Advanced Dynamics

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1 Introduction to particle dynamics

In the study of dynamics at an advanced level, it is important to consider many approaches and points of view in order that one may attain a broad theoretical perspective of the subject. As we proceed we shall emphasize those methods which are particularly effective in the analysis of relatively difficult problems in dynamics. At this point, however, it is well to review some of the basic principles in the dynamical analysis of systems of particles. In the process, the kinematics of particle motion will be reviewed, and many of the notational conventions will be established.

1.1 Particle motion

The laws of motion for a particle

Let us consider Newton's three laws of motion which were published in 1687 in his *Principia*. They can be stated as follows:

I. Every body continues in its state of rest, or of uniform motion in a straight line, unless compelled to change that state by forces acting upon it.
II. The time rate of change of linear momentum of a body is proportional to the force acting upon it and occurs in the direction in which the force acts.
III. To every action there is an equal and opposite reaction; that is, the mutual forces of two bodies acting upon each other are equal in magnitude and opposite in direction.

In the dynamical analysis of a system of particles using Newton's laws, we can interpret the word "body" to mean a particle, that is, a certain fixed mass concentrated at a point. The first two of Newton's laws, as applied to a particle, can be summarized by the *law of motion*:

$$\mathbf{F} = m\mathbf{a} \tag{1.1}$$

Here $\mathbf{F}$ is the total force applied to the particle of mass m and it includes both direct contact forces and field forces such as gravity or electromagnetic forces. The acceleration $\mathbf{a}$ of the particle must be measured relative to an *inertial* or *Newtonian* frame of reference. An example of an inertial frame is an xyz set of axes which is not rotating relative to the "fixed" stars and has its origin at the center of mass of the solar system. Any other reference frame which is not rotating but is translating at a constant rate relative to an inertial frame is itself an inertial frame. Thus, there are infinitely many inertial frames, all with constant translational velocities relative to the others. Because the relative velocities are constant, the acceleration of a given particle is the same relative to any inertial frame. The force $\mathbf{F}$ and mass m are also the same in all inertial frames, so Newton's law of motion is identical relative to all inertial frames.

Newton's third law, the law of action and reaction, has a corollary assumption that the interaction forces between any two particles are directed along the straight line connecting the particles. Thus we have the *law of action and reaction*:

When two particles exert forces on each other, these interaction forces are equal in magnitude, opposite in sense, and are directed along the straight line joining the particles.

The *collinearity* of the interaction forces applies to all mechanical and gravitational forces. It does not apply, however, to interactions between moving electrically charged particles for which the interaction forces are equal and opposite but not necessarily collinear. Systems of this sort will not be studied here.

An alternative form of the equation of motion of a particle is

$$\mathbf{F} = \dot{\mathbf{p}} \tag{1.2}$$

where the *linear momentum* of the particle is

$$\mathbf{p} = m\mathbf{v} \tag{1.3}$$

and $\mathbf{v}$ is the particle velocity relative to an inertial frame.

Kinematics of particle motion

The application of Newton's laws of motion to a particle requires that an expression can be found for the acceleration of the particle relative to an inertial frame. For example, the position vector of a particle relative to a *fixed* Cartesian frame might be expressed as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1.4}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are *unit vectors*, that is, vectors of unit magnitude which have the directions of the positive x, y, and z axes, respectively. When unit vectors are used to specify a vector in 3-space, the three unit vectors are always linearly independent and are nearly always mutually perpendicular. The velocity of the given particle is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{1.5}$$

and its acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \tag{1.6}$$

relative to the inertial frame.

A force $\mathbf{F}$ applied to the particle may be described in a similar manner.

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{1.7}$$

where (F_x, F_y, F_z) are the *scalar components* of $\mathbf{F}$. In general, the force components can be functions of position, velocity, and time, but often they are much simpler.

If one writes Newton's law of motion, (1.1), in terms of the Cartesian unit vectors, and then equates the scalar coefficients of each unit vector on the two sides of the equation, one obtains

$$\begin{aligned} F_x &= m\ddot{x} \\ F_y &= m\ddot{y} \\ F_z &= m\ddot{z} \end{aligned} \tag{1.8}$$

These three scalar equations are equivalent to the single vector equation. In general, the scalar equations are coupled through the expressions for the force components. Furthermore, the differential equations are often nonlinear and are not susceptible to a complete analytic solution. In this case, one can turn to numerical integration on a digital computer to obtain the complete solution. On the other hand, one can often use energy or momentum methods to obtain important characteristics of the motion without having the complete solution.

The calculation of a particle acceleration relative to an inertial Cartesian frame is straightforward because the unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are fixed in direction. It turns out, however, that because of system geometry it is sometimes more convenient to use unit vectors that are not fixed. For example, the position, velocity, and acceleration of a particle moving along a circular path are conveniently expressed using radial and tangential unit vectors which change direction with position.

As a more general example, suppose that an arbitrary vector $\mathbf{A}$ is given by

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 \tag{1.9}$$

where the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a mutually orthogonal set such that $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. This unit vector triad changes its orientation with time. It rotates as a rigid body with an angular velocity $\boldsymbol{\omega}$, where the direction of $\boldsymbol{\omega}$ is along the axis of rotation and the positive sense of $\boldsymbol{\omega}$ is in accordance with the right-hand rule.

