1

Electromagnetic field theory

1.1 Introduction

What is a field? Is it a scalar field or a vector field? What is the nature of a field? Is it a continuous or a rotational field? How is the magnetic field produced by a current-carrying coil? How does a capacitor store energy? How does a piece of wire (antenna) radiate or receive signals? How do electromagnetic fields propagate in space? What really happens when electromagnetic energy travels from one end of a hollow pipe (waveguide) to the other? The primary purpose of this text is to answer some of these questions pertaining to electromagnetic fields.

In this chapter we intend to show that the study of electromagnetic field theory is vital to understanding many phenomena that take place in electrical engineering. To do so we make use of some of the concepts and equations of other areas of electrical engineering. We aim to shed light on the origin of these concepts and equations using electromagnetic field theory.

Before we proceed any further, however, we mention that the development of science depends upon some quantities that cannot be defined precisely. We refer to these as fundamental quantities; they are **mass** (m), **length** (ℓ), **time** (t), **charge** (q), and **temperature** (T). For example, what is time? When did time begin? Likewise, what is temperature? What is hot or cold? We do have some intuitive feelings about these quantities but lack precise definitions. To measure and express each of these quantities, we need to define a system of units.

In the International System of Units (SI for short), we have adopted the units of kilogram (kg) for mass, meter (m) for length, second (s) for time, coulomb (C) for charge, and kelvin (K) for temperature. Units for all other quantities of interest are then defined in terms of these fundamental units. For example, the unit of current, the ampere (A), in terms of the fundamental units is coulombs per second (C/s). Therefore, the ampere is a derived unit. The newton (N), the unit of force, is also a derived unit; it can be expressed in terms of basic units as $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$. Units for some of the quantities that we will refer to in this

Table 1.1. Derived units for some electromagnetic quantities

Symbol	Quantity	Unit	Abbreviation
Y	admittance	siemen	S
ω	angular frequency	radian/second	rad/s
C	capacitance	farad	F
ρ	charge density	coulomb/meter^3	C/m^3
G	conductance	siemen	S
σ	conductivity	siemen/meter	S/m
W	energy	joule	J
F	force	newton	N
f	frequency	hertz	Hz
Z	impedance	ohm	Ω
L	inductance	henry	H
$\mathscr{F}$	magnetomotive force	ampere-turn	A°t
μ	permeability	henry/meter	H/m
ϵ	permittivity	farad/meter	F/m
P	power	watt	W
$\mathscr{R}$	reluctance	henry^{-1}	H^{-1}

Table 1.2. Unit conversion factors

From	Multiply by	To obtain
gilbert	0.79577	ampere-turn (At)
ampere-turn/cm	2.54	ampere-turn/inch
ampere-turn/inch	39.37	ampere-turn/meter
oersted	79.577	ampere-turn/meter
line (maxwells)	1×10^{-8}	weber (Wb)
gauss (lines/cm^2)	6.4516	line/inch^2
line/inch^2	0.155×10^{-4}	Wb/m^2 (tesla)
gauss	10^{-4}	Wb/m^2
inch	2.54	centimeter (cm)
foot	30.48	centimeter
meter	100	centimeter
square inch	6.4516	square cm
ounce	28.35	gram
pound	0.4536	kilogram
pound-force	4.4482	newton
ounce-force	0.278 01	newton
newton-meter	141.62	ounce-inch
newton-meter	0.73757	pound-feet
revolution/minute	$2\pi/60$	radian/second

text are given in Tables 1.1 and 1.3. Since English units are still being used in the industry to express some field quantities, it is necessary to convert from one unit system to the other. Table 1.2 is provided for this purpose.

Table 1.3. A partial list of field quantities

Variable	Definition	Type	Unit
$\vec{\mathbf{A}}$	magnetic vector potential	vector	Wb/m
$\vec{\mathbf{B}}$	magnetic flux density	vector	Wb/m^2 (T)
$\vec{\mathbf{D}}$	electric flux density	vector	C/m^2
$\vec{\mathbf{E}}$	electric field intensity	vector	V/m
$\vec{\mathbf{F}}$	Lorentz force	vector	N
I	electric current	scalar	A
$\vec{\mathbf{J}}$	volume current density	vector	A/m^2
q	free charge	scalar	C
$\vec{\mathbf{S}}$	Poynting vector	vector	W/m^2
$\vec{\mathbf{u}}$	velocity of free charge	vector	m/s
V	electric potential	scalar	V

