

Statics and Dynamics with Background Mathematics

A. P. Roberts

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa
<http://www.cambridge.org>

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First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typefaces Times 10.5/14 pt and Helvetica *System* L^AT_EX 2_ε [TB]

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

Roberts, A. P.

Statics and dynamics with background mathematics / A. P. Roberts.

p. cm.

Includes bibliographical references and index.

ISBN 0 521 81766 8 – ISBN 0 521 52087 8 (pbk.)

1. Statics. 2. Dynamics. I. Title.

QA821 .R64 2003

531'.12 – dc21 2002073607

ISBN 0 521 81766 8 hardback

ISBN 0 521 52087 8 paperback

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1 Forces

1.1 Force

In the study of statics we are concerned with two fundamental quantities: length or distance, which requires no explanation, and force. The quantity length can be seen with the eye but with force, the only thing that is ever seen is its effect. We can see a spring being stretched or a rubber ball being squashed but what is seen is only the effect of a force being applied and not the force itself. With a rigid body there is no distortion due to the force and in statics it does not move either. Hence, there is no visual indication of forces being applied.

We detect a force being applied to our human body by our sense of touch or feel. Again, it is not the force itself but its effect which is felt – we feel the movement of our stomachs when we go over a humpback bridge in a fast car; we feel that the soles of our feet are squashed slightly when we stand.

We have now encountered one of the fundamental conceptual difficulties in the study of mechanics. Force cannot be seen or measured directly but must always be imagined. Generally the existence of some force requires little imagination but to imagine all the different forces which exist in a given situation may not be too easy. Furthermore, in order to perform any analysis, the forces must be defined precisely in mathematical terms.

For the moment we shall content ourselves with a qualitative definition of force. ‘A force is that quantity which tries to move the object on which it acts.’ This qualitative definition will suffice for statical problems in which the object does not move but we shall have to give it further consideration when we study the subject of dynamics. If the object does not move, the force must be opposed and balanced by another force. If we push with our hand against a wall, we know that we are exerting a force; we also know that the wall would be pushed over if it were not so strong. By saying that the wall is strong we mean that the wall itself can produce a force to balance the one applied by us.

EXERCISE 1

Note down a few different forces and state whether, and if so how, they might be observed.

1.2 Forces of contact

Before giving a precise mathematical description of force, we shall discuss two general categories. We shall start with the type which is more easily imagined; this is that due to contact between one object and another.

In the example of pushing against a wall with one's hand, the wall and hand are in contact, a force is exerted by the hand on the wall and this is opposed by another force from the wall to the hand. In the same way, when we are standing on the ground we can feel the force of the ground on our feet in opposition to the force due to our weight transmitted through our feet to the ground. Sometimes we think of a force being a pull but if we analyse the situation, the force of contact from one object to another is still a push. For instance, suppose a rope is tied around an object so that the latter may be pulled along. When this happens, the force from the rope which moves the object is a push on the rear of the object.

Another form of contact force is that which occurs when a moving object strikes another one. Any player of ball games will be familiar with this type of force. It only acts for a short time and is called an impulsive force. It is given special consideration in dynamics but it also occurs in statics in the following sense. When the surface of an object is in contact with a gas, the gas exerts a pressure, that is a force spread over the surface. The pressure is caused by the individual particles of the gas bouncing against the surface and exerting impulsive forces. The magnitude of each force is so small but the frequency of occurrence is so high that the effect is that of a force continuously distributed over the whole surface.

Forces of contact need not be exerted normal to the surface of contact. It is also possible to exert what is called a tangential or frictional component of force. In this case the force is applied obliquely to the surface; we can think of part being applied normally, i.e. perpendicular to the surface, and part tangentially. The maximum proportion of the tangential part which may be applied depends upon the nature of the surfaces in contact. Ice skaters know how small the tangential component can be and manufacturers of motor car tyres know how high.

A fact which must be emphasized concerning contact forces is that the forces each way are always equal and opposite, i.e. action and reaction are equal and opposite. When you push against the wall with your hand, the force from your hand on the wall is equal and opposite to the force from the wall against your hand. The rule is true for any pair of contact forces.