The first time derivative of $\mathbf{A}$ is

$$\dot{\mathbf{A}} = \dot{A}_1\mathbf{e}_1 + \dot{A}_2\mathbf{e}_2 + \dot{A}_3\mathbf{e}_3 + A_1\dot{\mathbf{e}}_1 + A_2\dot{\mathbf{e}}_2 + A_3\dot{\mathbf{e}}_3 \tag{1.10}$$

where

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \qquad (i = 1, 2, 3) \tag{1.11}$$

Thus we obtain the important equation

$$\dot{\mathbf{A}} = (\dot{\mathbf{A}})_r + \boldsymbol{\omega} \times \mathbf{A} \tag{1.12}$$

Here $\dot{\mathbf{A}}$ is the time rate of change of $\mathbf{A}$, as measured in a nonrotating frame that is usually considered to also be inertial. $(\dot{\mathbf{A}})_r$ is the derivative of $\mathbf{A}$, as measured in a rotating frame in which the unit vectors are fixed. It is represented by the first three terms on the right-hand side of (1.10). The term $\boldsymbol{\omega} \times \mathbf{A}$ is represented by the final three terms of (1.10). In detail, if the angular velocity of the rotating frame is

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3 \tag{1.13}$$

then

$$\begin{aligned}\dot{\mathbf{A}} = {} & (\dot{A}_1 + \omega_2 A_3 - \omega_3 A_2)\,\mathbf{e}_1 + (\dot{A}_2 + \omega_3 A_1 - \omega_1 A_3)\,\mathbf{e}_2 \\ & + (\dot{A}_3 + \omega_1 A_2 - \omega_2 A_1)\,\mathbf{e}_3\end{aligned} \tag{1.14}$$

Velocity and acceleration expressions for common coordinate systems

Let us apply the general equation (1.12) to some common coordinate systems associated with particle motion.

Cylindrical coordinates

Suppose that the position of a particle P is specified by the values of its cylindrical coordinates (r, ϕ, z). We see from Fig. 1.1 that the position vector $\mathbf{r}$ is

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z \tag{1.15}$$

where we notice that r is not the magnitude of $\mathbf{r}$. The angular velocity of the $\mathbf{e}_r\mathbf{e}_\phi\mathbf{e}_z$ triad is

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{e}_z \tag{1.16}$$

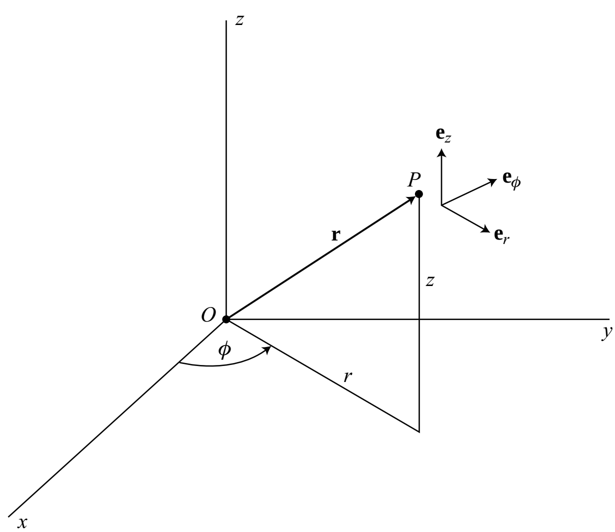

Figure 1.1.

so we find that $\dot{\mathbf{e}}_z$ vanishes and

$$\dot{\mathbf{e}}_r = \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\phi}\mathbf{e}_\phi \tag{1.17}$$

Thus, the velocity of the particle P is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \tag{1.18}$$

Similarly, noting that

$$\dot{\mathbf{e}}_\phi = \boldsymbol{\omega} \times \mathbf{e}_\phi = -\dot{\phi}\mathbf{e}_r \tag{1.19}$$

we find that its acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r} - r\dot{\phi}^2)\,\mathbf{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z \tag{1.20}$$

If we restrict the motion such that $\dot{z}$ and $\ddot{z}$ are continuously equal to zero, we obtain the velocity and acceleration equations for plane motion using *polar coordinates*.

Spherical coordinates

From Fig. 1.2 we see that the position of particle P is given by the spherical coordinates (r, θ, ϕ). The position vector of the particle is simply

$$\mathbf{r} = r\mathbf{e}_r \tag{1.21}$$

The angular velocity of the $\mathbf{e}_r\,\mathbf{e}_\theta\,\mathbf{e}_\phi$ triad is due to $\dot{\theta}$ and $\dot{\phi}$ and is equal to

$$\boldsymbol{\omega} = \dot{\phi}\cos\theta\;\mathbf{e}_r - \dot{\phi}\sin\theta\;\mathbf{e}_\theta + \dot{\theta}\;\mathbf{e}_\theta \tag{1.22}$$

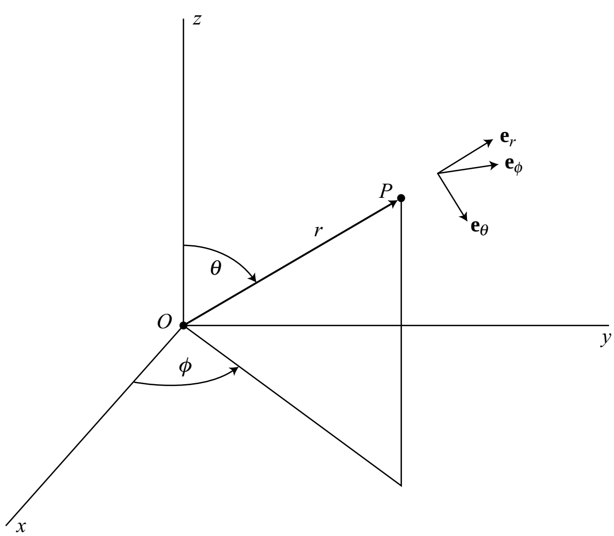

Figure 1.2.

