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Youssef Jabri
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Introduction

> The methods we will take up here are all variations on a basic result known to everyone who has done any walk in the hills: *the mountain pass lemma*.
>
> L. Nirenberg, Variational and topological methods in nonlinear problems, *Bull. Am. Math. Soc.*, **4**, 1981

Why a Book on the Mountain Pass Theorem?

The *mountain pass theorem* (henceforth abbreviated as MPT) is a "phenomenal result" that marks the beginning of a new approach to critical point theory. It constitutes a particularly interesting model for the abstract minimax principle known since the pioneering work of Ljusternik and Schnirelman in the 1940s. It is also the grandfather and the prototype of all the "postmodern" critical point results from the *linking* family. As early as it appeared, it attracted attention by raising up a lot of theoretical development and serving to solve a very large number of problems in many areas of nonlinear analysis.

The MPT has been intensively investigated. Indeed, there is actually a huge amount of references specifically devoted to its study or presenting one of its variants, generalizations, or applications. Its influence can be measured by the fact that you will rarely find a recent paper or book dealing with variational methods that do not cite it. Our aim here is to provide an expanded publication fully devoted to present some of its various forms and shed light on its numerous faces, comparing and classifying them when possible.

Who Should Read It?

The monograph may be used as a complementary textbook in a course on variational methods in nonlinear analysis. The reader is only supposed to be familiar with some elementary notions of topology and analysis. The first part, especially destined for beginners, aims to be a connection with the MPT starting from very simple notions of analysis. It consists of a very accessible expository of the basic background and main principles of critical point theory. We continue then gradually with more advanced

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topics including many very recent references, connecting the reader this way with up-to-date material not available anywhere in book literature.

More advanced readers working on critical point theory should also find it useful to have the large amount of information on the MPT that is scattered in the literature collected together and classified. They should appreciate as well the extended description in the final notes and comments of the items appearing in the very large bibliography on the MPT.

The Style and Format

In general, chapters begin with short abstracts, followed if necessary by the background material needed within, with pointers to the main references. Then the main results and their most important consequences and applications are given. We emphasize the basic ideas and principles.

> We premeditated to focus our attention on the abstract results rather than applications for three reasons. First, applications are very well documented in many recent books. (See a list following this discussion.) Second, the amount of details and technicalities they involve may sometimes hide the simple abstract ideas on which they rely. And third, this would have enlarged excessively the size of this book.

When judged to be too complicated or when requiring technical material not directly connected to the subject, the proofs are omitted and the corresponding results are only given in outline form for completeness. This is the case for the material involving algebraic topology notions or Morse or Morse-Conley theories, for example.

Chapters end in a systematic way with many complementary remarks and additional bibliographical references divided in blocks. There should be specific pointers (when appropriate) to

- alternative ways that could be used to present the material discussed in each chapter. This is an invitation to investigate the very rich existing literature.
- important "historical" contributions, so that the reader can trace the origin of these notions.
- the most recent developments, to follow the directions they are actually taking without having to go to other references.

This has been done with the intention to provide the reader with a comprehensive and as complete a reference as possible.

The Approach

The diversity and very large quantity of references dealing with the MPT makes it hard to find a pertinent and satisfactory classification that can serve as a plan. In particular, from a pure pedagogical point of view, chronological order does not seem to be a good choice because we noticed that ideas became easier to see with time and also that results are less burdened nowadays by technical details that may be omitted in a first contact.

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We remarked that the MPT was, for some reason, a fantastic testing tool used by anyone who has a new idea about a possible development in critical point theory, and we could not resist the temptation to try to present yet another look at minimax theorems in critical point theory through a special study of the MPT.

Our approach there is the following. In the first chapters, we will get a close look at the different ingredients involved in the elaboration of a critical point theorem, whereas in the subsequent chapters, we will discover how they are actually pushed to the limits, using each time some particular form of the MPT. This fantastic tool has grown so much so that this is indeed possible and this program really works! We could even treat some subjects that are not generally found in books on critical point theory.

