Econometric Exercises, Volume 1

Matrix Algebra

Matrix Algebra is the first volume of the Econometric Exercises Series. It contains exercises relating to course material in matrix algebra that students are expected to know while enrolled in an (advanced) undergraduate or a postgraduate course in econometrics or statistics. The book contains a comprehensive collection of exercises, all with full answers. But the book is not just a collection of exercises; in fact, it is a textbook, though one that is organized in a completely different manner than the usual textbook. The volume can be used either as a self-contained course in matrix algebra or as a supplementary text.

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Econometric Exercises

Titles in the Series (* = planned):

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* Simulation-Based Econometric Methods
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* Financial Econometrics
* Robustness
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Matrix Algebra

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
40 West 20th Street, New York, NY 10011-4211, USA

www.cambridge.org
Information on this title: www.cambridge.org/9780521822893

First published 2005

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data
Abadir, Karim M., 1964–
Matrix algebra / Karim M. Abadir, Jan R. Magnus.
p. cm. – (Econometric exercises ; 1)
Includes bibliographical references and index.
ISBN-13 978-0-521-82289-3
ISBN-10 0-521-82289-0
ISBN-13 978-0-521-53746-9 (pbk.)
ISBN-10 0-521-53746-0 (pbk.)
1. Matrices–Textbooks. I. Magnus, Jan R. II. Title. III. Series.
QA188.A195 2005
512.9′434 – dc22 2005002651

ISBN-13 978-0-521-82289-3 hardback
ISBN-10 0-521-82289-0 hardback

ISBN-13 978-0-521-53746-9 paperback
ISBN-10 0-521-53746-0 paperback

Cambridge University Press
0521822890 - Matrix Algebra
Karim M. Abadir and Jan R. Magnus
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To my parents, and to Kouka, Ramez, Naguib, Névine
To Gideon and Hedda

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Preface to the Series

The past two decades have seen econometrics grow into a vast discipline. Many different branches of the subject now happily coexist with one another. These branches interweave econometric theory and empirical applications, and bring econometric method to bear on a myriad of economic issues. Against this background, a guided treatment of the modern subject of econometrics in a of volumes of worked econometric exercises seemed a natural and rather challenging idea.

The present Series, *Econometric Exercises*, was conceived in 1995 with this challenge in mind. Now, almost a decade later it has become an exciting reality with the publication of the first installment of a series of volumes of worked econometric exercises. How can these volumes work as a tool of learning that adds value to the many existing textbooks of econometrics? What readers do we have in mind as benefiting from this Series? What format best suits the objective of helping these readers learn, practice, and teach econometrics? These questions we now address, starting with our overall goals for the Series.

Econometric Exercises is published as an organized set of volumes. Each volume in the Series provides a coherent sequence of exercises in a specific field or subfield of econometrics. Solved exercises are assembled together in a structured and logical pedagogical framework that seeks to develop the subject matter of the field from its foundations through to its empirical applications and advanced reaches. As the Schaum Series has done so successfully for mathematics, the overall goal of *Econometric Exercises* is to develop the subject matter of econometrics through solved exercises, providing a coverage of the subject that begins at an introductory level and moves through to more advanced undergraduate and graduate level material.

Problem solving and worked exercises play a major role in every scientific subject. They are particularly important in a subject like econometrics where there is a rapidly growing literature of statistical and mathematical technique and an ever-expanding core to the discipline. As students, instructors, and researchers, we all benefit by seeing carefully worked-out solutions to problems that develop the subject and illustrate its methods and workings. Regular exercises and problem sets consolidate learning and reveal applications of textbook material. Clearly laid out solutions, paradigm answers, and alternate routes to solution all develop problem-solving skills. Exercises train students in clear analytical thinking and help them in preparing for tests, and exams. Teachers, as well as students, find solved exercises useful in their classroom preparation and in designing problem sets, tests, and examinations. Worked problems and illustrative empirical applications appeal to researchers and professional economists wanting to learn about specific econometric techniques. Our intention for the *Econometric Exercises* Series is to appeal to this wide range of potential users.