We find that

$$
\begin{aligned}
\dot{\mathbf{e}}_r &= \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\theta}\mathbf{e}_\theta + \dot{\phi} \sin\theta \ \mathbf{e}_\phi \\
\dot{\mathbf{e}}_\theta &= \boldsymbol{\omega} \times \mathbf{e}_\theta = -\dot{\theta}\mathbf{e}_r + \dot{\phi} \cos\theta \ \mathbf{e}_\phi \\
\dot{\mathbf{e}}_\phi &= \boldsymbol{\omega} \times \mathbf{e}_\phi = -\dot{\phi} \sin\theta \ \mathbf{e}_r - \dot{\phi} \cos\theta \ \mathbf{e}_\theta
\end{aligned}
\tag{1.23}
$$

Then, upon differentiation of (1.21), we obtain the velocity

$$
\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\phi} \sin\theta \ \mathbf{e}_\phi \tag{1.24}
$$

A further differentiation yields the acceleration

$$
\begin{aligned}
\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta)\,\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta \cos\theta)\,\mathbf{e}_\theta \\
+ (r\ddot{\phi} \sin\theta + 2\dot{r}\dot{\phi} \sin\theta + 2r\dot{\theta}\dot{\phi} \cos\theta)\,\mathbf{e}_\phi
\end{aligned}
\tag{1.25}
$$

Tangential and normal components

Suppose a particle P moves along a given path in three-dimensional space. The position of the particle is specified by the single coordinate s, measured from some reference point along the path, as shown in Fig. 1.3. It is convenient to use the three unit vectors $(\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b)$ where $\mathbf{e}_t$ is tangent to the path at P, $\mathbf{e}_n$ is normal to the path and points in the direction of the center of curvature C, and the binormal unit vector is

$$
\mathbf{e}_b = \mathbf{e}_t \times \mathbf{e}_n \tag{1.26}
$$

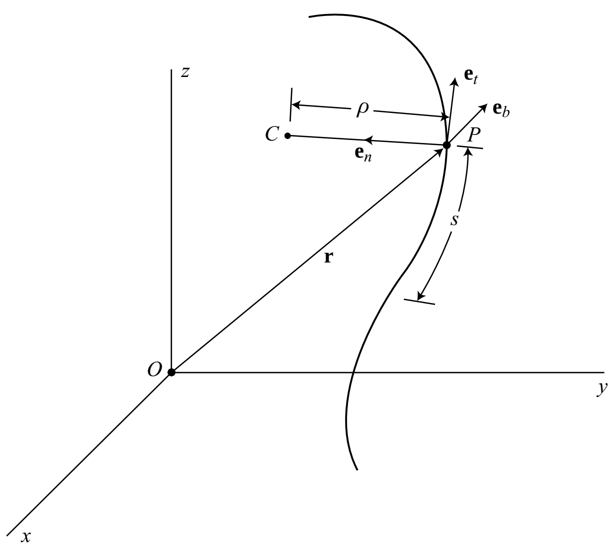

Figure 1.3.

The velocity of the particle is equal to its speed along its path, so

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{s}\mathbf{e}_t \tag{1.27}$$

If we consider motion along an infinitesimal arc of radius ρ surrounding P, we see that

$$\dot{\mathbf{e}}_t = \frac{\dot{s}}{\rho}\mathbf{e}_n \tag{1.28}$$

Thus, we find that the acceleration of the particle is

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{s}\mathbf{e}_t + \dot{s}\dot{\mathbf{e}}_t = \ddot{s}\mathbf{e}_t + \frac{\dot{s}^2}{\rho}\mathbf{e}_n \tag{1.29}$$

where ρ is the radius of curvature. Here $\ddot{s}$ is the tangential acceleration and $\dot{s}^2/\rho$ is the centripetal acceleration. The angular velocity of the unit vector triad is directly proportional to $\dot{s}$. It is

$$\omega = \omega_t\mathbf{e}_t + \omega_b\mathbf{e}_b \tag{1.30}$$

where ω_t and ω_b are obtained from

$$\begin{aligned} \dot{\mathbf{e}}_t &= \omega_b\mathbf{e}_n = \frac{\dot{s}}{\rho}\mathbf{e}_n \\ \dot{\mathbf{e}}_b &= -\omega_t\mathbf{e}_n = \dot{s}\frac{d\mathbf{e}_b}{ds} \end{aligned} \tag{1.31}$$

Note that $\omega_n = 0$ and also that $d\mathbf{e}_b/ds$ represents the *torsion* of the curve.

Relative motion and rotating frames

When one uses Newton's laws to describe the motion of a particle, the acceleration **a** must be *absolute*, that is, it must be measured relative to an inertial frame. This acceleration, of course, is the same when measured with respect to any inertial frame. Sometimes the motion of a particle is known relative to a rotating and accelerating frame, and it is desired to find its absolute velocity and acceleration. In general, these calculations can be somewhat complicated, but for the special case in which the moving frame A is not rotating, the results are simple. The absolute velocity of a particle P is

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} \tag{1.32}$$

where $\mathbf{v}_A$ is the absolute velocity of any point on frame A and $\mathbf{v}_{P/A}$ is the velocity of particle P relative to frame A, that is, the velocity recorded by cameras or other instruments fixed in frame A and moving with it. Similarly, the absolute acceleration of P is

$$\mathbf{a}_P = \mathbf{a}_A + \mathbf{a}_{P/A} \tag{1.33}$$

where we note again that the frame A is moving in pure translation.

Now consider the general case in which the moving xyz frame (Fig. 1.4) is translating and rotating arbitrarily. We wish to find the velocity and acceleration of a particle P relative

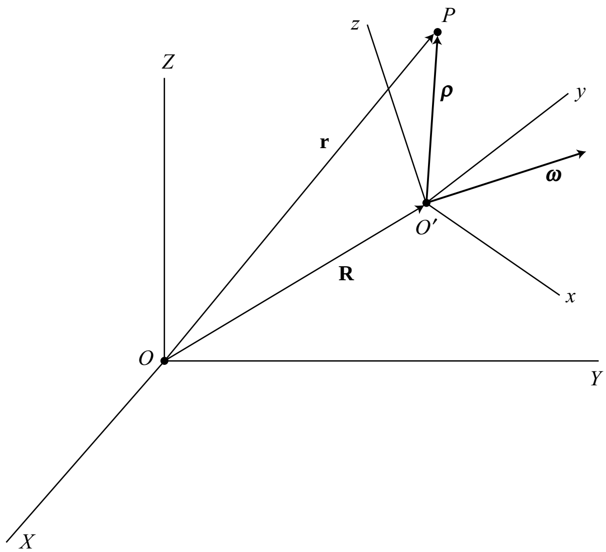

Figure 1.4.

