PLASTICITY AND GEOMECHANICS

R. O. DAVIS
University of Canterbury

A. P. S. SELVADURAI
McGill University

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13 pt *System* LaTeX 2ε [TB]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

ISBN 0 521 81830 3 hardback

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1.1 Introduction

How a material responds to load is an everyday concern for civil engineers. As an example we can consider a beam that forms some part of a structure. When loads are applied to the structure the beam experiences deflections. If the loads are continuously increased the beam will experience progressively increasing deflections and ultimately the beam will fail. If the applied loads are small in comparison with the load at failure then the response of the beam may be proportional, i.e. a small change in load will result in a correspondingly small change in deflection. This proportional behaviour will not continue if the load approaches the failure value. At that point a small increase in load will result in a very large increase in deflection. We say the beam has failed. The mode of failure will depend on the material from which the beam is made. A steel beam will bend continuously and the steel itself will appear to flow much like a highly viscous material. A concrete beam will experience cracking at critical locations as the brittle cement paste fractures. Flow and fracture are the two failure modes we find in all materials of interest in civil engineering. Generally speaking, the job of the civil engineer is threefold: first to calculate the expected deflection of the beam when the loads are small; second to estimate the critical load at which failure is incipient; and third to predict how the beam may respond under failure conditions.

Geotechnical engineers and engineering geologists are mainly interested in the behaviour of soils and rocks. They are often confronted by each of the three tasks mentioned above. Most problems will involve either foundations, retaining walls or slopes. The loads will usually involve the weight of structures that must be supported as well as the weight of the soil or rock itself. Failure may occur by flow or fracture depending on the soil or rock properties. The geo-engineer will generally be interested in the deformations that may occur when the loads are small, the critical load that will bring about failure and what happens if failure does occur.

When the loads are smaller than a critical value, the geotechnical engineer will often represent the soil or rock as an elastic material. This is an approximation but it can be used effectively to provide answers to the first question: what deformations will occur when loads are small? The approximation of soil as a linear elastic material has been explored in a number of textbooks including our own – *Elasticity and Geomechanics*.* For convenience we will refer to this book as *EG*. In *EG* we outlined the fundamentals of the classical or linear theory of elasticity and we investigated some simple applications useful in geotechnical engineering. The book you now hold is meant to be a logical progression from *EG*. *Plasticity and Geomechanics* carries the reader forward into the area of failure and flow. We will outline the mathematical theory of plasticity and consider some simple questions concerning collapse loads, post-failure deformations and why soils behave as they do when stresses become too severe. Like *EG* this book is not meant to be a treatise. It will hopefully provide a concise introduction to the fundamentals of the theory of plasticity and will provide some relatively simple applications that are relevant in geo-engineering.

As a matter of necessity some of the material from *EG* must be repeated here in order that this book may be self-contained. In the present chapter we will cover some fundamental ideas concerning deformation, strain and stress, together with the concept of equilibrium. Chapter 2 then outlines basic elastic behaviour and discusses aspects of inelastic behaviour in respect to soil and rock. The nomenclature used here is similar to that adopted in *EG*. Readers who feel they have a firm grasp of stress, strain and elasticity, especially those who may have spent some time with *EG*, may wish to omit this chapter, and parts of the next, and move more quickly to Chapter 3. In Chapter 3 the concept of *yielding* is introduced. This is the state at which the failure process is about to commence. In Chapter 4 we investigate the process of *plastic flow*. That is, we try to determine the rules that govern deformations occurring once yield has taken place. Chapter 5 considers two important theorems that provide bounds on the behaviour of a plastically deforming material. These theorems may be extremely useful in approximating the response of geotechnical materials in realistic loading situations without necessitating any elaborate mathematics. Chapter 6 briefly touches on the mathematics of finding exact solutions for a limited class of problems and, finally, Chapter 7 introduces certain modern developments in the use of plasticity specifically for soils. The main body of

* Complete references to cited works are given at the end of the chapter where they first appear.

the book is followed by appendices that offer a more rigourous development of several important aspects.

1.2 Soil mechanics and continuum mechanics

Even the most casual inspection of any real soil shows clearly the random, particulate, disordered character we associate with natural materials of geologic origin. The soil will be a mixture of particles of varying mineral (and possibly organic) content, with the pore space between particles being occupied by either water, or air, or both. There are many important virtues associated with this aspect of a soil, not least its use as an agricultural medium; but, when we approach soil in an engineering context, it will often be desirable to overlook its particulate character. Modern theories that model particulate behaviour directly do exist and we will discuss one in Chapter 7, but in nearly all engineering applications we idealise soil as a continuum: a body that may be subdivided indefinitely without altering its character.

The treatment of soil as a continuum has its roots in the eighteenth century when interest in geotechnical engineering began in earnest. Charles Augustus Coulomb, one of the founding fathers of soil mechanics, clearly implied the continuum description of soil for engineering purposes in 1773. Since then nearly all engineering theories of soil behaviour of practical interest have depended on the continuum assumption. This is true of nearly all the soil plasticity theories we discuss in this book.