EXERCISE 2

Note down some of the contact forces which you have experienced or which have been applied to objects with which you have been concerned during the day. For each contact force, note the equal and opposite force which opposed it.

1.3 Mysterious forces

It is not too difficult to imagine the contact forces already described from our everyday experience but what is it that prevents a solid object from bending, squashing or just falling apart under the action of such forces? Mysterious forces of attraction act between the separate molecules of the material binding them together in a particular way and resisting outside forces which try to disturb the pattern. These *intermolecular forces* constitute the strength of the material. Although we shall not be concerned with it here, knowledge of the strength of materials is of great importance to engineers when designing buildings, machinery, etc.

Another mysterious force which will concern us deeply is the *force of gravity*. The magnitude of the force of gravity acting on a particular object depends on the size and physical nature of the object. In our study this force will remain constant and it will always act vertically downwards, this being referred to as the *weight* of the object. This is sufficient for most earthbound problems but when studying artificial satellites and space-craft it is necessary to consider the full properties of gravity.

Gravity is a force of attraction between any two bodies. It needs no material for its transmission nor is it impeded or changed in any way by material placed in between the bodies in question. The magnitude of the force was given mathematical form by Sir Isaac Newton and published in his *Philosophiae Naturalis Principia Mathematica* in 1687. The force is proportional to the product of the masses of the two bodies divided by the square of the distance between them. We shall say more about mass when studying dynamics but it is a constant property of any body. The law of gravitation, i.e. the inverse square law, was deduced by correlating it with the elliptical motion of planets about the sun as focus. Newton proved that such motion would be produced by the inverse square law of attraction to the sun acting on each planet.

Given that a body generates an attractive force proportional to its mass, it is reasonable that an inverse square law with respect to distance should apply. The force acts in towards the body from all directions around. However, the force acts over a larger area as the distance from the body increases. Since the effort is spread out over a larger area we can expect the strength at any particular point in the area to decrease accordingly. Thus we expect the magnitude of the force to be inversely proportional to the area of the sphere with the body at the centre and the point at which the force acts being on the surface of the sphere. The area is proportional to the square of the radius of the sphere; hence the inverse square law follows.

Magnetic and electrostatic forces are also important mysterious forces. However, they will not be discussed here since we shall not be concerned with them in this text.

EXERCISE 3

Find the altitude at which the weight of a body is one per cent less than its weight at sea level. (Assume that the radius of the earth from sea level is 6370 km.)

EXERCISE 4

Find the percentage reduction in weight when the body is lifted from sea level to a height of 3 km.

1.4 Quantitative definition of force

In statics, force is that quantity which tries to move the object on which it acts. The magnitude of a force is the measure of its strength. It is then necessary to define basic units of measurement.

In lifting different objects we are very familiar with the concept of weight, which is the downward gravitational force on an object. It is tempting to use the weight of a particular object as the unit of force. However, weight varies with altitude (see Exercises 3 and 4) and also with latitude. To avoid this, a dynamical unit of force has been adopted.

The basic SI unit (Système International d'Unités) is the newton (symbol N). It is the force which would give a mass of one kilogramme (1 kg) an acceleration of one metre per second per second (1 m/s^2 or 1 m s^{-2}). The kilogramme is the mass of a particular piece of platinum–iridium. Of course, once a standard has been set, other masses can easily be evaluated by comparing relative weights. Incidentally, the mass of 1 kg is approximately the mass of one cubic decimetre of distilled water at the temperature (3.98°C) at which its density is maximum.

If you are more familiar with the pound-force (lbf) as the unit of force, then

$$1 \text{ lbf} = 4.449 \text{ N or } 1 \text{ N} = 0.2248 \text{ lbf.}$$

In quantifying a force, not only must its magnitude be given but also its direction of application, i.e. the direction in which it tries to move the object on which it acts. Having both magnitude and direction, force is a *vector* quantity. Sometimes it is convenient to represent a force graphically by an arrow (see Figure 1.1) which points in a direction corresponding to the direction of the force and has a length proportional to the magnitude of the force.