to the inertial *XYZ* frame in terms of its motion with respect to the noninertial *xyz* frame. Let the origin O' of the *xyz* frame have a position vector $\mathbf{R}$ relative to the origin O of the *XYZ* frame. The position of the particle P relative to O' is $\boldsymbol{\rho}$, so the position of P relative to *XYZ* is

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho} \tag{1.34}$$

The corresponding velocity is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + \dot{\boldsymbol{\rho}} \tag{1.35}$$

Now let us use the basic equation (1.12) to express $\dot{\boldsymbol{\rho}}$ in terms of the motion relative to the moving *xyz* frame. We obtain

$$\dot{\boldsymbol{\rho}} = (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{1.36}$$

where $\boldsymbol{\omega}$ is the angular velocity of the *xyz* frame and $(\dot{\boldsymbol{\rho}})_r$ is the velocity of P relative to that frame. In detail,

$$\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1.37}$$

and

$$(\dot{\boldsymbol{\rho}})_r = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{1.38}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors fixed in the *xyz* frame and rotating with it. From (1.35) and (1.36), the absolute velocity of P is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{\mathbf{R}} + (\dot{\boldsymbol{\rho}})_r + \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{1.39}$$

The expression for the inertial acceleration **a** of the particle is found by first noting that

$$\frac{d}{dt}(\dot{\rho})_r = (\ddot{\rho})_r + \omega \times (\dot{\rho})_r \tag{1.40}$$

$$\frac{d}{dt}(\omega \times \rho) = \dot{\omega} \times \rho + \omega \times ((\dot{\rho})_r + \omega \times \rho) \tag{1.41}$$

Thus, we obtain the important result:

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{R}} + \dot{\omega} \times \rho + \omega \times (\omega \times \rho) + (\ddot{\rho})_r + 2\omega \times (\dot{\rho})_r \tag{1.42}$$

where ω is the angular velocity of the *xyz* frame. The nature of the various terms is as follows. $\ddot{\mathbf{R}}$ is the inertial acceleration of O', the origin of the moving frame. The term $\dot{\omega} \times \rho$ might be considered as a tangential acceleration although, more accurately, it represents a changing tangential velocity $\omega \times \rho$ due to changing ω. The term $\omega \times (\omega \times \rho)$ is a centripetal acceleration directed toward an axis of rotation through O'. These first three terms represent the acceleration of a point coincident with P but fixed in the *xyz* frame. The final two terms add the effects of motion relative to the moving frame. The term $(\ddot{\rho})_r$ is the acceleration of P relative to the *xyz* frame, that is, the acceleration of the particle, as recorded by instruments fixed in the *xyz* frame and rotating with it. The final term $2\omega \times (\dot{\rho})_r$ is the *Coriolis acceleration* due to a velocity relative to the rotating frame. Equation (1.42) is particularly useful if the motion of the particle relative to the moving *xyz* frame is simple; for example, linear motion or motion along a circular path.

Instantaneous center of rotation

If each point of a rigid body moves in planar motion, it is useful to consider a *lamina*, or slice, of the body which moves in its own plane (Fig. 1.5). If the lamina does not move in pure translation, that is, if $\omega \neq 0$, then a point C exists in the lamina, or in an imaginary

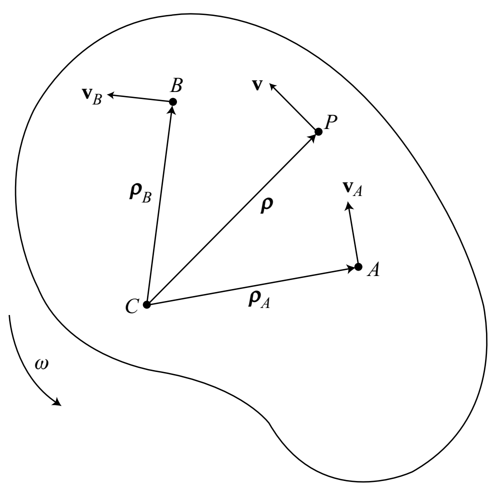

Figure 1.5.

extension thereof, at which the velocity is momentarily zero. This is the *instantaneous center of rotation*.

Suppose that arbitrary points A and B have velocities $\mathbf{v}_A$ and $\mathbf{v}_B$. The instantaneous center C is located at the intersection of the perpendicular lines to $\mathbf{v}_A$ and $\mathbf{v}_B$. The velocity of a point P with a position vector $\boldsymbol{\rho}$ relative to C is

$$\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho} \tag{1.43}$$

where $\boldsymbol{\omega}$ is the angular velocity vector of the lamina. Thus, if the location of the instantaneous center is known, it is easy to find the velocity of any other point of the lamina at that instant. On the other hand, the acceleration of the instantaneous center is generally not zero. Hence, the calculation of the acceleration of a general point in the lamina is usually not aided by a knowledge of the instantaneous center location.

If there is planar rolling motion of one body on another fixed body without any slipping, the instantaneous center lies at the contact point between the two bodies. As time proceeds, this point moves with respect to both bodies, thereby tracing a path on each body.