Relying on the continuum assumption, we can attribute familiar properties to all points in a soil body. For example, we can associate with any point $\boldsymbol{x}$ in the body a mass density ρ. In continuum mechanics we define ρ as the limiting ratio of an elemental mass ΔM and volume ΔV

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} \tag{1.1}$$

Of course we realise that were we to shrink the elemental volume ΔV to zero in a real soil we would find a highly variable result depending on whether the point coincides with the position occupied by a particle, or by water, or by air. Thus we interpret the density in (1.1) as a representative average value, as if the volume remains finite and of sufficient size to capture the salient qualities of the soil as a whole in the region of our point. Similar notions apply to other quantities of engineering interest. For example, there will be forces acting in the interior of the soil mass. In reality they will be unwieldy combinations of interparticle contact forces and hydrostatic forces. We will consider appropriate average forces and permit them to be supported by continuous surfaces. We can then consider the ratio of an elemental force on an elemental area and define stresses within the soil. It is elementary concepts such as these that we wish to elaborate in this chapter.

Although the concept of a continuum is elementary, it represents a powerful artifice, which enables the mathematical treatment of physical and mechanical phenomena in materials with complex internal structure such as soils. It allows us to take advantage of many mathematical tools in formulating theories of material behaviour for practical engineering applications.

1.3 Sign conventions

Before launching into our discussion of stress and strain, we will first consider the question of how signs for both quantities will be determined. In nearly all aspects of solid mechanics, tension is assumed to be positive. This includes both tensile stress and tensile strain. In geomechanics, on the other hand, most practitioners prefer to make compression positive, or at least to have compressive stress positive. This reflects the fact that particulate materials derive strength from confinement and confinement results from compressive stress. We will adopt the convention of compression being positive throughout this textbook.

Naturally, if compressive stress is considered to be positive then so must be compressive strain, and that requirement introduces an awkward aspect to the mathematical development of our subject. We can see the reason for this by considering a simple tension test as shown in Figure 1.1. In the figure a bar of some material is stretched by tensile forces T applied at each end. The axis of the bar is aligned with the coordinate axis x, and the end of the bar at the origin is fixed so that it cannot move. If the bar initially has length L, then application of the force T will be expected to cause an elongation of, say, Δ. Let the *displacement* of the bar be a function of x defined by $u = u(x) = \Delta(x/L)$. Physically the displacement tells us how far the particle initially located at x has moved, due to the force T. The extensional strain in the bar may be written as $\varepsilon = du/dx = \Delta/L$. If we were to adopt the solid mechanics convention of tension being positive, then the force T would be positive and so would be

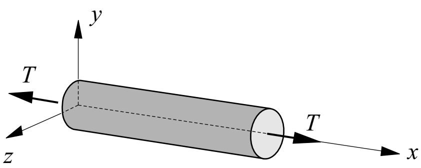

Figure 1.1. Prismatic bar in simple tension.

the extensional strain. Obviously all is well. On the other hand, if we wish to use the geomechanics convention that compression is positive, then the tensile force T is negative; but the strain, defined by $\varepsilon = du/dx$ remains positive. We could simply prescribe ε as a negative quantity, but that would not provide a general description for all situations. Instead we need some general method to correctly produce the appropriate sign for the strain.

There are two possible solutions to our problem. One approach is to redefine the extensional strain as $\varepsilon = -du/dx$. This will have the desired effect of making compressive strain always positive, but will have the undesirable effect of introducing negative signs in a number of equations where they may not be expected by the unwary and hence may cause confusion. The second solution is to agree from the outset that *positive displacements will always act in the negative coordinate direction*. If we adopt this convention, then the displacement of the bar is given by $u = u(x) = -\Delta(x/L)$. This second solution is the one we will adopt throughout the book. As a result nearly all the familiar equations of solid mechanics can be imported directly into our geomechanics context without any surprising negative signs. Moreover, there will be few opportunities where we must refer directly to the sign of the displacements, and so the convention of a positive displacement in the negative coordinate direction will mostly remain in the background. Specific comments will be made wherever we feel confusion might arise.

1.4 Deformation and strain

We begin by considering a continuum body with some generic shape similar to that shown in Figure 1.2. The body is placed in a reference system that we take to be a simple three-dimensional, rectangular Cartesian coordinate

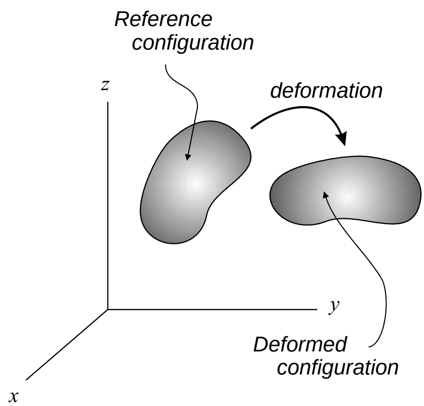

Figure 1.2. Reference and deformed configurations of body.

frame as shown in the figure. A deformation of the body results in it being moved from its original *reference configuration* to a new *deformed configuration*.

All deformations of a continuum are composed of two distinct parts. First there are *rigid motions*. These are deformations for which the shape of the body is not changed in any way. Two categories of rigid motion are possible, *rigid translation* and *rigid rotation*. A rigid translation simply moves the body from one location in space to another without changing its attitude in relation to the coordinate directions. A rigid rotation changes the attitude of the body but not its position.

The second part of our deformation involves all the changes of shape of the body. It may be stretched, or twisted, or inflated or compressed. These sorts of deformations result in *straining*. Strains are usually the most interesting aspect of a deformation.