Table 1.4. A partial list of relationships between various field quantities

$\vec{\mathbf{D}} = \epsilon\vec{\mathbf{E}}$	permittivity (ϵ)
$\vec{\mathbf{B}} = \mu\vec{\mathbf{H}}$	permeability (μ)
$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$	conductivity (σ), Ohm's law
$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{u}} \times \vec{\mathbf{B}})$	Lorentz force equation
$\nabla \cdot \vec{\mathbf{D}} = \rho$	Gauss's law (Maxwell's equation)
$\nabla \cdot \vec{\mathbf{B}} = 0$	Gauss's law (Maxwell's equation)
$\nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t}$	continuity equation
$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$	Faraday's law (Maxwell's equation)
$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{\mathbf{D}}}{\partial t}$	Ampère's law (Maxwell's equation)

1.2 Field concept

Prior to undertaking the study of electromagnetic fields we must define the concept of a **field**. When we define the behavior of a quantity in a given region in terms of a set of values, one for each point in that region, we refer to this behavior of the quantity as a field. The value at each point of a field can be either measured experimentally or predicted by carrying out certain mathematical operations on some other quantities.

From the study of other branches of science, we know that there are both scalar and vector fields. Some of the field variables we use in this text are given in Table 1.3. There also exist definite relationships between these field quantities, and some of these are given in Table 1.4.

The permittivity (ϵ) and the permeability (μ) are properties of the medium. When the medium is a vacuum or free space, their values are

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$$

$$\epsilon_0 = 8.851 \times 10^{-12} \approx 10^{-9}/36\pi \text{ F/m}$$

From the equations listed in Table 1.4, Maxwell was able to predict that electromagnetic fields propagate in a vacuum with the speed of light. That is,

$$c = (\mu_0 \epsilon_0)^{-1/2} \text{ m/s}$$

1.3 Vector analysis

Vector analysis is the language used in the study of electromagnetic fields. Without the use of vectors, the field equations would be quite unwieldy to write and onerous to remember. For example, the cross product of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ can be simply written as

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \vec{\mathbf{C}} \tag{1.1}$$

where $\vec{\mathbf{C}}$ is another vector. When expressed in scalar form, this equation yields a set of three scalar equtions. In addition, the appearance of these scalar equations depends upon the coordinate system. In the rectangular coordinate system, the previous equation is a concise version of the following three equations:

$$A_y B_z - A_z B_y = C_x \tag{1.2a}$$

$$A_z B_x - A_x B_z = C_y \tag{1.2b}$$

$$A_x B_y - A_y B_x = C_z \tag{1.2c}$$

You can easily see that the vector equation conveys the sense of a cross product better than its three scalar counterparts. Moreover, the vector representation is independent of the coordinate system. Thus, vector analysis helps us to simplify and unify field equations.

By the time a student is required to take the first course in electromagnetic theory, he/she has had a very limited exposure to vector analysis. The student may be competent to perform such vector operations as the gradient, divergence, and curl, but may not be able to describe the significance of each operation. The knowledge of each vector operation is essential to appreciate the development of electromagnetic field theory.

Quite often, a student does not know that (a) the unit vector that transforms a scalar surface to a vector surface is always normal to the surface, (b) a thin sheet (negligible thickness) of paper has two surfaces, (c) the direction of the line integral along the boundary of a surface depends upon the direction of the unit normal to that surface, and (d) there is a difference between an open surface and a closed surface. These concepts are important, and the student must comprehend the significance of each.

There are two schools of thought on the study of vector analysis. Some authors prefer that each vector operation be introduced only when it is needed, whereas others believe that a student must gain adequate proficiency in all vector operations prior to exploring electromagnetic field theory. We prefer the latter approach and for this reason have devoted Chapter 2 to the study of vectors.

1.4 Differential and integral formulations

Quite often a student does not understand why we present the same idea in two different forms: the differential form and the integral form. It must be pointed out that the integral form is useful to explain the significance of an equation, whereas the differential form is convenient for performing mathematical operations. For example, we express the equation of continuity of current in the differential form as

$$\nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t} \tag{1.3}$$

where $\vec{\mathbf{J}}$ is the volume current density and ρ is the volume charge density. This equation states that the divergence of current density at a point is equal to the rate at which the charge density is changing at that point. The usefulness of this equation lies in the fact that we can use it to calculate the rate at which the charge density is changing at a point when the current density is known at that point. However, to highlight the physical significance of this equation, we have to enclose the charge in a volume v and perform volume integration. In other words, we have to express (1.3) as

$$\int_v \nabla \cdot \vec{\mathbf{J}}\, dv = -\int_v \frac{\partial \rho}{\partial t}\, dv \tag{1.4}$$

