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978-0-521-82476-7 - Contact Geometry and Non-linear Differential Equations

Alexei Kushner, Valentin Lychagin and Vladimir Rubtsov

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