Example 1.1 A wheel of radius r rolls in planar motion without slipping on a fixed convex surface of radius R (Fig. 1.6a). We wish to solve for the acceleration of the contact point on the wheel. The contact point C is the instantaneous center, and therefore, the velocity of the wheel's center O' is

$$\mathbf{v} = r\omega\mathbf{e}_\phi \tag{1.44}$$

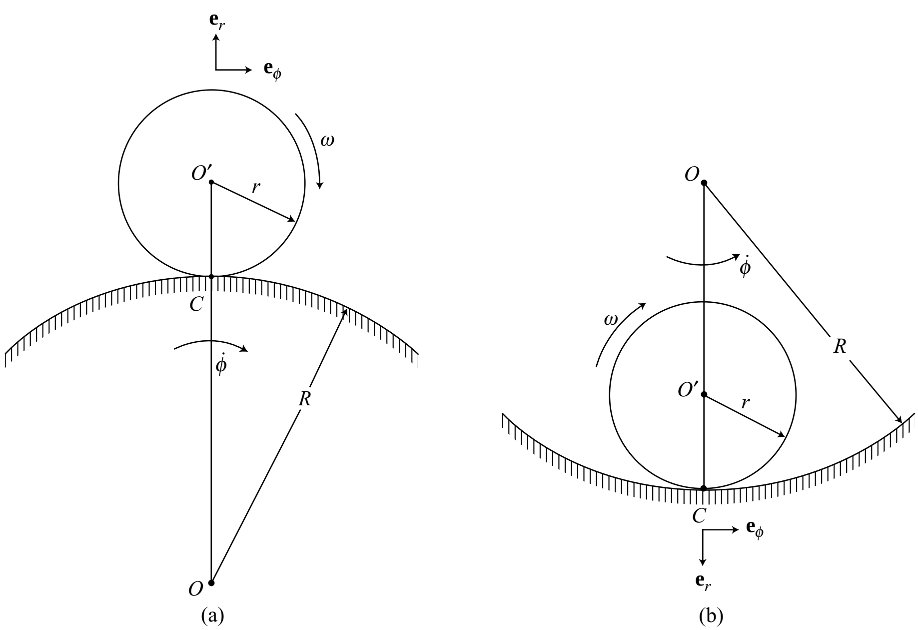

Figure 1.6.

In terms of the angular velocity $\dot{\phi}$ of the radial line OO', the velocity of the wheel is

$$(R+r)\dot{\phi} = r\omega \tag{1.45}$$

so we find that

$$\dot{\phi} = \frac{r\omega}{R+r} \tag{1.46}$$

To show that the acceleration of the contact point C is nonzero, we note that

$$\mathbf{a}_C = \mathbf{a}_{O'} + \mathbf{a}_{C/O'} \tag{1.47}$$

The center O' of the wheel moves in a circular path of radius $(R+r)$, so its acceleration $\mathbf{a}_{O'}$ is the sum of tangential and centripetal accelerations.

$$\begin{aligned}\mathbf{a}_{O'} &= (R+r)\ddot{\phi}\mathbf{e}_\phi - (R+r)\dot{\phi}^2\mathbf{e}_r \\ &= r\dot{\omega}\mathbf{e}_\phi - \frac{r^2\omega^2}{R+r}\mathbf{e}_r\end{aligned} \tag{1.48}$$

Similarly C, considered as a point on the rim of the wheel, has a circular motion about O', so

$$\mathbf{a}_{C/O'} = -r\dot{\omega}\mathbf{e}_\phi + r\omega^2\mathbf{e}_r \tag{1.49}$$

Then, adding (1.48) and (1.49), we obtain

$$\mathbf{a}_C = \left(r - \frac{r^2}{R+r}\right)\omega^2\mathbf{e}_r = \left(\frac{Rr}{R+r}\right)\omega^2\mathbf{e}_r \tag{1.50}$$

Thus, the instantaneous center has a nonzero acceleration.

Now consider the rolling motion of a wheel of radius r on a concave surface of radius R (Fig. 1.6b). The center of the wheel has a velocity

$$\mathbf{v}_{O'} = r\omega\mathbf{e}_\phi = (R-r)\dot{\phi}\mathbf{e}_\phi \tag{1.51}$$

so

$$\dot{\phi} = \frac{r\omega}{R-r} \tag{1.52}$$

In this case, the acceleration of the contact point is

$$\mathbf{a}_C = \mathbf{a}_{O'} + \mathbf{a}_{C/O'} \tag{1.53}$$

where

$$\begin{aligned}\mathbf{a}_{O'} &= (R-r)\,\ddot{\phi}\mathbf{e}_\phi - (R-r)\,\dot{\phi}^2\mathbf{e}_r \\ &= r\dot{\omega}\mathbf{e}_\phi - \frac{r^2\omega^2}{R-r}\mathbf{e}_r\end{aligned} \tag{1.54}$$

$$\mathbf{a}_{C/O'} = -r\dot{\omega}\mathbf{e}_\phi - r\omega^2\mathbf{e}_r \tag{1.55}$$

Thus, we obtain

$$\mathbf{a}_C = -\left(r + \frac{r^2}{R-r}\right)\omega^2\mathbf{e}_r = -\left(\frac{Rr}{R-r}\right)\omega^2\mathbf{e}_r \tag{1.56}$$

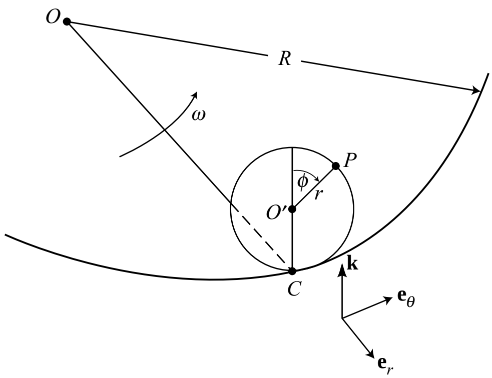

Figure 1.7.

Notice that very large values of $\mathbf{a}_{O'}$ and $\mathbf{a}_C$ can occur, even for moderate values of ω, if R is only slightly larger than r. This could occur, for example, if a shaft rotates in a sticky bearing.

Example 1.2 Let us calculate the acceleration of a point P on the rim of a wheel of radius r which rolls without slipping on a horizontal circular track of radius R (Fig. 1.7). The plane of the wheel remains vertical and the position angle of P relative to a vertical line through the center O' is ϕ.