The so-called "minimax methods" characterize a critical value c of a functional Φ by

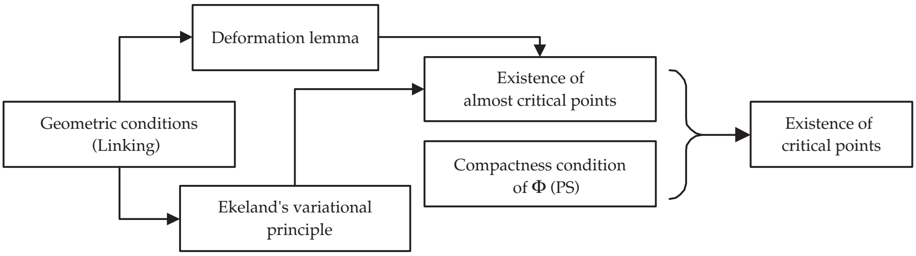
$$c = \inf_{K \in \mathcal{K}} \sup_{u \in K} \Phi(u).$$

The choice of the sets K must reflect some change in the topology of the (sub-)level sets $\Phi^\alpha = \{u;\ \Phi(u) < \alpha\}$ for the values α near to c, as we will see in Chapter 4.

If we carefully analyze minimax theorems, we will notice that independently of the level of smoothness ($\mathcal{C}^1$, Lipschitz continuity, etc.) of the functional they follow always the same scheme:

1. We require some geometric conditions where a relation appears between the values (levels) of the functional over sets that *link*.
2. Then, using either a (quantitative) deformation lemma or Ekeland's variational principle, we show that for some value c characterized by a minimax argument, there exists a Palais-Smale sequence of level c, that is, a sequence such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$.
3. Last, meaning some compactness condition of Palais-Smale type, we bring the amount of compactness required to conclude that c is a critical value.

The following figure describes this process. You must think of it as a graph in three dimensions, the third dimension (not represented) being the smoothness of Φ. This is very important, because the form of these principles and the techniques available vary dramatically according to the smoothness of Φ.



How do we get critical point theorems?

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As a constant in this book, we try always to go from the more elementary to the more sophisticated, gradually adding new elements, our aim being to exhibit clearly the exact role of each assumption and how it intervenes in the proof. In general, consecutive results will seem to be natural and facile generalizations of each other. This will indeed be visible in the first part. Chapters constitute different faces of the MPT. Although there is some logical ordering, they are independent and must not necessarily be read linearly. Nevertheless, Chapters 2, 3, and 4, relating respectively to (a first contact with) the Palais-Smale condition and both Ekeland's variational principle and the deformation lemma, are of critical importance. They present basic features of critical point theory and should be well known before any further reading.

Beginners *should not* linger, in a first reading, on the final comments and bibliographical remarks of the first chapters because they are linked with more advanced topics discussed in the chapters that follow. They should come back after mastering the new material.

How Is the Book Organized?

As we said before, the first part is initiative; it is intended to present the basics of critical point theory. After Chapter 1, where we briefly expose a historical description of the subject, we get a first and brief contact in Chapter 2, with a compactness condition on functionals that plays a central role in critical point theory, known as the Palais-Smale (PS) condition. It was introduced by R. Palais and S. Smale in the 1960s to allow the calculus of variations in the large to deal with mappings on general Banach manifolds. Chapters 3 and 4 recall two fundamental results in critical point theory. Chapter 3 is on Ekeland's variational principle while Chapter 4 is on the deformation lemma. They are behind the scenes in the statements and proofs of all the abstract results that will be covered in this monograph.

