1 Mathematics and its philosophy

Mathematics is the queen of the sciences and arithmetic is the queen of mathematics. She often condescends to render service to astronomy and other natural sciences, but in all relations, she is entitled to first rank.

Carl Friedrich Gauss (1777–1855)[1]

Mathematics occupies a unique and privileged position in human inquiry. It is the most rigorous and certain of all of the sciences, and it plays a key role in most, if not all, scientific work. It is for such reasons that the great German mathematician Carl Friedrich Gauss (1777–1855) pronounced mathematics to be the queen of the sciences. But the subject matter of mathematics is unlike that of any of the other branches of science. Mathematics seems to be the study of mathematical entities – such as numbers, sets, and functions – and the structural relationships between them. Mathematical entities, if there are such things, are very peculiar. They are abstract: they do not have spatiotemporal location and do not have causal powers. Moreover, the methodology of mathematics is apparently unlike the methodology of other sciences. Mathematics seems to proceed via a-priori means using deductive proof, as opposed to the a-posteriori methods of experimentation and induction found in the rest of science. And, on the face of it at least, mathematics is not revisable in the way that the rest of our science is. Once a mathematical theorem is proven, it stands forever. Mathematics may well be the queen of the sciences, but she would seem to be an eccentric and obstinate queen.

The philosophy of mathematics is the branch of philosophy charged with trying to understand this queen. We investigate the limits of mathematics, the subject matter of mathematics, the relationship between mathematics

[1] Sartorius von Waltershausen, *Gauss zum Gadächtniss*, 1856, p. 79. Quoted in Robert Edouard Moritz, *Memorabilia Mathematica*, 1914, p. 271.

and the rest of science, the logic of mathematical proofs, and the significance of the language of mathematics to mathematical practice. These are all important topics, and we address each of them in this book. They are significant for both philosophy and for mathematics. For example, understanding one of the paradigmatic cases of secure, a-priori knowledge is crucial to the branch of philosophy concerned with knowledge and its acquisition: epistemology. The importance of philosophy of mathematics to mathematics is also clear. Apart from anything else, philosophy sheds light on what mathematics is about. No self-respecting branch of science should be in the position of not knowing what its primary object of study is. More importantly, it may well be that the very methodology of mathematics hangs on the answers to some of the philosophical questions that impose themselves upon us. A brief look at the history of the relationship between mathematics and philosophy of mathematics will help illustrate the importance of philosophy of mathematics for both philosophy and mathematics.

1.1 Skipping through the big isms

The first half of the twentieth century was a golden age for philosophy of mathematics. It started with a philosopher, Bertrand Russell (1872–1970), proving that the foundational mathematical theory, set theory, was inconsistent. This led to a crisis in the foundations of mathematics and an intense period of debate. The debate and subsequent development of new set theories involved major philosophers of the time, such as Frank P. Ramsey (1903–30), Ludwig Wittgenstein (1889–1951), Gottlob Frege (1848–1925), Edmund Husserl (1859–1938), Charles Sanders Peirce (1839–1914), and of course Russell and his collaborator Alfred North Whitehead (1861–1947). Leading figures in mathematics were also involved. These included Hermann Weyl (1885–1955), Henri Poincaré (1854–1912), Kurt Gödel (1906–78), David Hilbert (1862–1943), L. E. J. Brouwer (1881–1966), Ernst Zermelo (1871–1953), and Alfred Tarski (1901–83).[2] The participants in these debates are major figures and household names (in my household, at least). There

[2] The distinction between philosophers and mathematicians here is somewhat arbitrary; many of these people should rightly be thought of as both philosophers and mathematicians. And, of course, there were many other major figures involved in these debates –

is no doubt about it, these must have been heady times – times when philosophy of mathematics really mattered, and everybody knew it.