One way to characterise any deformation is to assign a *displacement vector* to every point in the body. The displacement vector joins the position of a point in the reference configuration to its position in the deformed configuration. We represent the vector by

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) \tag{1.2}$$

where $\boldsymbol{x}$ denotes the position of any point within the body and t denotes time. A typical displacement vector is shown in Figure 1.3. Since there is a displacement vector associated with every point in the body, we say there is a *displacement vector field* covering the body. In our x, y, z coordinate frame, $\boldsymbol{u}$ has components denoted by u_x, u_y, u_z. Each component is, in general, a function of position

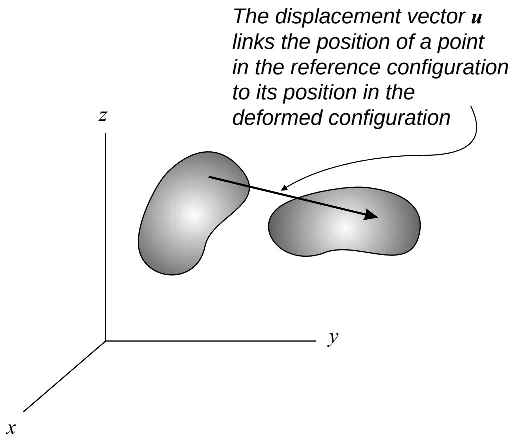

Figure 1.3. The displacement vector.

and time, and, according to our sign convention, components acting in negative coordinate directions will be considered to be positive.

If we know the displacement vector field, then we have complete knowledge of the deformation. Of course, part of the displacement field may be involved with rigid motions while the remainder results from straining. Our first task is to separate the two.

We begin by taking spatial derivatives of the components of the displacement vector. We arrange the derivatives into a 3 × 3 matrix called the displacement gradient matrix, $\nabla \boldsymbol{u}$.* If we are working in a three-dimensional rectangular Cartesian coordinate system we can represent $\nabla \boldsymbol{u}$ in an array as follows:

$$\nabla \boldsymbol{u} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & \dfrac{\partial u_z}{\partial z} \end{bmatrix} \tag{1.3}$$

Note the use of partial derivatives. Note also that the derivatives of $\boldsymbol{u}$ will *not* be affected by rigid translations. This might suggest we could use (1.3) as a measure of strain. But rigid rotations will give rise to non-zero derivatives of $\boldsymbol{u}$, so we need to introduce one more refinement. We use the *symmetric part* of $\nabla \boldsymbol{u}$. Let

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] \tag{1.4}$$

We call $\boldsymbol{\varepsilon}$ the *strain matrix*. Note that the superscript T indicates the transpose of the displacement gradient matrix. Also note that $\boldsymbol{\varepsilon}$ is a symmetric matrix. As its name implies, $\boldsymbol{\varepsilon}$ represents the straining that occurs during our deformation. Just as is the case with the displacement vector, $\boldsymbol{\varepsilon}$ is also a function of both position $\boldsymbol{x}$ and time t.

We write the components of $\boldsymbol{\varepsilon}$ as follows:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{1.5}$$

The diagonal components of $\boldsymbol{\varepsilon}$ are referred to as *extensional strains*,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{1.6}$$

* We use the symbol ∇ to denote the del operator $\frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}$ denote the triad of unit base vectors.

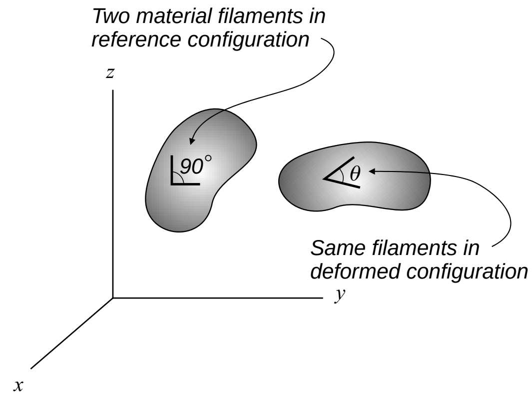

Figure 1.4. Physical meaning of shearing strain.

Each of these represents the change in length per unit length of a material filament aligned in the appropriate coordinate direction.

The off-diagonal components of $\boldsymbol{\varepsilon}$ are called *shear strains*

$$
\begin{aligned}
\varepsilon_{xy} = \varepsilon_{yx} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \\
\varepsilon_{yz} = \varepsilon_{zy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
\varepsilon_{zx} = \varepsilon_{xz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)
\end{aligned}
\tag{1.7}
$$

These strains represent one-half the increase* in the initially right angle between two material filaments aligned with the appropriate coordinate directions in the reference configuration. For example, consider two filaments aligned with the x- and y-directions in the reference configuration as shown in Figure 1.4. After the deformation the attitude of the filaments may have changed and the angle between them is now θ. Then $2\varepsilon_{xy} = 2\varepsilon_{yx} = \theta - \pi/2$. The presence of the factor of $\frac{1}{2}$ in (1.7) is important to ensure that the strain matrix will give the correct measure of straining in different coordinate systems. Often the change in an initially right angle (rather than one-half the change) is referred to as the *engineering shear strain*. It is usually denoted by the Greek letter gamma, γ. Obviously if we know one of the shear strains defined in (1.7), then we can determine the corresponding engineering shear strain.

* In solid mechanics the shear strain represents the *decrease* in the right angle. We have the *increase* because of the assumption that compression is positive and our sign convention for displacements.

