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To Mom and Dad

Contents

Preface

Although the title of this book is "Computational Algebraic Geometry", it could also be titled "Snapshots of Commutative Algebra via Macaulay 2". The aim is to bring algebra, geometry, and combinatorics to life by examining the interplay between these areas; it also provides the reader with a taste of algebra different from the usual beginning graduate student diet of groups and field theory. As background the prerequisite is a decent grounding in abstract algebra at the level of [56]; familiarity with some topology and complex analysis would be nice but is not indispensable. The snapshots which are included here come from commutative algebra, algebraic geometry, algebraic topology, and algebraic combinatorics. All are set against a backdrop of homological algebra. There are several reasons for this: first and foremost, homological algebra is the common thread which ties everything together. The second reason is that many computational techniques involve homological algebra in a fundamental way; for example, a recurring motif is the idea of replacing a complicated object with a sequence of simple objects. The last reason is personal – I wanted to give the staid and abstract constructs of homological algebra (e.g. derived functors) a chance to get out and strut their stuff. This is said only half jokingly – in the first class I ever had in homological algebra, I asked the professor what good Tor was; the answer that Tor is the derived functor of tensor product did not grip me. When I complained to my advisor, he said "Ah, but you can give a two line proof of the Hilbert syzygy theorem using Tor – go figure it out". What an epiphany it was! Note to student: if you don't know what homological algebra and derived functors are, one point of this book is to give a hands-on introduction to these topics.

Of course, to understand anything means being able to compute examples, so oftentimes rather than dwelling on details best left to specialized texts (e.g. showing simplicial homology is indeed a topological invariant) we plunge blithely forward into computations (both by hand and by computer) in order to get a feel for how things work. This engineering mentality may be bothersome to the fastidious reader, but the first word in the title is not "Theoretical" but "Computational". We work mostly in the category of graded rings and modules, so the geometric setting is usually projective space. One unifying theme is the study of finite free resolutions; in particular, lots of the geometric invariants we study can be read off from a free resolution. Advances in computing and algorithms over the last twenty years mean that these gadgets are actually computable, so we can get our hands dirty doing lots of examples. By the end of the book the reader should feel comfortable talking about the degree and genus of a curve, the dimension and Hilbert polynomial of a variety, the Stanley–Reisner ring of a simplicial complex (and simplicial homology) and such abstract things as Ext, Tor, and regularity. Overall, the book is something of an algebra smorgasbord, moving from an appetizer of commutative algebra to homological methods. Of course, homological algebra would be less tasty without a garnish of history, so we add a dash of algebraic topology and a pinch of simplicial complexes and combinatorics. For dessert, we give Stanley's beautiful application of these methods to solve a combinatorial problem (the upper bound conjecture for spheres).

One of the wonderful things about computational algebra is that it is very easy to generate and test ideas. There are numerous exercises where the reader is asked to write scripts to test open research conjectures; the idea is to get folks thinking about open problems at an early stage. It is also exciting to find (albeit a couple years too late!) a counterexample to a published conjecture; the reader gets a chance to do this. In short, the exercises are geared at convincing students that doing research mathematics does not consist solely of ruminating alone in a darkened room, but also of rolling up one's sleeves, writing some code, and having the computer do the legwork.

Rather than giving examples of scripts in pseudocode, I have chosen to use a specific computer algebra package (Macaulay 2, by Dan Grayson and Mike Stillman). Macaulay 2 is free, easy to use, fast and flexible. Another virtue of Macaulay 2 is that the syntax is pretty straightforward. Thus, Macaulay 2 scripts *look like* pseudocode, but the reader can have the satisfaction of typing in scripts and seeing them run. Macaulay 2 works over finite fields of characteristic ≤ 32749, also over $\mathbb{Q}$ and certain other fields of characteristic zero. The examples in this book are often computed over finite fields. As Eisenbud notes in [32] "Experience with the sort of computation we will be doing shows that working over $\mathbb{Z}/p$, where p is a moderately large prime, gives results identical to the results we would get in characteristic 0".