Sadly, the excitement of these times didn't last. The debates over the foundations of mathematics bogged down. After a very productive 30 or 40 years, very little progress was made thereafter, and, by and large, both philosophers and mathematicians became tired of the philosophy of mathematics. At least, they became tired of the major movements of the first half of the twentieth century – 'the big isms' we'll get to shortly – and purely foundational issues in mathematics. Philosophy of mathematics kept going, of course – philosophy always does – but it had lost its urgency and, to some extent, its *raison d'être*.

It is very easy, as a student of philosophy of mathematics, to spend one's time looking back to the debates and developments of the first half of the twentieth century. But the philosophy of mathematics has moved on, and it is once again relevant and engaged with mathematical practice. The aim of this book is to get beyond the first half of the twentieth century and explore the issues capturing the attention of contemporary philosophers of mathematics. I will thus relegate most of the historical material to this short section, where we look at three of 'the big isms', and to the following chapter.[3] In Chapter 2 we consider some of the important mathematical results about the limits of mathematics. Although most of the results are from the first half of the twentieth century, they still loom large in contemporary philosophy of mathematics and thus deserve a more extensive treatment. My apologies to anyone who is disappointed by the relatively superficial treatment of the early twentieth-century philosophy of mathematics. While very good discussions of these topics abound, entry-level accounts of contemporary philosophy of mathematics are rare.

Below I give the briefest outline of three of the major movements in the philosophy of mathematics from the early twentieth century. Each of these has its charms; they each take one particular aspect of mathematical methodology as central to understanding mathematics. I should add that the three positions outlined below are historically very important, but they

too many to list here. The interested student is encouraged to read about the relevant history; it is a fascinating story, involving many noteworthy characters.

[3] The fourth 'ism', Platonism, is still very prominent in the contemporary literature so earns a chapter to itself: Chapter 3.

are not merely of historical interest – there are modern defenders of versions of each of them. It's just that the discussions of these positions no longer take centre stage.

1.1.1 Formalism

This view takes mathematical notation and its manipulation to be the core business of mathematics. In its purest form, formalism is the view that mathematics is nothing more than the manipulation of meaningless symbols. So-called *game formalism* is the view that mathematics is much like chess. The pieces of a chess set do not represent anything; they are just meaningless pieces of wood, metal, or whatever, defined by the rules that govern the legal moves that they can participate in. According to game formalism, mathematics is like this. The mathematical symbols are nothing more than pieces in a game and can be manipulated according to the rules. So, for example, elementary calculus may tell us that $d(ax^2 + bx + c)/dx = 2ax + b$. This is taken by formalism to mean that the right-hand side of the equation can be reached by a series of legal mathematical 'moves' from the left-hand side. As a result of this, in future mathematical 'games' one is licensed to replace the symbols '$d(ax^2 + bx + c)/dx$' with the symbols '$2ax + b$'. That too becomes a legal move in the game of mathematics. There are more sophisticated versions of formalism, but that's the basic idea. There is a question about whether the 'pieces' of the game are the actual mathematical symbol tokens, or whether it is the symbol types. That is, is this instance of 'π' different from, or the same as, this one: 'π'? They are two different tokens of the same type. Formalists need to decide where they stand on this and other such issues. Different answers give rise to different versions of formalism.

Formalism faces a number of difficulties, including accounting for the usefulness of mathematics in applications. But for now we just want to get a sense of what formalism is and why it was, at one time, a serious contender as a philosophy of mathematics. For a start, and as I've already mentioned, formalism takes notation seriously.[4] Indeed, it takes mathematics as being primarily about the notation. In so doing, it avoids problems associated with other accounts of mathematics, whereby the notation is

[4] See Chapter 8 for more on the importance of notation to mathematics.