Let us choose the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{k}$, as shown. They rotate about a vertical axis at an angular rate ω which is the rate at which the contact point C moves along the circular path. Since the center O' and C move along parallel paths with the same speed, we can write

$$v_{O'} = r\dot{\phi} = R\omega \tag{1.57}$$

from which we obtain

$$\boldsymbol{\omega} = \frac{r}{R}\dot{\phi}\mathbf{k} \tag{1.58}$$

Choose C as the origin of a moving frame which rotates with the angular velocity $\boldsymbol{\omega}$.

To find the acceleration of P, let us use the general equation (1.42), namely,

$$\mathbf{a} = \ddot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + (\ddot{\rho})_r + 2\boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_r \tag{1.59}$$

The acceleration of C is

$$\ddot{\mathbf{R}} = -R\omega^2\mathbf{e}_r + R\dot{\omega}\mathbf{e}_\theta = -\frac{r^2\dot{\phi}^2}{R}\mathbf{e}_r + r\ddot{\phi}\mathbf{e}_\theta \tag{1.60}$$

The relative position of P with respect to C is

$$\boldsymbol{\rho} = r\sin\phi\ \ \mathbf{e}_\theta + r(1+\cos\phi)\mathbf{k} \tag{1.61}$$

From (1.58) we obtain

$$\dot{\boldsymbol{\omega}} = \frac{r}{R}\ddot{\phi}\mathbf{k} \tag{1.62}$$

Then

$$\dot{\boldsymbol{\omega}} \times \rho = -\frac{r^2}{R}\ddot{\phi} \sin\phi \, \mathbf{e}_r \tag{1.63}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = -\frac{r^3}{R^2}\dot{\phi}^2 \sin\phi \, \mathbf{e}_\theta \tag{1.64}$$

Upon differentiating (1.61), with $\mathbf{e}_\theta$ and $\mathbf{k}$ held constant, we obtain

$$(\dot{\boldsymbol{\rho}})_r = r\dot{\phi} \cos\phi \, \mathbf{e}_\theta - r\dot{\phi} \sin\phi \, \mathbf{k} \tag{1.65}$$

and

$$2\boldsymbol{\omega} \times (\dot{\boldsymbol{\rho}})_r = -\frac{2r^2}{R}\dot{\phi}^2 \cos\phi \, \mathbf{e}_r \tag{1.66}$$

Also,

$$(\ddot{\boldsymbol{\rho}})_r = (r\ddot{\phi} \cos\phi - r\dot{\phi}^2 \sin\phi)\mathbf{e}_\theta - (r\ddot{\phi} \sin\phi + r\dot{\phi}^2 \cos\phi)\mathbf{k} \tag{1.67}$$

Finally, adding terms, the acceleration of P is

$$\mathbf{a} = -\left[\frac{r^2}{R}\ddot{\phi} \sin\phi + \frac{r^2}{R}\dot{\phi}^2(1 + 2\cos\phi)\right]\mathbf{e}_r + \left[r\ddot{\phi}(1 + \cos\phi) - r\dot{\phi}^2\left(1 + \frac{r^2}{R^2}\right)\sin\phi\right]\mathbf{e}_\theta$$
$$- (r\ddot{\phi} \sin\phi + r\dot{\phi}^2 \cos\phi)\mathbf{k} \tag{1.68}$$

Example 1.3 A particle P moves on a plane spiral having the equation

$$r = k\theta \tag{1.69}$$

where k is a constant (Fig. 1.8). Let us find an expression for its acceleration. Also solve for the radius of curvature of the spiral at a point specified by the angle θ.

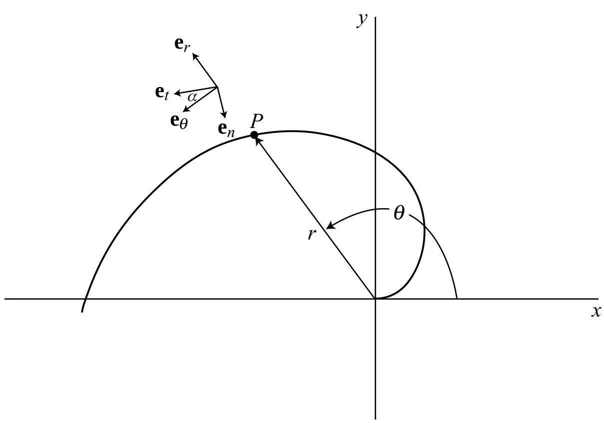

Figure 1.8.

First note that the unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta)$ rotate with an angular velocity

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k} \tag{1.70}$$

where the unit vector $\mathbf{k}$ points out of the page. We obtain

$$\begin{aligned} \dot{\mathbf{e}}_r &= \boldsymbol{\omega} \times \mathbf{e}_r = \dot{\theta}\mathbf{e}_\theta \\ \dot{\mathbf{e}}_\theta &= \boldsymbol{\omega} \times \mathbf{e}_\theta = -\dot{\theta}\mathbf{e}_r \end{aligned} \tag{1.71}$$

The position vector of P is

$$\mathbf{r} = r\mathbf{e}_r \tag{1.72}$$

and its velocity is

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta \tag{1.73}$$

The acceleration of P is

$$\begin{aligned} \mathbf{a} = \dot{\mathbf{v}} &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + r\ddot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta \\ &= (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \\ &= (k\ddot{\theta} - k\theta\dot{\theta}^2)\mathbf{e}_r + (k\theta\ddot{\theta} + 2k\dot{\theta}^2)\mathbf{e}_\theta \end{aligned} \tag{1.74}$$