We can now apply the divergence theorem to transform the volume integral on the left-hand side into a closed surface integral. We can also interchange the operations of integration and differentiation on the right-hand side of equation (1.4). We can now obtain

$$\oint_s \vec{\mathbf{J}} \cdot \vec{d\mathbf{s}} = -\frac{\partial}{\partial t} \int_v \rho\, dv \tag{1.5}$$

This equation is an integral formulation of (1.3). The integral on the left-hand side represents the net outward current I through the closed surface s bounding volume v. The integral on the right-hand side yields the charge q inside the volume v. This equation, therefore, states that the *net outward current through a closed surface bounding a region is equal to the rate at which the charge inside the region is decreasing with time.* In other words,

$$I = -\frac{dq}{dt} \tag{1.6}$$

which is a well-known circuit equation when the negative sign is omitted.

The details of the preceding development are given in Chapter 4. We used this example at this time just to show that (1.3) and (1.5) are the same and that they embody the same basic idea.

1.5 Static fields

Once again we face the dilemma of how to begin the presentation of electromagnetic field theory. Some authors believe in starting with the presentation of Maxwell's equations as a basic set of postulates and then summarizing the results of many years of experimental observations of electromagnetic effects. We, however, think that the field theory should always be developed by making maximum possible use of the concepts previously discussed in earlier courses in physics. For this reason we first discuss static fields.

In the study of electrostatics, or static electric fields, we assume that (a) all charges are fixed in space, (b) all charge densities are constant in time, and (c) the charge is the source of the electric field. Our interest is to determine (a) the electric field intensity at any point, (b) the potential distribution, (c) the forces exerted by the charges on other charges, and (d) the electric energy distribution in the region. We will also explore how a capacitor stores energy. To do so, we will begin our discussion with Coulomb's law and Gauss's law and formulate such well-known equations as Poisson's equation and Laplace's equation in terms of potential functions. We will show that the electric field at any point is perpendicular to an equipotential surface and emphasize its ramifications. Some of the equations pertaining to electrostatic fields are given in Table 1.5 (see below).

Table 1.5. Electrostatic field equations

Coulomb's law:	$\vec{\mathbf{F}} = q\vec{\mathbf{E}}$		
Electric field:	$\vec{\mathbf{E}} = \dfrac{Q\vec{\mathbf{a}}_R}{4\pi\epsilon R^2}$	or	$\vec{\mathbf{E}} = \dfrac{1}{4\pi\epsilon}\displaystyle\int_v \dfrac{\rho\vec{\mathbf{a}}_R}{R^2}\,dv$
Gauss's law:	$\nabla\cdot\vec{\mathbf{D}} = \rho$	or	$\displaystyle\oint_s \vec{\mathbf{D}}\cdot\vec{ds} = Q$
Conservative $\vec{\mathbf{E}}$ field:	$\nabla\times\vec{\mathbf{E}} = 0$	or	$\displaystyle\oint_c \vec{\mathbf{E}}\cdot\vec{d\ell} = 0$
Potential function:	$\vec{\mathbf{E}} = -\nabla V$	or	$V_{ba} = -\displaystyle\int_a^b \vec{\mathbf{E}}\cdot\vec{d\ell}$
Poisson's equation:	$\nabla^2 V = -\dfrac{\rho}{\epsilon}$		
Laplace's equation:	$\nabla^2 V = 0$		
Energy density:	$w_e = \frac{1}{2}\vec{\mathbf{D}}\cdot\vec{\mathbf{E}}$		
Constitutive relationship:	$\vec{\mathbf{D}} = \epsilon\vec{\mathbf{E}}$		
Ohm's law:	$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$		