The second part begins with some "elementary" versions of the MPT appropriate enough to introduce its different aspects. Chapter 5 describes a finite dimensional ancestor of the MPT due to Courant (1950), very similar to the version of Ambrosetti-Rabinowitz both in its statement and its proof. Then Chapter 6 presents, in a topological adaptation of the concepts of Chapter 5, a purely topological version of the MPT. We continue then in Chapter 7 with the Ambrosetti-Rabinowitz MPT, the result properly known as *the* MPT in the literature. We also present two models of *standard* applications of the MPT to variational problems for illustration. The final chapter in this part, Chapter 8, contains some of the earliest variants of the MPT, including a dual form. It also presents some details of one of the first extensions of the MPT to higher dimensions, destined essentially to provide a more appropriate tool to treat some particular kinds of semilinear elliptic equations.

In the third part, we should relate more deeply the *topological* consideration involved in the MPT. Chapter 9 gives a detailed account of the results that accumulated gradually during years concerning what should be the right geometry to answer in the affirmative the question of the "limiting case" in the MPT. We will have the opportunity to see non-linear analysts at work on an exciting example. Chapter 10 is a continuation of Chapter 2. It focuses on the asymptotic behavior of functionals satisfying (PS) when some control is imposed on the level sets and presents some second-order information on functionals

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that satisfy the geometric conditions of the MPT. Chapter 11 discusses in detail the symmetric MPT, a multiplicity result in which the functional is supposed to be invariant under the action of a group of symmetries. It also discusses some extensions: the fountain theorem and its dual form and a procedure that inductively uses the MPT to obtain multiplicity results without passing by any *Index theory*. In Chapter 12, we describe the structure of the critical set in the situation of the MPT without requiring any nondegeneracy condition. In Chapter 13, we first present a minimax theorem that uses a "weighted" form of the (PS) condition. Then, we recall a very interesting procedure, attributed to Corvellec, for deducing new critical point theorems with weighted (PS) condition from older ones with the standard (PS) condition just by performing a change of metric.

The fourth part is devoted to some versions of the MPT that can be described as *nonsmooth* in many senses. They are motivated by applications to variational problems for functionals lacking regularity. Chapter 14 is consecrated to a situation of functionals $\Phi + \Psi$ considered as *semismooth*, where Φ is a $\mathcal{C}^1$-functional and Ψ is proper, lower semicontinuous, and convex. In Chapter 15, we present a version of the MPT for locally Lipschitz functionals on Banach spaces. Chapter 16 goes further in nonsmoothness. We consider *continuous functionals* defined on *metric spaces*.

The fifth part is devoted to some speculations about the mountain pass geometry. In Chapter 17, a special extension of critical point theory to smooth functionals defined on *convex sets* is recalled briefly, and a corresponding version of the MPT with two different proofs is then given. The first proof is based on an appropriate form of the deformation lemma, while the second uses Ekeland's variational principle. While in Chapter 18, some variational methods in ordered Banach spaces are investigated. In particular, a variant of the MPT in order intervals in the spirit of some pioneering work by Hofer that exploited the natural *ordering*, intrinsic to semilinear elliptic problems, is given. In Chapter 19, we review the notion of linking that proved to be very important in critical point theory. This is a unified formulation of the geometric conditions that appear, among other results, in the MPT. We will see various definitions of this notion that led to many new results extending the MPT in different contexts: homotopical, homological, local, isotopic, and so forth. Chapter 20 is devoted to the "intrinsic MPT" and to one of its metric extensions. And in Chapter 21, we present some bounded variants of the MPT where the minimaxing paths are confined in a bounded region.

In the sixth and last part (Technical Climbs), we take the risk to go a little farther from the main road to discover the neighboring landscape. We investigate some topics that require the user to have a more advanced level and broader interests. In Chapter 22, we present three numerical implementations of the MPT. We present first a "mountain pass algorithm" that begins to be widely used. Then, we describe a partially interactive algorithm for computing unstable solutions of differential equations and a third algorithm used in quantum chemistry. In Chapter 23, we expose two approaches relying on the MPT to investigate the stability of the multiplicity results obtained by the symmetric MPT when the symmetry is broken. While in Chapter 24, we indicate how the MPT was used by Rabinowitz to treat some bifurcation problems. The last chapter in this part, Chapter 25, contains a series of short descriptions of many interesting variants of the MPT and some atypical or ingenious applications that did not find a place in the text in its actual form. This may be done in forthcoming editions.