The radius of curvature at P can be found by first establishing the orthogonal unit vectors $(\mathbf{e}_t, \mathbf{e}_n)$ and then finding the normal component of the acceleration. The angle α between the unit vectors $\mathbf{e}_t$ and $\mathbf{e}_\theta$ is obtained by noting that

$$\tan\alpha = \frac{v_r}{v_\theta} = \frac{\dot{r}}{r\dot{\theta}} = \frac{k\dot{\theta}}{k\theta\dot{\theta}} = \frac{1}{\theta} \tag{1.75}$$

and we see that

$$\begin{aligned} \sin\alpha &= \frac{1}{\sqrt{1+\theta^2}} \\ \cos\alpha &= \frac{\theta}{\sqrt{1+\theta^2}} \end{aligned} \tag{1.76}$$

The normal acceleration is

$$a_n = -a_r\cos\alpha + a_\theta\sin\alpha \tag{1.77}$$

where, from (1.74),

$$\begin{aligned} a_r &= k\ddot{\theta} - k\theta\dot{\theta}^2 \\ a_\theta &= k\theta\ddot{\theta} + 2k\dot{\theta}^2 \end{aligned} \tag{1.78}$$

Thus, we obtain

$$a_n = \frac{k\dot{\theta}^2}{\sqrt{1+\theta^2}}(2+\theta^2) \tag{1.79}$$

From (1.29), using tangential and normal components, we find that the normal acceleration is

$$a_n = \frac{\dot{s}^2}{\rho} = \frac{v^2}{\rho} = \frac{v_r^2 + v_\theta^2}{\rho} = \frac{k^2\dot{\theta}^2(1+\theta^2)}{\rho} \tag{1.80}$$

where ρ is the radius of curvature. Comparing (1.79) and (1.80), the radius of curvature at P is

$$\rho = \frac{k(1+\theta^2)^{3/2}}{2+\theta^2} \tag{1.81}$$

Notice that ρ varies from $\frac{1}{2}k$ at $\theta = 0$ to r for very large r and θ.

1.2 Systems of particles

A system of particles with all its interactions constitutes a dynamical system of great generality. Consequently, it is important to understand thoroughly the principles which govern its motions. Here we shall establish some of the basic principles. Later, these principles will be used in the study of rigid body dynamics.

Equations of motion

Consider a system of N particles whose positions are given relative to an inertial frame (Fig. 1.9). The ith particle is acted upon by an external force $\mathbf{F}_i$ and by $N-1$ internal

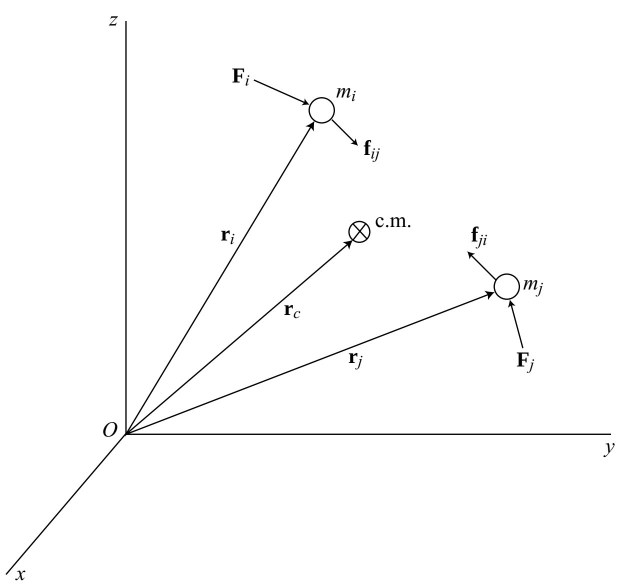

Figure 1.9.

interaction forces $\mathbf{f}_{ij}$ $(j \neq i)$ due to the other particles. The equation of motion for the ith particle is

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \tag{1.82}$$

The right-hand side of the equation is equal to the total force acting on the ith particle, external plus internal, and we note that $\mathbf{f}_{ii} = 0$; that is, a particle cannot act on itself to influence its motion.

Now sum (1.82) over the N particles.

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{F}_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} \tag{1.83}$$

Because of Newton's law of action and reaction, we have

$$\mathbf{f}_{ji} = -\mathbf{f}_{ij} \tag{1.84}$$

and therefore

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{f}_{ij} = 0 \tag{1.85}$$

The *center of mass* location is given by

$$\mathbf{r}_c = \frac{1}{m} \sum_{i=1}^{N} m_i \mathbf{r}_i \tag{1.86}$$

where the total mass m is

$$m = \sum_{i=1}^{N} m_i \tag{1.87}$$

Then (1.83) reduces to

$$m \ddot{\mathbf{r}}_c = \mathbf{F} \tag{1.88}$$

where the total external force acting on the system is

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i \tag{1.89}$$

This result shows that the motion of the center of mass of a system of particles is the same as that of a single particle of total mass m which is driven by the total external force $\mathbf{F}$.

The translational or linear momentum of a system of N particles is equal to the vector sum of the momenta of the individual particles. Thus, using (1.3), we find that

$$\mathbf{p} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \tag{1.90}$$

where each particle mass m_i is constant. Then, for the system, the rate of change of momentum is

$$\dot{\mathbf{p}} = \sum_{i=1}^{N} \dot{\mathbf{p}}_i = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \mathbf{F} \tag{1.91}$$

in agreement with (1.88). Note that if $\mathbf{F}$ remains equal to zero over some time interval, the linear momentum remains constant during the interval. More particularly, if a component of $\mathbf{F}$ in a certain fixed direction remains at zero, then the corresponding component of $\mathbf{p}$ is conserved.