taken to be standing for mathematical objects.[5] Formalism also places great importance on stating what the legal manipulations of the symbols are and which symbols are legitimate. This approach sits very well with a great deal of mathematics, especially axiomatic theories such as set theory and group theory.[6] The axioms of these theories function as the specification of both the legal manipulations in question and the objects of manipulation. And the formalist's suggestion that there is nothing more to these theories is not altogether mad. For example, in set theory the membership relation $\in$ really does seem to be a primitive notion, defined implicitly by the theory in which it resides. Just as the question of what a bishop in chess is can be answered in full by explaining the rules of chess and the role a bishop plays in the game. There is nothing more to say in either case, or so goes the formalist line of thought. As we shall see in the next chapter, it is generally thought that the most sophisticated version of a theory along these lines was put to rest by Gödel's Incompleteness Theorems. In any case, formalism has few supporters these days. But the other big isms are in better shape.[7]

1.1.2 Logicism

This view of mathematics takes the a-priori methodology of mathematics as central. According to logicism, mathematics is logic. That's the slogan, at least; spelling out what this slogan amounts to is more difficult, but the basic idea is that mathematical truths can, in some sense, be reduced to truths about logic. The position is epistemologically motivated: logical knowledge is thought to be more basic and less mysterious than mathematical knowledge. Given the German mathematician Richard Dedekind's (1831–1916) reduction of real numbers to sequences of rational numbers[8] and other known reductions in mathematics, it was tempting to see basic arithmetic as the foundation of mathematics. Moreover, if arithmetic were

[5] We will encounter the problems with such realist accounts of mathematics in due course.

[6] See p. 88 for the axioms of group theory.

[7] See Curry (1951) for a classic defence of formalism and Weir (2010) for an interesting modern attempt to resuscitate the position.

[8] Dedekind's idea was to identify real numbers with the limits of sequences of rational numbers – so-called *Dedekind cuts*.

to turn out to be derivable from logic, then we'd have a compelling account of the nature of mathematics. Logicism was first proposed and developed in detail by Gottlob Frege. Unfortunately Frege system was inconsistent. He included the now-infamous Basic Law V as one of his logical axioms.[9] This rather innocuous-looking axiom about the extensions of predicates was shown by Bertrand Russell to lead to a contradiction. But many thought that Frege was onto something. Indeed, Russell was one of them. He, in collaboration with Whitehead, pushed the logicist programme forward, but the further this programme was developed, the less the basic machinery looked as if it deserved to be called 'logic'.

The allure of logicism and the considerable achievements of Frege live on, though. The contemporary descendant of this programme is *neo-logicism*. The neo-logicist programme takes as its point of departure the fact that Frege did not really need anything so strong as his problematic Basic Law V in order to get most of what he wanted. Basic Law V can be replaced with Hume's principle: the number of Fs is equal to the number of Gs iff the Fs and the Gs can be placed in one–one correspondence. Strictly speaking, this principle is not a law of logic, but it's very, very close. With Hume's principle in hand and helping oneself to second-order logic,[10] the core of Frege's project can be carried out.[11]

1.1.3 Intuitionism

This view of mathematics takes proof in mathematics very seriously. Indeed, according to intuitionism, proof and constructions are all there is. (Intuitionism is sometimes called *constructivism* for this reason.) Accordingly, mathematics is not taken to be about some pre-existing realm of mathematical objects. Intuitionism has it that mathematical objects need to be constructed before one can sensibly speak about them. This has

9 Basic Law V states that the value-ranges of two functions f and g are the same iff $\forall x(f(x) = g(x))$.

10 Second-order logic is logic that allows quantification over predicates as well as over individuals. First-order logic is logic that quantifies only over individuals. There is some debate over whether second-order logic really is logic or merely disguised set theory.

11 See, for example, Boolos (1987, 1998), Burgess (2005), Hale and Wright (2001), Wright (1983), and Zalta (1999, 2000) for modern neo-logicist approaches. The classic original logicist treatises are Frege (1967, 1974), and Whitehead and Russell (1910, 1912, 1913).

ramifications for both the style of proof that is acceptable in mathematics and the domains of mathematical objects one can work with. Unless there is a procedure for delivering the mathematical objects in question, they are committed to the flames. All but the smallest, most well-behaved infinities are rejected. But most notable is that many proofs of classical mathematics are not valid by intuitionistic lights.