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For the convenience of the reader, we include at the end an appendix where we recall some definitions and basic properties of Sobolev spaces. We also investigate Nemystskii operators. Mastering these two topics is essential to treat nonlinear differential problems by critical point theorems in general and by the MPT in particular. Finally, we give a *large* bibliography on the subject, whose size might be explained by the growing interest stirred up by this specific area of analysis. The subject is so healthy that it is impossible to be exhaustive and any attempt in this direction gives only a momentary snapshot that may become obsolete in little time. An index is also given to help in navigating the book.

We would like to mention a certain number of very interesting and useful books on variational methods and critical point theory focusing on some particular aspects that appeared these past years, enhancing the existing bibliography and confirming that the theory is passing through a new age. We cite, among others, the following excellent references [43, 74, 197, 205, 315, 360, 411, 425, 517, 534, 623, 628, 654, 700, 748, 816, 882, 957, 982]. Of course standard books consecrated to nonlinear functional analysis, to cite only a few of them [49, 121, 131, 145, 285, 430, 520, 641, 825, 983, 984, 986], are also an indispensable part of the bibliography to be consulted by any serious "critical point theorist" and should certainly not be neglected.

Conventions and Notations

The typographical conventions used are standard. Theorems, lemmata, corollaries, and propositions are numbered consecutively and the counter they use is reset each chapter. For example, Theorem 5.3 refers to the third theorem in the fifth chapter. Equations are also numbered according to their appearance in chapters; for example, (6.4) refers to the fourth equation in the sixth chapter. The Notes at the end of the different chapters are also numbered according to their order of appearance within each chapter with the symbol ◇ before their number; for example ◇ 4.3. The sans serif font is used to report quotation from the existing literature.

By the end of this mountaineering expedition, I hope sincerely that you will feel all the beauty and elegance of the subject and all the pleasure experienced by the mathematicians who discovered the different versions of the MPT during their climbs.

Acknowledgments

I would like to express my gratitude and recognition to the kind people who transmitted their enthusiasm to me and expressed their interest in the project of writing this monograph. I. Ekeland, P.H. Rabinowitz, and M. Willem deserve a special mention. I am indebted also to all those who kindly sent me materials in connection with the subject. Of course, none else is to blame for the misprints and errors which are to be found.

Should you have any remark, suggestion, or correction, please do not hesitate to contact me.

Oujda, April 2001

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Retrospective

. . . it was Riemann who aroused great interest in them [problems of the calculus of variations] by proving many interesting results in function theory by assuming Dirichlet's principle . . .

C.B. Morrey Jr., *Multiple integrals in the calculus of variations,* Springer-Verlag, 1966.

Variational and topological methods have proved to be powerful tools in the resolution of concrete nonlinear boundary value problems appearing in many disciplines where classical methods may fail. This is the case in particular for critical point theory, which became very successful these past years. Its success is due, in addition to its theoretical interest, to the large number of problems it handles.

To understand how the interest arose in this discipline, let us recall some of the main evolutions of its underlying principles in a series of historical events.

An Algorithm for Finding Extrema by Fermat

In a pure chronological order, the first *variational* treatments may be traced to the Greeks, who were interested in isoperimetric problems. Hero of Alexandria discovered in 125 B.C. that the light reflected by a mirror follows the shortest possible path. Fermat proved in 1650 that the light follows the path that takes the *least* time to go from one point to an other.

