Flexagons Inside Out

LESLIE PHILIP (Les) POOK was born in Middlesex, England in 1935. He obtained a BSc in metallurgy from the University of London in 1956. He started his career at Hawker Siddeley Aviation Ltd, Coventry, in 1956. In 1963 he moved to the National Engineering Laboratory, East Kilbride, Glasgow. In 1969, while at the National Engineering Laboratory he obtained a PhD in mechanical engineering from the University of Strathclyde. Dr Pook moved to University College London in 1990. He retired formally in 1998 but remained affiliated to University College London as a Visiting Professor. He is a Fellow of the Institution of Mechanical Engineers and a Fellow of the Institute of Materials, Minerals and Mining.

Dr Pook has wide experience of both research and practical engineering problems involving metal fatigue and fracture mechanics. In these fields he has published four books and over a hundred papers. Present professional interests include the fatigue behaviour of cracks in complex stress fields, finite element analysis, and language editing of translated technical material. His leisure activities have for many years included recreational mathematics, horology, in which he has published one paper, and gardening. He regards his extensive DIY activities, especially plumbing, as a chore.

Les married his wife, Ann, in 1960. They have a daughter, Stephanie, and a son, Adrian.

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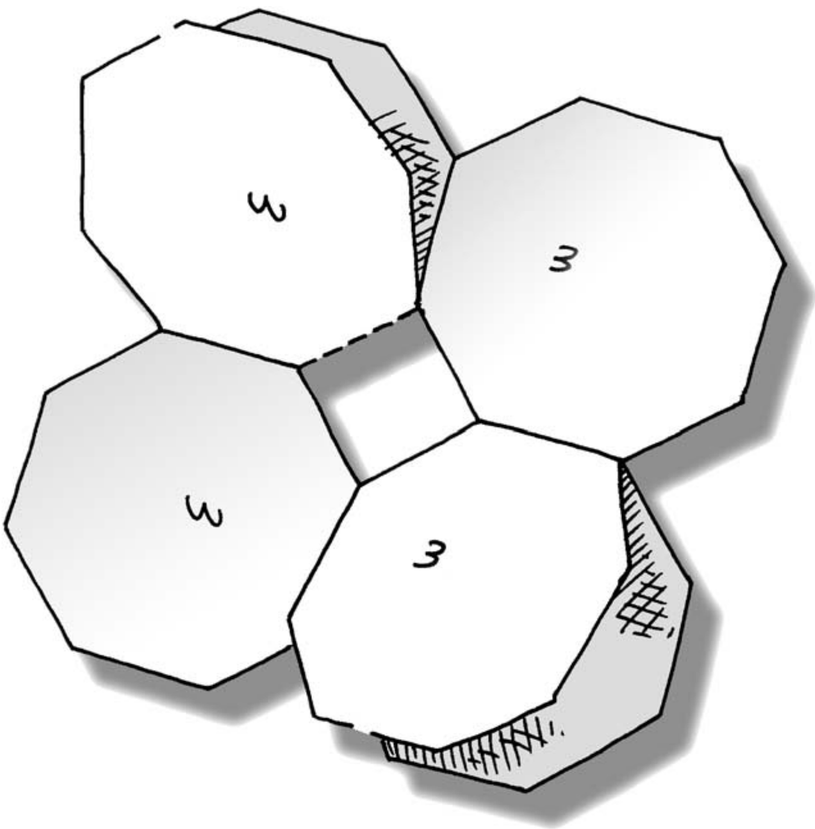

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Preface

Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture.

Bertrand Russell, *Mysticism and Logic*.

Flexagons are hinged polygons that have the intriguing property of displaying different pairs of faces when they are 'flexed'. Workable paper models of flexagons are easy to make, and entertaining to manipulate. Flexagons have a surprisingly complex mathematical structure and just how a flexagon works is not obvious on casual examination of a paper model. The aesthetic appeal of flexagons is in their dynamic behaviour rather than the static appeal of, say, polyhedra. One of the attractions of flexagons is that it is possible to explore their dynamic behaviour experimentally as well as theoretically. Flexagons may be appreciated at three different levels: firstly as toys or puzzles, secondly as a recreational mathematics topic, and finally as the subject of serious mathematical study. Well made models of simple flexagons can be bought, but they are not widely available.

I first became interested in flexagons in the late 1960s through reading two of Martin Gardner's books. At that time I made numerous paper models of flexagons, and carried out some theoretical analysis. I did try writing up some of the material for publication, but editors showed a marked lack of enthusiasm and I lost interest. In 1997, while browsing in a library I came across a paper on flexagons, and this revived my interest. Fortunately, my collection of flexagon models and associated notes had survived three house moves.

I carried out a literature search as part of my renewed investigations. This showed that there is only a limited amount of published information on flexagons. Furthermore, this information is scattered and some items are difficult to locate. There may well be significant items which I have missed. The only books I found devoted entirely to flexagons are at the toy or puzzle level. These usually contain cut out nets for paper models of flexagons, but making these up destroys the book. At the recreational mathematics level I found several books containing sections or chapters on flexagons, but the information included tends to be fragmentary, and is sometimes difficult to appreciate fully without some prior knowledge of flexagons. At this level I did not find any books devoted entirely to flexagons.

The first paper at the serious mathematics level was published in 1957 and a comprehensive report on flexagons was issued in 1962. I only recently obtained a copy of this report and it turned out to include quite a lot of material which I had rediscovered for myself. At this level the subsequent literature is sparse. The main features of flexagons have been understood for half a century, but a long standing, and still not fully resolved, problem is how best to describe their structure and dynamic behaviour. For this reason publications on flexagons at the serious mathematics level tend either to be not very informative or to need close reading by a competent mathematician in order to fully appreciate their content.

Contemplation of my collection of publications on flexagons, together with the results of my own early and recent investigations, showed that I had enough information to write a book on flexagons at the recreational mathematics level. Helpful comments made by anonymous reviewers of the first draft of the book encouraged me to proceed. This book is the result. It is assumed that the reader has some knowledge of elementary geometry. No previous knowledge of flexagons is assumed, so the book is suitable as an introduction to flexagons at the toy or puzzle level. In general, detailed proofs are long and tedious so they are not included. Where there is uncertainty over the accuracy of a conclusion this is made clear in the text.

There is an infinite number of possible types of flexagon so no book on flexagons can be comprehensive. The material included is very much a personal choice. It is arranged roughly in order of increasing difficulty rather than in a strictly logical order which would be appropriate for a formal textbook. The terminology used is mostly based on that used by previous authors but to keep the text concise it was found that some new terms were needed. Specialised terms are listed in the index so that definitions can be easily located.

A feature of the book is a collection of nets, with assembly instructions, for a wide range of paper models of flexagons. They are printed full size and laid out so that they can be photocopied. Some of the flexagons are difficult either to make or to manipulate, and this is noted in captions. The flexagons have been chosen to complement the text. Some of the nets were specially designed using methods described in the book.

Notation

c Number of sides on the inscribed polygon of a flexagon figure.

f Number of faces on a flexagon.

n Number of sectors on a main position of a flexagon.

$2n$ Number of polygons on a face of a main position of a flexagon. Number of polyhedra on a hyperface of a main position of a flexahedron.

s Number of sides on a polygon. Number of sides on the circumscribing polygon of a flexagon figure.

$\{s\}$ Schläfli symbol for a regular polygon.

$\langle s, c\rangle$ Flexagon symbol for a regular flexagon.