Each volume of the Series follows the same general template. Chapters begin with a short outline that emphasizes the main ideas and overviews the most relevant theorems and results. The introductions are followed by a sequential development of the material by solved examples and applications, and computer exercises where these are appropriate. All problems are solved and they are graduated in difficulty with solution techniques evolving in a logical, sequential fashion. Problems are asterisked when they require more creative solutions or reach higher levels of technical difficulty. Each volume is self-contained. There is some commonality in material across volumes in the Series in order to reinforce learning and to make each volume accessible to students and others who are working largely, or even completely, on their own.

Content is structured so that solutions follow immediately after the exercise is posed. This makes the text more readable and avoids repetition of the statement of the exercise when it is being solved. More importantly, posing the right question at the right moment in the development of a subject helps to anticipate and address future learning issues that students face. Furthermore, the methods developed in a solution and the precision and insights of the answers are often more important than the questions being posed. In effect, the inner workings of a good solution frequently provide benefit beyond what is relevant to the specific exercise.

Exercise titles are listed at the start of each volume, following the Table of Contents, so that readers may see the overall structure of the book and its more detailed contents. This organization reveals the exercise progression, how the exercises relate to one another, and where the material is heading. It should also tantalize readers with the exciting prospect of advanced material and intriguing applications.

The Series is intended for a readership that includes undergraduate students of econometrics with an introductory knowledge of statistics, first and second year graduate students of econometrics, as well as students and instructors from neighboring disciplines (like statistics, psychology, or political science) with interests in econometric methods. The volumes generally increase in difficulty as the topics become more specialized.

The early volumes in the Series (particularly those covering matrix algebra, statistics, econometric models, and empirical applications) provide a foundation to the study of econometrics. These volumes will be especially useful to students who are following the first year econometrics course sequence in North American graduate schools and need to prepare for graduate comprehensive examinations in econometrics and to write an applied econometrics paper. The early volumes will equally be of value to advanced undergraduates studying econometrics in Europe, to advanced undergraduates and honors students in the Australasian system, and to masters and doctoral students in general. Subsequent volumes will be of interest to professional economists, applied workers, and econometricians who are working with techniques in those areas, as well as students who are taking an advanced course sequence in econometrics and statisticians with interests in those topics.

The *Econometric Exercises* Series is intended to offer an independent learning-by-doing program in econometrics and it provides a useful reference source for anyone wanting to learn more about econometric methods and applications. The individual volumes can be used in classroom teaching and examining in a variety of ways. For instance, instructors can work through some of the problems in class to demonstrate methods as they are introduced, they can illustrate theoretical material with some of the solved examples, and they can show real data applications of the methods by drawing on some of the empirical examples. For examining purposes, instructors may draw freely from the solved exercises in test preparation. The systematic development of the subject in individual volumes will make the material easily accessible both for students in revision and for instructors in test preparation.

In using the volumes, students and instructors may work through the material sequentially as part of a complete learning program, or they may dip directly into material where they are experiencing difficulty, in order to learn from solved exercises and illustrations. To promote intensive study, an instructor might announce to a class in advance of a test that some questions in the test will be selected from a certain chapter of one of the volumes. This approach encourages students to work through most of the exercises in a particular chapter by way of test preparation, thereby reinforcing classroom instruction.

Further details and updated information about individual volumes can be obtained from the *Econometric Exercises* website,

http://us.cambridge.org/economics/ee/econometricexercises.htm

The website also contains the basic notation for the Series, which can be downloaded along with the LaTeX style files.

As Series Editors, we welcome comments, criticisms, suggestions, and, of course, corrections from all our readers on each of the volumes in the Series as well as on the Series itself. We bid you as much happy reading and problem solving as we have had in writing and preparing this Series.

York, Tilburg, New Haven
July 2004

Karim M. Abadir
Jan R. Magnus
Peter C. B. Phillips

Preface

This volume on matrix algebra and its companion volume on statistics are the first two volumes of the *Econometric Exercises* Series. The two books contain exercises in matrix algebra, probability, and statistics, relating to course material that students are expected to know while enrolled in an (advanced) undergraduate or a postgraduate course in econometrics.