A little time before, in 1637, he published without proof in a small treatise entitled *Methodus ad Disquirendam Maximam et Minimam* an algorithm for finding the extrema of algebraic functions. It may be described as follows:

> We want to find a maximum or a minimum of a function f whose variable is A. We replace A with $A + E$ in the expression of f (E plays the role of a little Δx) and suppose that $f(A + E) \approx f(A)$. Then, we divide each term by E and eliminate all terms where E appears (i.e., we take $E = 0$). The values for which the result vanishes correspond then to a minimum or a maximum.
>
> This algorithm will certainly be more clear with an example. Let us consider a rectangle $\mathcal{R}$ with sides A and B, and perimeter $P = 2(A + B)$. The area of $\mathcal{R}$ is

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$AB = A(P/2 - A)$. We want to find the lengths of A and B for which $\mathcal{R}$ has a maximal area, for a fixed parameter P.
Set

$$f(A) = A\left(\frac{P}{2} - A\right).$$

Then, $f(A + E) = (AP + EP)/2 - E^2 - X^2 - 2XE$. When taking

$$f(A) = f(A + E),$$

we get

$$0 = \frac{EP}{2} - E^2 - 2XE.$$

Dividing then by E, we get

$$0 = \frac{P}{2} - E - 2X.$$

Take $E = 0$. Then, $X = P/4$, i.e.,

$$X = Y.$$

So, $\mathcal{R}$ is the *square* of side $P/4$.

The procedure of Fermat turns out to be just evaluating

$$\lim_{E \to 0} \frac{f(A + E) - f(A)}{E}$$

and looking for the extrema of f at the points where the derivative of f vanishes.

Notice that at that time, nobody knew what a *limit* or a *derivative* was.

Appearance of Calculus

Calculus and derivatives were first discovered in connection with the study of the variation of functions (a concept which was also not yet well comprehended), simultaneously and independently by two exceptional mathematicians: Newton and Leibnitz (see, for example, [124, 467]).

The approach of Newton, in 1672, relied on kinematics. He imagined an *auxiliary* moving point M following the curve describing the real function to study, like a car moving on a road. He supposed that the "speed" of the projection of M on the X-axis moves *uniformly*, and he noticed that, as a consequence, M should move forward slowly when the curve is flat and quickly elsewhere. And instead of following the point M in its trajectory on the curve, Newton discovered that he would learn as much while following its projection on the Y-axis. The advantage was that he would have to work on a line, which was the only type of curve that one could really treat in those times. The speed with which one explores the X-coordinates does not have an absolute sense and is used only to give some mental support.

The approach of Leibnitz does not rely on kinematics and is more abstract than that of Newton. It is essentially the one we use nowadays. In 1684 he had a publication that

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appeared in *Acta eriditorum*, entitled *Nova methodus proximamis et minimis, itemque tangentibus, qua nec irrationales quantiates moratur* (A new method for maxima and minima, and also tangents, which can be used with fractional and irrational quantities too), which gave some general rules of calculus for *differentials* using the symbol d. He presented among other things, the formulas he had already obtained in 1677:

$$d(xy) = x dy + y dx, \qquad d(x/y) = (y dx - x dy)/y^2, \qquad dx^n = nx^{n-1}.$$

As geometrical applications, he studied tangents, minima, and maxima. In particular, he gave the conditions $dv = 0$ for a minimum or a maximum.

Nevertheless, these two founders of modern analysis did not convince the whole mathematical community. The reason was that they did not get control of a concept at the heart of this process: the *limit*. To be accepted by all, calculus had to wait until 1820, when Cauchy gave the final and unassailable definition of this notion.

Meanwhile, in 1743, Euler submitted "*A method for finding curves possessing certain properties of maximum or minimum* [376]". And, in 1744, he published the first book on the calculus of variations, in which he expressed his conviction that the nature acts everywhere following some rule of maximum or minimum[1]:

> . . . je suis convincu que partout la nature agit selon quelque principe d'un maximum ou minimum . . .

This book was a source of inspiration for the mathematicians who came later (according to [882]).

Dirichlet Principle at the Roots of Modern Critical Point Theory

Critical point theory is concerned with *variational problems*. These are problems $(\mathcal{P})$ such that there exists a smooth functional Φ whose critical points are solutions of $(\mathcal{P})$.