To understand why, think about the theorem of classical logic known as the *law of excluded middle*: for every proposition P, the disjunction of P and its negation, $(P \vee \neg P)$, is true.[12] This law is well motivated in cases where we may be ignorant of the facts of the matter, but where there *are* facts of the matter. For example, the exact depth of the Mariana Trench in the Pacific Ocean at its deepest point at exactly 12.00 noon GMT on 1 January 2011 is unknown, I take it. But there is a fact of the matter about the depth of this trench at this time. It was, for example, either greater than 11,000 m or it was not. Contrast this with cases where there is plausibly no fact of the matter. Many philosophers think that future contingent events are good examples of such indeterminacies. Take, for example, the height of the tallest building in the world at 12.00 noon GMT on 1 January 2031. According to the line of thought we're considering here, the height of this building is not merely unknown, the relevant facts about this building's height are not yet settled. The facts in question will be settled in 2031, but right now there is no fact of the matter about the height of this building. Accordingly, excluded middle is thought to fail here. It is not, for example, true that either this building is taller than 850 m or not.

Now consider mathematics, as understood by the intuitionists. For them, mathematics is all about the construction of mathematical objects and proofs concerning them. Let's focus on the proofs. Consider some mathematical statement S that is neither proven nor proven to be false. If one does not recognise some objective, external sense of truth, and instead takes proof to be all there is to it, excluded middle fails for S. In particular, excluded middle cannot be used in the process of proving S. Double-negation elimination also fails. After all, proving that there is no proof that there can't be a proof of S is not the same thing as having a proof of S. The rejection of double-negation elimination undermines an important form of

[12] Excluded middle should be carefully distinguished from its semantic counterpart, *bivalence*: every proposition is either true or false.

proof in classical mathematics known as *reductio ad absurdum*. This style of proof starts by assuming the negation of S, then proceeds to draw a contradiction from this assumption, thus concluding simply S.[13] In intuitionistic logic, this is all fine until the last step. According to the intuitionist, all that has been shown is $\neg\neg S$ and it is a further unjustified step to conclude S from this. Some other classical forms of proof are intuitionistically invalid. These include various existence proofs that show that some object must exist but do not deliver a construction of the object in question (e.g., the proof of the Tarski–Banach Theorem in section 9.1.1 is not intuitionistically valid). Intuitionism is thus a more radical philosophy of mathematics than the others we've seen so far, in that it demands a change in mathematical practice. It requires a new logic, with many traditional proofs of mathematical results no longer accepted.[14]

1.2 Charting a course to contemporary topics

The agenda for contemporary philosophy of mathematics was shaped by Paul Benacerraf in a couple of landmark papers. In the first of the papers (Benacerraf 1983a, originally published in 1965), Benacerraf outlines an underdetermination problem for the project of reducing all of mathematics to set theory. Such underdetermination or non-uniqueness problems had been around for some time, but Benacerraf's presentation was compelling, and its relevance to a popular position in philosophy of mathematics was firmly established. The second and third problems (Benacerraf 1983b, originally published in 1973) are presented as a challenge that any credible philosophy of mathematics must meet: (i) allow for a semantics that is uniform across both mathematical and non-mathematical discourse and (ii) provide a plausible epistemology for mathematics. As Benacerraf went on to show, it is difficult to satisfy both parts of this challenge simultaneously. Any philosophy of mathematics that meets one part of the challenge typically has serious difficulties meeting the other part.