EXERCISE 5

Consider an aeroplane (see Figure 1.2) flying along at constant speed and height. Since there is no acceleration, forces should balance out in the same way that they do in statics. Draw vectors which might correspond to (a) the weight of the aeroplane, (b) the thrust from its engines and (c) the force from the surrounding air on the aeroplane which is a combination of lift and drag (lift/drag).

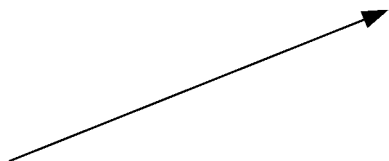


Figure 1.1. Force vector.



Figure 1.2. Simple sketch of an aeroplane.

1.5 Point of application

In studying forces acting on a rigid body, it is necessary to know the points of the body to which the forces are applied. For instance, consider a horizontal force applied to a stone which is resting on horizontal ground. If the force is strong enough the stone will move, but whether it moves by toppling or slipping depends on where the force is applied.

Forces rarely act at a single point of a body. Usually the force is spread out over a surface or volume. If the stone mentioned above is pushed with your hand, then the force from your hand is spread out over the surface of contact between your hand and the stone. The force from the ground which is acting on the stone is spread out over the surface of contact with the ground. The gravitational force acting on the stone is spread out over the whole volume of the stone. In order to perform the analysis in minute detail it would be necessary to consider each small force acting on each small element of area and on each small element of volume. However, since we are only considering rigid bodies, we are not concerned with internal stress. Thus we can replace many small forces by one large force. In our example, the small forces from the small elements of area of contact of your hand are represented by a single large force acting on the stone. Similarly, we have a single large force acting from the ground. Also, for the small gravitational forces acting on all the small elements of volume of the stone, we have instead a single force equal to the weight of the stone acting at a point in the stone which is called the *centre of gravity*.

The derivation of the points of action of these equivalent resultant forces will be discussed later. For the time being we shall assume that the representation is valid so that we can study the example of the stone as though there were only three forces acting on it, one from your hand, one from the ground and one from gravity.

EXERCISE 6

Continue Exercise 5 by drawing in the three force vectors on a rough sketch of the aeroplane.

1.6 Line of action

In the answer to Exercise 6, it appears that the three resultant forces of weight, thrust and lift/drag all act at the same point. Of course this may not be so but it does not

matter provided the lines of action of the three resultant forces intersect at one point. For instance, this point need not coincide with the centre of gravity but it must be in the same vertical line as the centre of gravity.

Thus, with a rigid body the effect of a force is the same for any point of application along its line of action. This property is referred to as the *principle of transmissibility*. If two non-parallel but coplanar forces act on a body, it is convenient to imagine them to be acting at the point of intersection of their lines of action.

EXERCISE 7

Suppose that a smooth sphere is held on an inclined plane by a string which is fastened to a point on the surface of the sphere at one end and to a point on the plane at the other end. Sketch the side view and draw in the force vectors at the points of intersection of their lines of action.

EXERCISE 8

Do the same as in Exercise 7 for a ladder leaning against a wall, assuming that the lines of action of the three forces (weight and reactions from wall and ground) are concurrent.

Problems 1 and 2.

1.7 Equilibrium of two forces

A force tries to move its point of application and it will move it unless there is an equal and opposite counterbalancing force. When you push a wall with your hand, the wall will move unless it is strong enough to produce an equal and opposite force on your hand. If you are holding a dog with a lead, you will only remain stationary if you pull on the lead with the same amount of force as that exerted by the dog. By considering such physical examples we can see that for two forces to balance each other, they must be equal in magnitude and opposite in direction.

Yet another property is also required for the balance to exist. Suppose we have a large wheel mounted on a vertical axle. If one person pushes the wheel tangentially along the rim on one side and another person pushes on the other side, the wheel will start to move if the two pushes are equal in magnitude and opposite in direction. In fact two forces only balance each other if not only are they equal in magnitude and opposite in direction but also have the same line of action. When you are holding the dog, the line of the lead is the line of action of both the force from your hand and the force from the dog.