The *abstract process* followed in modern critical point theorems has its roots in the Dirichlet principle. Dirichlet postulated at Göttingen that, given an open bounded set Ω in the plane and a continuous function $h : \partial\Omega \to \mathbb{R}$, the boundary value problem

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \\ \quad u = h & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

admits a *smooth* solution u that minimizes the functional[2]

$$\Phi(u) = \int_\Omega \sum_{i=1}^{2} (D_i h(x))^2 \, dx \tag{1.2}$$

in the set of smooth functions defined on Ω that are equal to h on $\partial\Omega$. This principle was called the Dirichlet principle by Riemann in his thesis in 1851. He used it as a basis for his theory of analytic functions of a complex variable. "He studied the properties

[1] Note also a similar quotation by Maupertuis in Chapter 25.

[2] By *functional* we mean a function defined on a space whose elements are functions, and by *smooth* that it is continuous on $\overline{\Omega}$ and that its Laplacian exists in the usual sense using Fréchet derivatives, so that $u \in \mathcal{C}^2(\Omega; \mathbb{R}) \cap \mathcal{C}(\overline{\Omega}; \mathbb{R})$. This particular one is known as the *Dirichlet integral*.

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of analytic functions by investigating harmonic functions in the plane," to quote Brézis and Browder [150].

The *Euler equation* corresponding to (1.2) is the equation (1.1). This appellation is due to the fact that Euler discovered the first general necessary condition $f'(u) = 0$ which must be satisfied by a smooth functional f at an extremum u. The condition was known to hold for polynomials since Fermat.

And any *smooth minimizer* of (1.2), such that $u = h$ on $\partial\Omega$, is a solution of (1.1). This very important principle was already observed for the Laplace operator, some time before Dirichlet did, by Green in 1833. The idea was defended by Gauss in 1839 (in his study of magnetism) and (the future Lord Kelvin) W. Thomson in 1847. (The reference [642] is entirely dedicated to the history of Dirichlet principle.)

Weierstrass pointed out in 1870, that the existence of the minimum is not assured in spite of the fact that the functional Φ may be bounded from below. The subtle difference between minimum and *infimum*, not yet perceived in these early times, was made. He proved that the functional

$$\Psi(u) = \int_{-1}^{1} (x.u'(x))^2\, dx$$

possesses an infimum but does not admit any minimum in the set

$$\mathfrak{C} = \left\{u \in \mathcal{C}^1[-1, 1];\ u(-1) = 0, u(1) = 1\right\}.$$

Indeed, if we consider the sequence

$$u_n = \frac{1}{2} + \frac{\arctan(x/n)}{2\arctan(1/n)}, \qquad n = 1, 2, \ldots,$$

then, $u_n \in \mathfrak{C}$ and $\Psi(u_n) \to 0$. If some u was a minimum, then $xu'(x) = 0$ on $[-1, 1]$. Therefore, $u =$ constant, in contradiction with $u(-1) = 0$ and $u(1) = 1$.

Another nice counterexample to the Dirichlet principle, attributed to Courant [279], is the following. Consider the (one-dimensional) integral

$$\Phi(u) = \int_0^1 \left(1 + (u'(x))^2\right)\, dx,$$

for $\Omega =]0, 1[$, where the admissible functions u are those in $\mathcal{C}^1([0, 1]; \mathbb{R})$ with $u(0) = 0$ and $u(1) = 1$.

Stating correctly and justifying the Dirichlet principle became a challenge for the mathematicians in the second half of the 19$^{\text{th}}$ century. After many partially successful tentative attempts to solve the problem by many mathematicians, Arzela used his famous compactness theorem in 1897 to treat the problem, under some conditions, and was not far from succeeding. Only few times after, following Arzela's ideas, the Dirichlet principle was established rigorously in certain important cases by Hilbert [470], Lebesgue [556], and others in what is considered the beginning of the *direct methods of the calculus of variations*.

Tonelli is the author of the three volumes "Fondamenti di calcolo delle variazioni" [923] in 1921–23, one of the main references used by the mathematicians of the 1930s.

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