I include here a mea culpa. This book grew from a dilemma – to give students a tapa of advanced algebra means that one would like to include snippets from

commutative algebra	**algebraic geometry**	**topology and combinatorics**
Atiyah–Macdonald [3]	*Cox–Little–O'Shea* [23]	*Fulton* [41]
Balcerzyk–Jozefiak [6]	*Griffiths* [48]	*Munkres* [71]
Bruns–Herzog [21]	*Harris* [52]	*Spanier* [87]
Eisenbud [28]	*Hartshorne* [53]	*Stanley* [88]
Matsumura [64]	*Miranda* [69]	*Sturmfels* [92]
Sharp [84]	*Reid* [78]	*Weibel* [98]
Vasconcelos [95]	*Shafarevich* [82]	*Ziegler* [100]
⋮	⋮	⋮

This book should be thought of as an advertisement for other, more advanced texts (or, perhaps, texts where details omitted here are carefully worked out!); there is nothing here that cannot be found elsewhere. What I hope is novel is the emphasis on working with a keyboard at hand to try out computations, the choice of topics, and the commingling of algebra, combinatorics, topology, and geometry. There are all sorts of gaps (some even by design!); for example the Nullstellensatz is not proved, nor is Nakayama's lemma; and little is said about smoothness. The most egregious example of this occurs in Chapter 9, which gives a synopsis of algebraic curves. Since the sketch of Riemann–Roch uses residues, a one-hour turbo lecture on complex analysis is included as an appendix. But generally I have tried to resist the temptation to be completely comprehensive, hoping rather to be convincing without bogging down in detail. The two introductory algebraic geometry texts listed above (Cox–Little–O'Shea and Reid) are nice complementary readings. A good way for readers to begin this book is to flip to Appendix A, which gives a warm-up review of algebra concepts and an introduction to basic Macaulay 2 commands.

These notes grew out of a class taught to junior mathematics majors at Harvard in fall of 2000. I thank Harvard for providing a great postdoctoral experience, the N.S.F. for providing funding, and my students for being such a lively, engaged, hardworking and fun group; Richard Stanley was kind enough to cap the course with a guest lecture. I also thank all the folks from whom I've learned over the years – both in print (see above texts!) and in person. Many people were kind enough to provide feedback on drafts of

this book: Marcelo Aguiar, Harold Boas, Al Boggess, Jorge Calvo, Renzo Cavalieri, David Cox, Jim Coykendall, John Dalbec, Marvin Decker, Alicia Dickenstein, David Eisenbud, Bahman Engheta, Chris Francisco, Tony Geramita, Leah Gold, Mark Gross, Brian Harbourne, Mel Hochster, Morten Honsen, Graham Leuschke, Paulo Lima-Filho, John Little, Diane Maclagan, Juan Migliore, Rick Miranda, Alyson Reeves, Vic Reiner, Bill Rulla, Sean Sather-Wagstaff, Fumitoshi Sato, Jessica Sidman, Greg Smith, Jason Starr, Peter Stiller, Emil Straube, Alex Suciu, Hugh Thomas, Stefan Tohaneanu, Will Traves, Adam Van Tuyl, Pete Vermeire, Lauren Williams, and Marina Zompatori. To them, many, many thanks. It goes without saying that any blunders are a result of ignoring their advice. Updates to reflect changes to Macaulay 2, corrections, and (eventually) solutions to the problems will be posted at: `http://us.cambridge.org/mathematics/`

I owe much to Mike Stillman – teacher, mentor, and friend – who introduced me to most of the material here. I hope that the notes convey some of the enthusiasm and joy in mathematics that Mike imparted to me. To acknowledge my debt (and pay back some small portion!), all author royalties from this book go to the Cornell mathematics department graduate teaching excellence fund.