There are two main camps in philosophy of mathematics and each has a serious problem with one or other of these challenges. Realist or

13 See section 9.1.9 for an example of such a proof in mathematics.

14 For more on intuitionism see Heyting (1971; 1983), Dummett (1983), and Brouwer (1983).

Platonist philosophies of mathematics[15] hold that at least some mathematics is objectively true and is about a realm of abstract mathematical entities. Mathematics is taken at face value and the semantics here is the same as elsewhere. Mathematical realism has no problem with the first of Benacerraf's challenges but, notoriously, has serious difficulties providing a plausible epistemology. Anti-realist positions, on the other hand, hold that there are no such abstract mathematical entities. The anti-realist thus has no epistemic problems, but these positions typically fall foul of the first of Benacerraf's challenges.

The three Benacerraf problems, along with a few others we'll encounter, are the rocks on which many philosophies of mathematics founder. The challenge is to chart a course past these difficulties to arrive at a credible philosophy of mathematics. So let's get better acquainted with the main obstacles.

1.2.1 Uniform semantics

The requirements for a uniform semantics is just that one should not give special semantic treatment to mathematical discourse. If a mathematical statement such as '$\sqrt{2}$ is irrational' is taken to be true, the semantics should be the same as for other true sentences such as 'Jupiter is a gas giant'. The latter is true by virtue of the existence of Jupiter and it having the property of being a large planet composed primarily of the gases hydrogen and helium. Under a uniform semantics, '$\sqrt{2}$ is irrational' is true by virtue of the existence of the number $\sqrt{2}$ and it having the property of not being expressible in the form a/b, where a and b are integers. The requirement of providing a uniform semantics leads very naturally from truth of mathematical statements to the existence of mathematical objects. Mathematical realism thus has a very natural answer to this challenge. It is anti-realism that has difficulties here. For example, if your view is that what makes '$\sqrt{2}$ is irrational' true is something about a social agreement to assent to such claims or to the existence of a proof of an appropriate kind, then the requirement for uniform semantics requires that you do the same for the sentence above about Jupiter. Either you give a deviant semantics across the board or you use the usual semantics in mathematics as well. Of course

15 I will use the terms 'Platonism' and 'mathematical realism' interchangeably.

you may decide to treat the semantics of mathematics differently and violate the requirement for uniform semantics, but then you at least owe an explanation why mathematics comes in for such special treatment.

1.2.2 The epistemic problem

The epistemic problem is very simple: provide an account of how we come by mathematical knowledge. The problem was originally cast in terms of the causal theory of knowledge. This theory holds that for an agent A to know some proposition P, A must believe that P, and the fact that makes P true must cause A's belief that P. Thus construed, the epistemic challenge was to show how mathematical knowledge could be reconciled with the causal theory of knowledge. For Platonist accounts of mathematics, this was nearly impossible, for it would mean coming in causal contact with mathematical entities: the number 7, for instance, would need to cause my belief that 7 is prime. But surely numbers do not have causal powers. Indeed, it would seem that numbers are the wrong kind of thing to be causing anything, let alone beliefs. This leads many to be wary of, if not outright reject, Platonism.

But there are problems with the argument, thus construed. For a start, why should we accept the causal theory of knowledge? After all, this theory was formulated with empirical knowledge in mind and was not intended to deal with mathematical knowledge. It is question-begging to require the Platonist to provide a causal account of mathematical knowledge. If anything should be rejected here, it should be the causal theory of knowledge. In any case, the causal theory of knowledge did eventually fall from grace. The reasons for this were various, but its inability to account for mathematical knowledge was chief among its deficiencies. Still, many seem to think there's something to Benacerraf's challenge which survives the demise of the causal theory of knowledge. W. D. Hart puts the point thus:

> [I]t is a crime against the intellect to try to mask the problem of naturalizing the epistemology of mathematics with philosophical razzle-dazzle. Superficial worries about the intellectual hygiene of causal theories of knowledge are irrelevant to and misleading from this problem, for the problem is not so much about causality as about the very possibility of natural knowledge of abstract objects. (Hart 1977, pp. 125–6)

What is the worry about abstract objects? What is it about abstract objects that suggests that it's impossible to have knowledge of them? In