Table 1.6. Magnetostatic field equations

Force equation:	$\vec{\mathbf{F}} = q\vec{\mathbf{u}} \times \vec{\mathbf{B}}$	or	$d\vec{\mathbf{F}} = I\,\vec{d\ell} \times \vec{\mathbf{B}}$
Biot–Savart law:	$\vec{d\mathbf{B}} = \dfrac{\mu}{4\pi}\dfrac{I\,\vec{d\ell} \times \vec{\mathbf{a}}_r}{r^2}$		
Ampère's law:	$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}$	or	$\oint_c \vec{\mathbf{H}} \cdot \vec{d\ell} = I$
Gauss's law:	$\nabla \cdot \vec{\mathbf{B}} = 0$	or	$\oint_s \vec{\mathbf{B}} \cdot \vec{d\mathbf{s}} = 0$
Magnetic vector potential:	$\vec{\mathbf{B}} = \nabla \times \vec{\mathbf{A}}$	or	$\vec{\mathbf{A}} = \dfrac{\mu}{4\pi}\displaystyle\int_c \dfrac{I\,\vec{d\ell}}{r}$
Magnetic flux:	$\Phi = \displaystyle\int_s \vec{\mathbf{B}} \cdot \vec{d\mathbf{s}}$	or	$\Phi = \oint_c \vec{\mathbf{A}} \cdot \vec{d\ell}$
Magnetic energy:	$w_m = \frac{1}{2}\vec{\mathbf{B}} \cdot \vec{\mathbf{H}}$		
Poisson's equation:	$\nabla^2\vec{\mathbf{A}} = -\mu\vec{\mathbf{J}}$		
Constitutive relationship:	$\vec{\mathbf{B}} = \mu\vec{\mathbf{H}}$		

We already know that a charge in motion creates a current. If the movement of the charge is restricted in such a way that the resulting current is constant in time, the field thus created is called a magnetic field. Since the current is constant in time, the magnetic field is also constant in time. The branch of science relating to constant magnetic fields is called magnetostatics, or static magnetic fields. In this case, we are interested in the determination of (a) magnetic field intensity, (b) magnetic flux density, (c) magnetic flux, and (d) the energy stored in the magnetic field. To this end we will begin our discussion with the Biot-Savart law and Ampère's law and develop all the essential equations. From time to time we will also stress the correlation between the static electric and magnetic fields. Some of the important equations that we will either state or formulate in magnetostatics are given in Table 1.6.

There are numerous practical applications of static fields. Both static electric and magnetic fields are used in the design of many devices. For example, we can use a static electric field to accelerate a particle and a static magnetic field to deflect it. This scheme can be employed in the design of an oscilloscope and/or an ink-jet printer. We have devoted Chapter 6 to address some of the applications of static fields. Once a student has mastered the fundamentals of static fields, he/she should be able to comprehend their applications without further guidance from the instructor. The instructor may decide to highlight the salient features of each application and then treat it as a reading assignment. The discussion of real-life applications of the theory makes the subject interesting.

1.6 Time-varying fields

In the study of electric circuits, you were introduced to a differential equation that yields the voltage drop $v(t)$ across an inductor L when it carries a current $i(t)$. More often than not, the relationship is stated without proof as follows:

$$v = L\frac{di}{dt} \tag{1.7}$$

Someone with a discerning mind may have wondered about the origin of this equation. It is a consequence of a lifetime of work by Michael Faraday (1791–1867) toward an understanding of a very complex phenomenon called magnetic induction.

We will begin our discussion of time-varying fields by stating *Faraday's law of induction* and then explain how it led to the development of generators (sources of three-phase energy), motors (the workhorses of the industrialized world), relays (magnetic controlling mechanisms), and transformers (devices that transfer electric energy from one coil to another entirely by induction). One of the four well-known Maxwell equations is, in fact, a statement of Faraday's law of induction. At this time it will suffice to say that Faraday's law relates the induced electromotive force (emf) $e(t)$ in a coil to the time-varying magnetic flux $\Phi(t)$ linking that coil as

$$e = -\frac{d\Phi}{dt} \tag{1.8}$$

The significance of the negative sign (*Lenz's law*) and the derivation of (1.7) from (1.8) will be discussed in detail in this text.

We will also explain why Maxwell felt it necessary to modify Ampère's law for time-varying fields. The inclusion of displacement current (current through a capacitor) enabled Maxwell to predict that fields should propagate in free space with the velocity of light. The modification of Ampère's law is considered to be one of the most significant contributions by James Clerk Maxwell (1831–1879) in the area of electromagnetic field theory.

Faraday's law of induction, the modified Ampère law, and the two Gauss laws (one for the time-varying electric field and the other for the time-varying magnetic field) form a set of four equations; these are now called *Maxwell's equations.* These equations are given in Table 1.4. Evident from these equations is the fact that time-varying electric and magnetic fields are intertwined. In simple words, a time-varying magnetic field gives rise to a time-varying electric field and vice versa.

The modification of Ampère's law can also be viewed as a consequence of the equation of continuity or conservation of charge. This equation is also given in Table 1.4.