An important aspect of the definition of the strain matrix in (1.4) is the requirement that the displacement derivatives remain small during the deformation. Sometimes the matrix $\boldsymbol{\varepsilon}$ is referred to as the *small* strain matrix. The name is meant to imply that the components of $\boldsymbol{\varepsilon}$ are only a correct measure of the actual straining so long as the components of $\nabla \boldsymbol{u}$ are much smaller in magnitude than 1. More complex definitions of strain are required in the case where deformation gradient components have large magnitudes. If the components of $\nabla \boldsymbol{u}$ are $\ll 1$, then products of the components can be ignored and the small-strain definition (1.4) results. There are substantial advantages associated with the small-strain matrix $\boldsymbol{\varepsilon}$ because it is a linear function of the displacement derivatives, while the large-strain measures are not. Because of this fact we may find that $\boldsymbol{\varepsilon}$ is used in some situations where it is not strictly applicable. Simple solutions are often good solutions, even if they are technically only approximations, and in geotechnical engineering the virtue of simplicity may justify a considerable loss of rigour.

Arising from the small-strain approximation is another measure of strain, the *volumetric strain, e.* It represents the change in volume per unit volume of the material in the reference configuration. It is defined as the sum of the three extensional strains:

$$e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \nabla \cdot \boldsymbol{u} \tag{1.8}$$

Here $\nabla \cdot \boldsymbol{u}$ represents the divergence of the vector $\boldsymbol{u}$.* There are a number of instances where the sum of the diagonal terms of a matrix gives a useful result. Because of this we define an operator called the *trace*, abbreviated as *tr*, which gives the sum. Thus (1.8) could also be written as $e = tr(\boldsymbol{\varepsilon})$.

In classical plasticity theory where metals are the primary material of interest, it is usual to assume that the material is incompressible and hence e is always zero. This is often not the case for soils, at least when they are permitted to drain. In undrained situations a fully saturated soil may be nearly incompressible, but if drainage can occur volume change is likely. In keeping with our definition of extensional strain, compressive volumetric strain will be considered to be positive.

Finally, note that all of the development above is based on the assumption that we are using a rectangular or Cartesian coordinate frame. At times it may be more convenient to use cylindrical or spherical coordinates. In that case there will be some subtle differences in many of the results given thus far. Appendix A

* It is the scalar quantity defined by $\nabla \cdot \boldsymbol{u} = (\frac{\partial}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial z}\hat{\boldsymbol{\jmath}}) \cdot (u_x\hat{\boldsymbol{\imath}} + u_y\hat{\boldsymbol{\jmath}} + u_z\hat{\boldsymbol{k}}) = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$.

outlines how one moves from rectangular to cylindrical or spherical coordinates and summarises the main results in non-Cartesian coordinate frames.

1.5 Strain compatibility

An important concept with regard to deformation and strain is the idea of *strain compatibility*. In simplest terms this is the physically reasonable requirement that when an intact body deforms, it does so without the development of *gaps or overlaps*. To be a little more precise, consider a point in the reference configuration, and construct some small neighbourhood of surrounding points. If we examine that same point in the deformed configuration, then we would hope to find the same neighbouring points surrounding it and, moreover, we would expect them to have similar relationships to the central point. That is, if neighbouring points α and β are arranged in the reference configuration so that α is closer and β more distant from the central point, then that arrangement should prevail in the deformed configuration as well.

Another way to look at this concept is to consider the definition of the strain matrix itself (1.4). We see that six independent components of strain are obtained from three independent components of displacement. If the displacement vector field is fully specified, then there is clearly no difficulty in determining the strains, but what if the problem is turned around? Suppose the six components of strain are specified. Is it then possible to integrate (1.4) to determine the three displacements uniquely? In general it is not. Moving from strains to displacements we find that the problem is over-determined, i.e. we have more equations than unknowns.

The great French mathematician Barré de Saint-Venant solved the general problem of strain compatibility in 1860. He showed that the strain components must satisfy a set of six *compatibility equations* shown in (1.9). A derivation of these equations may be found in Appendix A of *EG*. The derivation shows how equations (1.9) given below ensure that (1.4) can be integrated to yield single-valued and continuous displacements:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = -\frac{\partial^2 \varepsilon_{yz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{zx}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial z} \tag{1.9}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = -\frac{\partial^2 \varepsilon_{zx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{xy}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial x}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = -\frac{\partial^2 \varepsilon_{xy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{yz}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial y}$$

Finally, it is perhaps worth noting that the compatibility conditions impose a kinematic constraint on the strains in a continuum where the mechanical behaviour is as yet unspecified.

1.6 Forces and tractions

We approach the concept of stress through considering the forces that act on an exterior boundary or inside the body. We are aware that there are two distinct types of forces: *contact forces* and *body forces*. Body forces are forces caused by outside influences such as gravity or magnetism. They are associated with the volume or mass of the body and they are fully specified at the outset of any problem. Contact forces are associated with surfaces, either surfaces inside the body or segments of the exterior bounding surface of the body. Contact forces result from the action of the body on itself, such as the tension that exists inside a stretched rubber band or from specified boundary conditions such as an applied load on the upper surface of a beam.