When we started writing this volume, our aim was to provide a collection of interesting exercises with complete and rigorous solutions. In fact, we wrote the book that we — as students — would have liked to have had. Our intention was not to write a textbook, but to supply material that could be used together with a textbook. But when the volume developed we discovered that we did in fact write a textbook, be it one organized in a completely different manner. Thus, we do provide and prove theorems in this volume, because continually referring to other texts seemed undesirable. The volume can thus be used either as a self-contained course in matrix algebra or as a supplementary text.

We have attempted to develop new ideas slowly and carefully. The important ideas are introduced algebraically and sometimes geometrically, but also through examples. It is our experience that most students find it easier to assimilate the material through examples rather than by the theoretical development only.

In proving the more difficult theorems, we have always divided them up in smaller questions, so that the student is encouraged to understand the structure of the proof, and also will be able to answer at least some of the questions, even if he/she can not prove the whole theorem. A more difficult exercise is marked with an asterisk ($*$).

One approach to presenting the material is to prove a general result and then obtain a number of special cases. For the student, however, we believe it is more useful (and also closer to scientific development) to first prove a simple case, then a more difficult case, and finally the general result. This means that we sometimes prove the same result two or three times, in increasing complexity, but nevertheless essentially the same. This gives the student who could not solve the simple case a second chance in trying to solve the more general case, after having studied the solution of the simple case.

We have chosen to take real matrices as our unit of operation, although almost all results are equally valid for complex matrices. It was tempting — and possibly would have been more logical and aesthetic — to work with complex matrices throughout. We have resisted this temptation, solely for educational reasons. We emphasize from time to time that results are also valid for complex matrices. Of course, we explicitly need complex matrices in some important cases, most notably in decomposition theorems involving eigenvalues.

Occasionally we have illustrated matrix ideas in a statistical or econometric context, realizing that the student may not yet have studied these concepts. These exercises may be skipped at the first reading.

In contrast to statistics (in particular, probability theory), there only exist a few books of worked exercises in matrix algebra. First, there is Schaum's Outline Series with four volumes: *Matrices* by Ayres (1962), *Theory and Problems of Matrix Operations* by Bronson (1989), *3000 Solved Problems in Linear Algebra* by Lipschutz (1989), and *Theory and Problems of Linear Algebra* by Lipschutz and Lipson (2001). The only other examples of worked exercises in matrix algebra, as far as we are aware, are Proskuryakov (1978), Prasolov (1994), Zhang (1996, 1999), and Harville (2001).

Matrix algebra is by now an established field. Most of the results in this volume of exercises have been known for decades or longer. Readers wishing to go deeper into the material are advised to consult Mirsky (1955), Gantmacher (1959), Bellman (1970), Hadley (1961), Horn and Johnson (1985, 1991), Magnus (1988), or Magnus and Neudecker (1999), among many other excellent texts.

We are grateful to Josette Janssen at Tilburg University for expert and cheerful typing in LaTeX, to Jozef Pijnenburg for constant advice on difficult LaTeX questions, to Andrey Vasnev for help with the figures, to Sanne Zwart for editorial assistance, to Bertrand Melenberg, William Mikhail, Maxim Nazarov, Paolo Paruolo, Peter Phillips, Gabriel Talmain, undergraduates at Exeter University, PhD students at the NAKE program in Utrecht and at the European University Institute in Florence, and two anonymous referees, for their constructive comments, and to Scott Parris and his staff at Cambridge University Press for his patience and encouragement. The final version of this book was completed while Jan spent six months as a Jean Monnet fellow at the European University Institute in Florence.

Updates and corrections of this volume can be obtained from the *Econometric Exercises* website,

http://us.cambridge.org/economics/ee/econometricexercises.htm

Of course, we welcome comments from our readers.

York, Tilburg Karim M. Abadir
July 2004 Jan R. Magnus