When a particle having a charge q is moving with a velocity $\vec{\mathbf{u}}$ in a region where there exist a time-varying electric field ($\vec{\mathbf{E}}$) and a magnetic field ($\vec{\mathbf{B}}$), it experiences a force ($\vec{\mathbf{F}}$) such that

$$\vec{\mathbf{F}} = q(\vec{\mathbf{E}} + \vec{\mathbf{u}} \times \vec{\mathbf{B}}) \tag{1.9}$$

We will refer to this equation as the *Lorentz force equation.*

With the help of the four Maxwell equations, the equation of continuity, and the Lorentz force equation we can now explain all the effects of electromagnetism.

1.7 Applications of time-varying fields

Among the numerous applications of electromagnetic field theory, we will consider those pertaining to the transmission, reception, and propagation of energy. This selection of topics is due to the fact that the solution of Maxwell's equations always leads to waves. The nature of the wave depends upon the medium, the type of excitation (source), and the boundary conditions.

The propagation of a wave may either be in an unbounded region (fields exist in an infinite cross section, such as free space) or in a bounded region (fields exist in a finite cross section, such as a waveguide or a coaxial transmission line).

Although most of the fields transmitted are in the form of spherical waves, they may be considered as plane waves in a region far away from the transmitter (radiating element, such as an antenna). How far "far away" is depends upon the wavelength (distance traveled to complete one cycle) of the fields. Using plane waves as an approximation, we will derive wave equations from Maxwell's equations in terms of electric and magnetic fields. The solution of these wave equations will describe the behavior of a plane wave in an unbounded medium. We will simplify the analysis by imposing restrictions such that (a) the wave is a uniform plane wave, (b) there are no sources of currents and charges in the medium, and (c) the fields vary sinusoidally in time. We will then determine (i) the expressions for the fields, (ii) the velocity with which they travel in a region, and (iii) the energy associated with them. We will also show that the medium behaves as if it has an impedance; we refer to this as *intrinsic impedance.* The intrinsic impedance of free space is approximately 377 Ω.

Our discussion of uniform plane waves will also include the effect of interface between two media. Here we will discuss (a) how much of the energy of the incoming wave is transmitted into the second medium or reflected back into the first medium, (b) how the incoming wave and reflected wave combine to form a standing wave, and (c) the condition necessary for total reflection.

We devote Chapter 9 to the discussion of transmission of energy from one end to the other via a transmission line. We will show that when one end of the transmission line is excited by a time-varying source, the transmission of energy is in the form of a wave. The wave equations in this case will be in terms of the voltage and the current at any point along the transmission line. The solution of these wave equations will tell us that a finite time is needed for the wave to reach the other end, and for practical transmission lines, the wave attenuates exponentially with the distance. The attenuation is due to the resistance and conductance of the transmission line. This results in a loss in energy along the entire length of the transmission line. However, at power frequencies (50 or 60 Hz) there is a negligible loss in energy due to radiation because the spacing between the conductors is extremely small in comparison with the wavelength.

As the frequency increases so does the loss of signal along the length of the transmission line. At high frequencies, the energy is transmitted from one point to another via waveguides. Although any hollow conductor can be used as a waveguide, the most commonly used waveguides have rectangular or circular cross sections. We will examine the necessary conditions that must be satisfied for the fields to exist, obtain field expressions, and compute the energy at any point inside the waveguide. The analysis involves the solution of the wave equation inside the waveguide subjected to external boundary conditions. The analysis is complex; thus, we will confine our discussion to a rectangular waveguide. Although the resulting equations appear to be quite involved and difficult to remember, we must not forget that they are obtained by simply applying the boundary conditions to a general solution of the wave equation.

A transmission line can be used to transfer energy from very low frequencies (even dc) to reasonably high frequencies. The waveguide, on the other hand, has a lower limit on the frequency called the *cutoff frequency.* The cutoff frequency depends upon the dimensions of the waveguide. Signals below the cutoff frequency cannot propagate inside the waveguide. Another major difference between a transmission line and a waveguide is that the transmission line can support the *transverse electromagnetic* (TEM) mode. In practice, both coaxial and parallel wire transmission lines use the TEM mode. However, such a mode cannot exist inside the waveguide. Why this is so will be explained in Chapter 10. The waveguide can support two different modes, the *transverse electric mode* and the *transverse magnetic mode.* The conditions for the existence of these modes will also be discussed.

The last application of Maxwell's equations that we will discuss in this text deals with electromagnetic radiation produced by time-varying sources of finite dimensions. The very presence of these sources adds