For the time being we will concentrate our attention on contact forces. Every contact force is associated with a surface, so we consider a small element of surface dA embedded somewhere inside our continuum body. If we magnify the element as shown in Figure 1.5 then we can see its associated contact force as a vector $d\boldsymbol{F}$. Presumably $d\boldsymbol{F}$ results from the action of the body on itself since dA lies in the interior of the body. We then define the *surface traction vector*, $\boldsymbol{T}$, as the limiting value of the ratio of force and area.

$$\boldsymbol{T} = \lim_{dA \to 0} \frac{d\boldsymbol{F}}{dA} \tag{1.10}$$

We are aware of course that in the context of a real soil the limiting process must be treated with considerable care. We are concerned with a continuum, or at least a continuum approximation of the real material. In a real soil we would not wish to shrink dA to zero area, rather to terminate the limiting process at some point giving a reasonable representation of the soil structure.

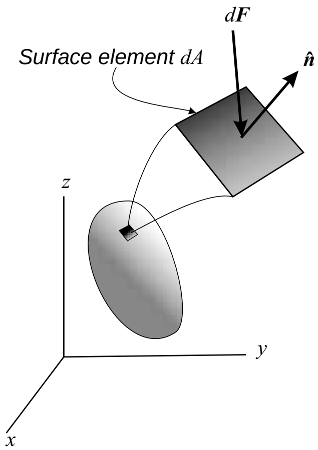

Figure 1.5. Traction vector acting on a surface element.

Note that the traction vector $\boldsymbol{T}$ is directly associated with the particular surface element we have chosen. If we choose a different surface element at the same point in the body, we will generally find a different traction vector. Therefore we see that the *orientation* of the surface element plays an important role. Since there are infinitely many possible orientations for our surface, there are infinitely many traction vectors operating at any given point in the body. This fact raises significant problems with regard to the description of stress. A number of eminent researchers in the eighteenth century were unsure of how stress might be easily characterised in all but simple problems. As it turns out, the problem is not difficult. We will only need to know tractions on three surfaces in order to fully prescribe the traction on any other surface.

1.7 The stress matrix

In 1823 the French mathematician Augustin Cauchy showed how we may solve the problem of determining the traction vector for a given surface. First we need to identify the orientation of the surface we are interested in. This is accomplished by the construction of a unit vector $\hat{\boldsymbol{n}}$ normal to the surface as shown in Figure 1.5. Then Cauchy showed that the product of a 3×3 square matrix $\boldsymbol{\sigma}^T$ with the vector $\hat{\boldsymbol{n}}$ gives the traction $\boldsymbol{T}$ acting on the surface,

$$\boldsymbol{T} = \boldsymbol{\sigma}^T \hat{\boldsymbol{n}} \tag{1.11}$$

This equation is derived in detail in Appendix C of *EG*. In equation (1.11) the superscript T indicates the transpose of the matrix. The matrix $\boldsymbol{\sigma}$ is called the

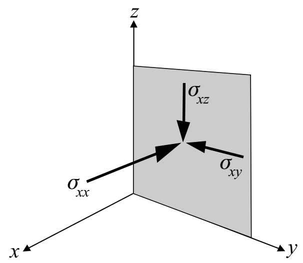

Figure 1.6. Components of the stress matrix acting on a surface perpendicular to the x-direction.

stress matrix. Its component form looks like this

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{1.12}$$

Each of the components, σ_{xx}, σ_{xy}, etc. is a component of a particular surface traction vector. For example, the components of the first row of $\boldsymbol{\sigma}$ are precisely the components of the traction vector that acts on a surface which is perpendicular to the x-axis as shown in Figure 1.6. This follows immediately if we note that the unit normal vector to the surface is $\hat{\boldsymbol{n}} = [1, 0, 0]^T$. Similarly, the second and third rows of the $\boldsymbol{\sigma}$ matrix are composed of, respectively, the components of traction vectors acting on surfaces perpendicular to the y- and z-axes. The subscripts of the stress matrix components identify which component of which surface traction is being represented. The xx-component, σ_{xx}, is the x-component of the traction acting on the surface perpendicular to the x-direction. Similarly, σ_{xy} is the y-component of that same traction. The yz-component, σ_{yz}, is the z-component of the traction acting on the surface perpendicular to the y-direction.

Note that in Figure 1.6 the stress matrix components are drawn pointing in the opposite direction to the coordinate axes. Because of this σ_{xx} appears to be a compressive stress. This is the usual sign convention in geomechanics where compression is positive.

The diagonal components of $\boldsymbol{\sigma}(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ are called the normal stress components, or simply the normal stresses. They act normal to the three surfaces perpendicular to the three coordinate directions. The off-diagonal components, $\sigma_{xy}, \sigma_{yz}, \ldots$ are called the shear stress components, or simply shear stresses. They act tangential to the three surfaces. Cauchy also showed that, in the absence of internal couples, the shear stresses must be complementary and hence the stress matrix is symmetric, i.e. $\sigma_{xy} = \sigma_{yx}, \sigma_{xz} = \sigma_{zx}, \sigma_{zy} = \sigma_{yz}$.

Because of this fact, the transpose of $\boldsymbol{\sigma}$ in (1.11) is not really important. We choose not to omit it, however, since the understanding of the physical meaning of the stress components springs directly from the equation.