Angular momentum

The angular momentum of a single particle of mass m_i about a fixed reference point O (Fig. 1.10) is

$$\mathbf{H}_i = \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = \mathbf{r}_i \times \mathbf{p}_i \tag{1.92}$$

which has the form of a moment of momentum. Upon summation over N particles, we find that the angular momentum of the system about O is

$$\mathbf{H}_O = \sum_{i=1}^{N} \mathbf{H}_i = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i \tag{1.93}$$

Now consider the angular momentum of the system about an arbitrary reference point P. It is

$$\mathbf{H}_p = \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i \tag{1.94}$$

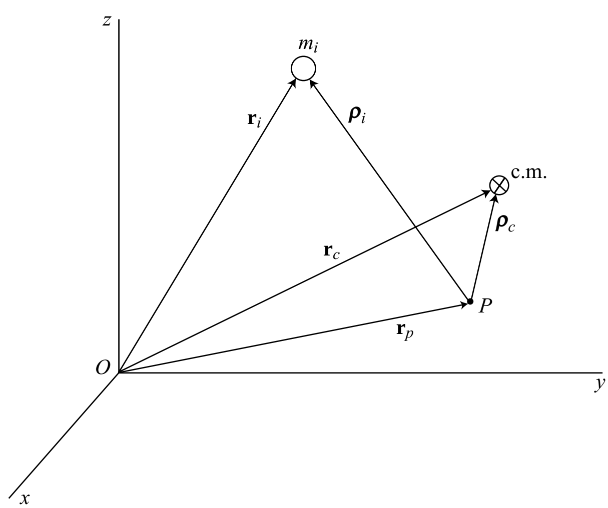

Figure 1.10.

Notice that the velocity $\dot{\boldsymbol{\rho}}_i$ is *measured relative to the reference point* P rather than being an absolute velocity. The use of relative versus absolute velocities in the definition of angular momentum makes no difference if the reference point is either fixed or at the center of mass. There is a difference, however, in the form of the equation of motion for the general case of an accelerating reference point P, which is not at the center of mass. In this case, the choice of relative velocities yields simpler and physically more meaningful equations of motion.

To find the angular momentum relative to the center of mass, we take the reference point P at the center of mass ($\boldsymbol{\rho}_c = 0$) and obtain

$$\mathbf{H}_c = \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i \tag{1.95}$$

where $\boldsymbol{\rho}_i$ is now the position vector of particle m_i relative to the center of mass.

Now let us write an expression for $\mathbf{H}_c$ when P is not at the center of mass. We obtain

$$\begin{aligned} \mathbf{H}_c &= \sum_{i=1}^{N} (\boldsymbol{\rho}_i - \boldsymbol{\rho}_c) \times m_i (\dot{\boldsymbol{\rho}}_i - \dot{\boldsymbol{\rho}}_c) \\ &= \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i - \boldsymbol{\rho}_c \times m \dot{\boldsymbol{\rho}}_c \end{aligned} \tag{1.96}$$

where

$$\sum_{i=1}^{N} m_i \boldsymbol{\rho}_i = m \boldsymbol{\rho}_c \tag{1.97}$$

Then, recalling (1.94), we find that

$$\mathbf{H}_p = \mathbf{H}_c + \boldsymbol{\rho}_c \times m \dot{\boldsymbol{\rho}}_c \tag{1.98}$$

This important result states that the angular momentum about an *arbitrary point* P is equal to the angular momentum about the center of mass plus the angular momentum due to the relative translational velocity $\dot{\boldsymbol{\rho}}_c$ of the center of mass. Of course, this result also applies to the case of a fixed reference point P when $\dot{\boldsymbol{\rho}}_c$ is an absolute velocity.

Now let us differentiate (1.93) with respect to time in order to obtain an equation of motion. We obtain

$$\dot{\mathbf{H}}_O = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \ddot{\mathbf{r}}_i \tag{1.99}$$

where, from Newton's law,

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \tag{1.100}$$

and we note that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{r}_i \times \mathbf{f}_{ij} = 0 \tag{1.101}$$

since, by Newton's third law, the internal forces $\mathbf{f}_{ij}$ occur in equal, opposite, and collinear pairs. Hence we obtain an equation of motion in the form

$$\dot{\mathbf{H}}_O = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i = \mathbf{M}_O \tag{1.102}$$

where $\mathbf{M}_O$ is the applied moment about the fixed point O due to forces external to the system.

In a similar manner, if we differentiate (1.95) with respect to time, we obtain

$$\dot{\mathbf{H}}_c = \sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \ddot{\boldsymbol{\rho}}_i \tag{1.103}$$

where $\boldsymbol{\rho}_i$ is the position vector of the ith particle relative to the center of mass. From Newton's law of motion for the ith particle,

$$m_i(\ddot{\mathbf{r}}_c + \ddot{\boldsymbol{\rho}}_i) = \mathbf{F}_i + \sum_{j=1}^{N} \mathbf{f}_{ij} \tag{1.104}$$

Now take the vector product of $\boldsymbol{\rho}_i$ with both sides of this equation and sum over i. We find that

$$\sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \ddot{\mathbf{r}}_c = 0 \tag{1.105}$$

since

$$\sum_{i=1}^{N} m_i \boldsymbol{\rho}_i = 0 \tag{1.106}$$

for a reference point at the center of mass. Also,

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{\rho}_i \times \mathbf{f}_{ij} = 0 \tag{1.107}$$

because the internal forces $\mathbf{f}_{ij}$ occur in equal, opposite, and collinear pairs. Hence we obtain

$$\sum_{i=1}^{N} \boldsymbol{\rho}_i \times m_i \ddot{\boldsymbol{\rho}}_i = \sum_{i=1}^{N} \boldsymbol{\rho}_i \times \mathbf{F}_i = \mathbf{M}_c \tag{1.108}$$

and, from (1.103) and (1.108),

$$\dot{\mathbf{H}}_c = \mathbf{M}_c \tag{1.109}$$

where $\mathbf{M}_c$ is the external applied moment about the center of mass.

At this point we have found that the basic rotational equation

$$\dot{\mathbf{H}} = \mathbf{M} \tag{1.110}$$

applies in each of two cases: (1) the reference point is fixed in an inertial frame; or (2) the reference point is at the center of mass.