burchard drew the figures for the book. Burchard provided invaluable help revising Chapter 8, and Laugesen revised and added material to Chapter 9. Al had prepared the first section of Chapter 10, and Laugesen and Jeffrey Langford wrote the remainder. David Drasin and Allen Weitsman wrote Chapter 11, using Al's earlier account (Baernstein, 2002) as a guide.

The extended period required to finish the book was warranted, we feel, by the mathematical vision embedded in Al's manuscript. We hope his monograph

[1] Supported by the Simons Foundation (#429422 to Richard Laugesen).

will serve as a foundation for further research in symmetrization and its applications.

Cambridge University Press has always been patient and supportive. Special thanks go to Juan Manfredi, who contributed to and encouraged the group effort at every stage. We warmly thank Judy Baernstein for decades of friendship, and for the hospitality she showed to all mathematicians making pilgrimages to Washington University in St. Louis.

— David Drasin (Purdue University),
Richard S. Laugesen (University of Illinois),
June 2018