1.8 Principal stresses

At any point in the body there will always be at least three surfaces on which the shear stresses $\sigma_{xy}, \sigma_{yz}, \ldots$ will vanish. These are the *principal surfaces* or *principal planes*. To see how this comes about note that if there is no shear stress on a surface the traction vector $\boldsymbol{T}$ must be parallel to the unit normal vector $\hat{\boldsymbol{n}}$. Then using (1.11) we see that

$$\boldsymbol{T} = \boldsymbol{\sigma}^T \hat{\boldsymbol{n}} = \alpha\, \hat{\boldsymbol{n}} \tag{1.13}$$

where α is a scalar multiplier. We can rearrange this result to obtain

$$(\boldsymbol{\sigma} - \alpha \mathbf{I})\, \hat{\boldsymbol{n}} = 0 \tag{1.14}$$

where $\mathbf{I}$ denotes the identity matrix and we have used the fact that $\boldsymbol{\sigma}$ is a symmetric matrix. Equation (1.14) gives three homogeneous linear equations. We know from linear algebra that there will either be no solutions, infinitely many solutions or a unique solution for any system of homogeneous linear equations. The condition for the existence of a unique solution is

$$\det(\boldsymbol{\sigma} - \alpha \mathbf{I}) = 0 \tag{1.15}$$

So we have an eigenvalue problem. If we expand the determinant in (1.15) we find the following *characteristic equation*:

$$-\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3 = 0 \tag{1.16}$$

where the coefficients I_1, I_2 and I_3 are functions of the stress matrix components $\sigma_{xx}, \sigma_{xy}, \ldots$. This cubic equation will have three roots (or three eigenvalues) for the multiplier α. Referring back to (1.13) we see that the roots will be the physical magnitudes of the traction $\boldsymbol{T}$ on each of the surfaces where there is no shear stress. We call these the *principal stresses* and denote them by σ_1, σ_2 and σ_3. Compression is taken to be positive here as everywhere in our development.

The greatest and least principal stress are called the *major principal stress* and *minor principal stress*, respectively. The remaining stress is called the *intermediate principal stress*. In some applications it is convenient to agree to number the principal stresses so that σ_1 is the major principal stress while σ_3 is the minor principal stress. This is a common convention but it may not always be the preferred option and we will not apply any particular rule to how σ_1, σ_2

and σ_3 may be related. In some circumstances we may have the conventional definition of $\sigma_1 \geq \sigma_2 \geq \sigma_3$, but at other times it may be more convenient to have $\sigma_3 \geq \sigma_2 \geq \sigma_1$, or one of the other four possible permutations of the three indices.

If we now substitute each of σ_1, σ_2 or σ_3 back in (1.13) to replace α, we can solve for the corresponding eigenvectors $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3$. These three vectors are called the principal directions. They define the three *principal surfaces*, i.e. the surfaces on which $\boldsymbol{T}$ and $\hat{\boldsymbol{n}}$ are parallel and therefore the surfaces that support no shear. A theorem from linear algebra assures us that the eigenvectors will be mutually orthogonal, hence the principal surfaces will also be mutually orthogonal. This can be a particularly useful result. It means that we can always find *some* coordinate system, say x', y', z', such that the coordinate directions are parallel to the principal directions (or perpendicular to the principal surfaces). In that coordinate system the stress matrix will have this simple form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{1.17}$$

That is, in this particular coordinate system, surfaces that are perpendicular to the coordinate axes support no shear. They are the principal surfaces.

Another interesting point arises here. Note that regardless of what coordinate system we happen to use, the principal stresses are independent entities. The components of $\boldsymbol{\sigma}$ at a point will, in general, be different in different coordinate systems, but the three principal stresses that we determine by finding the roots of (1.16) will always be the same. They are unique quantities associated with the particular point of interest in the continuum. We say that the principal stresses are *invariant* under a coordinate transformation. Invariants are often useful quantities owing to their independence from our choice of coordinate directions. This can be especially useful when it comes to creating descriptions of how materials behave. Obviously a material cannot know what coordinate directions we have chosen to use for its description. Therefore it would be unwise to create a model for the material stress–strain response that depended on the coordinate axis directions. But if we model the material using invariant quantities such as principal stresses, then there is no connection between the material model and the chosen frame of reference.

Of course σ_1, σ_2 and σ_3 are not the only invariant quantities associated with the stress matrix. It also follows from (1.16) that the three coefficients I_1, I_2 and I_3 must also be invariants. This must be true since, if we were to substitute one of the invariant principal stresses for the quantity α, the equation would be satisfied. If the principal stresses do not depend on the choice of coordinates, then neither can the coefficients I_1, I_2 and I_3. We call I_1, I_2 and I_3 the *principal stress invariants*. They are related to the components of the stress matrix by the following equations:

$$
\begin{aligned}
I_1 &= tr(\boldsymbol{\sigma}) \\
I_2 &= \frac{1}{2}[(tr\boldsymbol{\sigma})^2 - tr(\boldsymbol{\sigma}^2)] \\
I_3 &= \det(\boldsymbol{\sigma})
\end{aligned}
\tag{1.18}
$$

where we recall that the trace operator *tr* gives the sum of the diagonal components of the matrix. In the event that our coordinate system happened to align with the principal directions, and the stress matrix had the simple form shown in (1.17), the above equations would become

$$
\begin{aligned}
I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\
I_3 &= \sigma_1\sigma_2\sigma_3
\end{aligned}
\tag{1.19}
$$

Of course these equations are always true regardless of the choice of coordinate system. The sum of two invariant quantities will itself be invariant, as will the product of two invariants. For that matter any combination of invariants will also be an invariant. Equation (1.19) is simply the universal relationship between the principal stresses and the principal stress invariants. Note that the dimensions of the three principal invariants are [*stress*], [$stress^2$] and [$stress^3$].

One other invariant quantity that is often defined is the *mean stress* or *pressure*, denoted by p. It is equal to one-third of the first invariant, $I_1/3$. Thus $p = (\sigma_1 + \sigma_2 + \sigma_3)/3 = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3$. In the theory of elasticity, tensile stress is commonly taken as positive and the pressure is defined as the negative of $I_1/3$ so that positive pressure is compressive. We have no need of that definition since we have made compressive stress positive from the outset.

1.9 Mohr circles

Next, suppose we want to consider the stress state in a body at a specified point. Let us assume that the components of the stress matrix are known. In that case (1.11) applies and we can determine the traction $\boldsymbol{T}$ acting on any surface passing through the point. We could characterise the stress state by simply writing out the stress matrix, or we could list the principal stresses and the principal directions. In either case six independent numbers would be required.* If we

* Why are only six numbers needed to describe the three principal stresses and three principal directions? See Exercise 1.5.

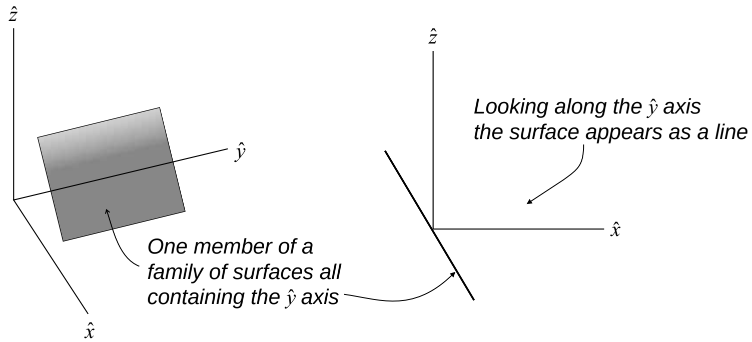

Figure 1.7. One of infinitely many surface elements generated by the $\hat{y}$-axis.

wished, we could visualise the stress state as a point in a six-dimensional space. However, there is another way to characterise the stress. We can create a simple graphical representation called the Mohr stress circle. The Mohr stress circle, or simply Mohr circle, is so important in relation to the theory of plasticity that Appendix B is completely devoted to its development. Only the major points will be described here to ensure that this introductory chapter remains brief.

Again suppose that the components of the stress matrix are known for some particular point in the body. Then we could solve the eigenvalue problem (1.14) to find the principal directions $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3$. These three vectors form the basis for a coordinate system that we might represent by $\hat{x}, \hat{y}, \hat{z}$ as shown in Figure 1.7. We know that the principal surfaces must be perpendicular to these coordinate directions. Suppose we now consider a family of surfaces composed of all the surfaces that are perpendicular to the $(\hat{x}, \hat{z})$-plane. One particular surface is shown in Figure 1.7. Any other member of the family could be obtained by rotating that surface about the $\hat{y}$-axis. The $\hat{y}$-axis is called a *generator* for the family of surfaces. We can use (1.11) to ascertain the traction vector $\boldsymbol{T}$ for each surface of our family. This will give us infinitely many traction vectors, but we won't worry about that point for the moment. Each traction vector $\boldsymbol{T}$ will have components in the $\hat{x}$- and $\hat{z}$-directions, but the component in the $\hat{y}$-direction will always be zero. This is a consequence of using the principal directions as our coordinate system.

To obtain a Mohr stress circle, we now plot the components of all the traction vectors for all the surfaces of our family. However, we do not plot the traction components acting in the $\hat{x}$ and $\hat{z}$ directions. Instead we plot the components that act normal and tangential to the surface on which $\boldsymbol{T}$ acts. To be more precise, consider the surface shown in Figure 1.7. If we arrange our view point so that we look directly down the $\hat{y}$-axis, we see the situation shown in Figure 1.8. In that figure the $\hat{y}$-axis is perpendicular to the plane of the figure and we see

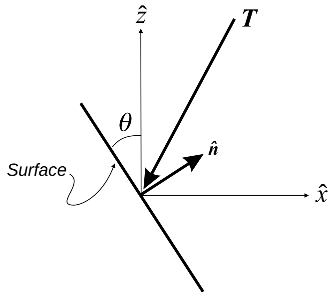

Figure 1.8. Traction vector acting on the surface element in Figure 1.7.

it as a point at the origin. Our surface appears as a line. Both the normal vector to the surface $\hat{\boldsymbol{n}}$ and the traction vector $\boldsymbol{T}$ are shown and both lie in the plane of the figure.

If we use the angle θ shown in Figure 1.8 to identify the particular surface, then the unit normal vector components can be written as

$$\hat{\boldsymbol{n}} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} \tag{1.20}$$

Also, since the coordinate axes are parallel to the principal directions, the stress matrix will have the form (1.17). Then (1.11) gives the following result for the components of $\boldsymbol{T}$ in the $\hat{x}$- and $\hat{z}$-directions:

$$\boldsymbol{T} = \begin{bmatrix} \sigma_1 \sin\theta \\ \sigma_3 \cos\theta \end{bmatrix} \tag{1.21}$$

Now let σ and τ identify the components of $\boldsymbol{T}$ that act normal and tangential to our surface. We find σ by taking the inner product of $\boldsymbol{T}$ and $\hat{\boldsymbol{n}}$

$$\sigma = \boldsymbol{T} \cdot \hat{\boldsymbol{n}} = \sigma_1 \sin^2\theta + \sigma_3 \cos^2\theta \tag{1.22}$$

It is similarly easy to show that

$$\tau = (\sigma_1 - \sigma_3) \sin\theta \cos\theta \tag{1.23}$$

The final step is to plot τ against σ for all the surfaces as θ varies between 0 and π.* The result is a circle, the Mohr circle. A typical Mohr circle is shown in Figure 1.9. Each point on the circumference of the circle identifies the normal and tangential components of the traction vector acting on one particular member of our family of surfaces. We refer to the points on the circle circumference as *stress points*.

* Note that there is no need to let θ run to 2π since a rotation of only π radians brings us back to our starting surface.

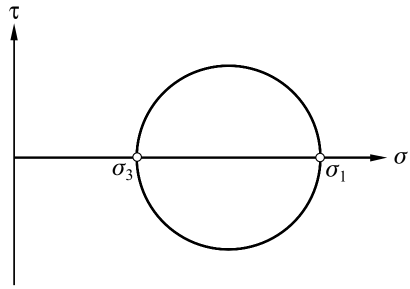

Figure 1.9. Mohr stress circle.

The centre of the circle must lie on the σ-axis. The circle crosses the σ-axis at the points that correspond to the two surfaces that support zero shear: the principal surfaces. As a result the diameter of the circle is the principal stress difference, in this case $(\sigma_1 - \sigma_3)$. The greatest and least shear stresses are equal to the positive and negative values of the circle radius $(\sigma_1 - \sigma_3)/2$. If once again we think of physically rotating the surface shown in Figure 1.8, then a rotation of π radians will result in the corresponding stress point moving completely around the circle and returning to its original starting point. In Appendix B the exact relationship between any surface and its corresponding stress point is developed in full.

The Mohr circle in Figure 1.9 contains all the stress information for all the surfaces of our family. Obviously, however, there are many other surfaces we have not yet considered. We could easily go through the same procedures for surfaces generated by the $\hat{x}$-axis and this would give another Mohr circle. Since the $\hat{x}$-axis corresponds to the $\hat{\boldsymbol{n}}_1$ principal direction, the resulting circle would cross the σ-axis at the principal stresses σ_2 and σ_3. Similarly, if we considered surfaces generated by the $\hat{z}$-axis, we would obtain a third circle spanning the principal stresses σ_2 and σ_1. The three circles might look like those sketched in Figure 1.10. Note how the circles join at the principal stresses and how each circle spans two of the principal stress values. We have drawn the figure as if $\sigma_1 > \sigma_2 > \sigma_3$, i.e. the usual convention used for numbering principal stresses, but we realise that any other numbering, such as $\sigma_2 > \sigma_3 > \sigma_1$, is equally possible.

Now we have exhausted all the obvious possibilities for surfaces. We have considered all the surfaces that are generated by each of the three principal directions and this has led to three Mohr circles. What about all of the other possible surfaces that are not generated by the principal directions but instead are oriented at non-right angles to the principal surfaces? These surfaces will generally have traction vectors that have non-zero components in all three of

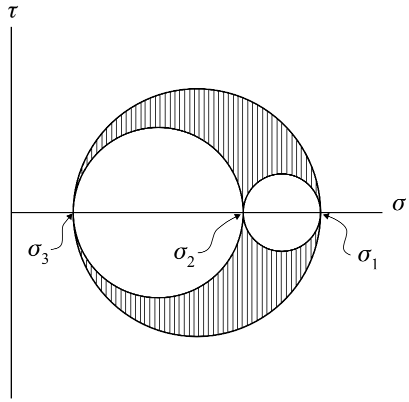

Figure 1.10. Mohr stress circle for the three-dimensional stress state.

the coordinate directions $\hat{x}$, $\hat{y}$ and $\hat{z}$. If we determine their components normal and tangential to their respective surfaces, and plot the components σ and τ we find that the resulting points exactly fill the regions between the three Mohr circles. That is, the stress points associated with these remaining surfaces all fall within the hatched regions in Figure 1.10. The three circles plus the interior points represent the entire stress state graphically.

Often, because of symmetry about the σ-axis, only the upper half of the Mohr stress circle is drawn. Also only the outermost circle is frequently shown. This reflects the fact that the most extreme stress states are represented by points on the outermost circle. Regardless of these details, the Mohr circle is an extremely useful tool. It allows one to visualise the entire stress state at any point in a body easily and it permits an intuitive grasp of stress that is not possible by considering formal equations such as (1.11). Later in the book when we consider yield criteria the Mohr circle will be a very valuable tool.

1.10 The effective stress principle

A concept familiar to all geotechnical engineers is the effective stress principle. It was formulated by one of the founding fathers of soil mechanics, Karl Terzaghi, in 1925. Terzaghi realised that in a saturated soil the solid particle skeleton must play a much more important role than the pore water. This is particularly true in regard to shearing stresses since the pore water can carry no shear stress at all. All shearing stresses are supported by the solid particle skeleton. The situation with normal stresses, however, is not quite so clear.