When the three conditions hold, we say that the two forces are in *equilibrium*. If a rigid body is acted on by only two such forces, the body will not move and we say that the body is in equilibrium. When a stone rests in equilibrium on the ground, the resultant contact force from the ground is equal, opposite and collinear to the resultant gravitational force acting on the stone.

EXERCISE 9

Suppose that a rigid straight rod rests on its side on a smooth horizontal surface. Let two horizontal forces of equal magnitude be applied to the rod simultaneously, one at either end. Consider what will happen to the rod immediately after the forces have been applied for a few different situations regarding the directions in which the separate forces are applied. Show that there will be only two possible situations in which the rod will remain in equilibrium.

1.8 Parallelogram of forces (vector addition)

If two non-parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act at a point A, they have a combined effect equivalent to a single force $\mathbf{R}$ acting at A. The single force $\mathbf{R}$ is called the *resultant* and it may be found as follows. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be represented in magnitude and direction by two sides of a parallelogram meeting at A. Then $\mathbf{R}$ is represented in magnitude and direction by the diagonal of the parallelogram from A, as shown in Figure 1.3. This is an empirical result referred to as the parallelogram law.

The parallelogram law may be illustrated by the following experiment. Take three different known weights of magnitudes W_1 , W_2 and W_3 , and attach W_1 and W_2 to either end of a length of string. Drape the string over two smooth pegs set a distance apart about the same height. Then attach W_3 with a small piece of string to a point A of the other string between the two pegs. Finally, allow W_3 to drop gently and possibly move sideways until an equilibrium position is established (see Figure 1.4).

Now measure the angles to the horizontal made by the sections of string between the two pegs and A. Make an accurate drawing of the strings which meet at A and mark off distances proportional to W_1 and W_2 as shown in Figure 1.5. Since the pegs are smooth,

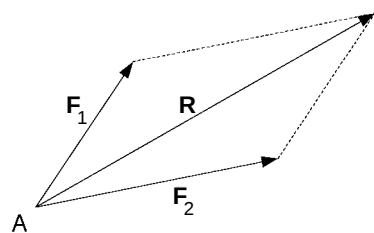


Figure 1.3. Parallelogram of forces.

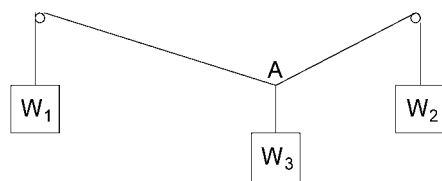


Figure 1.4. String over two smooth pegs.

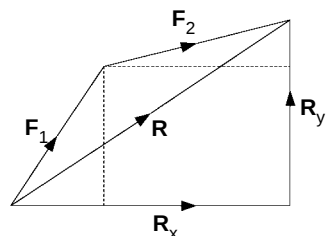


Figure 1.8. Addition of Cartesian components in vector addition.

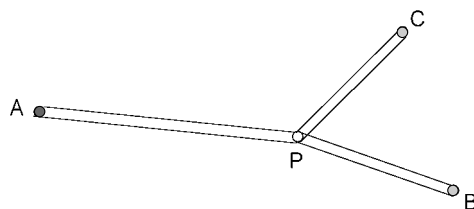


Figure 1.9. Three elastic bands used to demonstrate the parallelogram law.

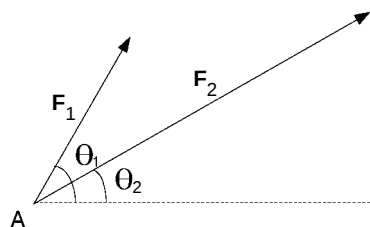


Figure 1.10. Two forces F_1 and F_2 acting at a point A.

can be seen diagrammatically by drawing in the Cartesian components as illustrated in Figure 1.8.

EXERCISE 10

Use a piece of cotton thread to tie together three identical elastic bands. Having measured the unstretched length of the bands, peg them out as indicated in Figure 1.9, so that each band is in a stretched state but not beyond the elastic limit. The points A, B and C represent the fixed positions of the pegs but the point P takes up its equilibrium position pulled in three directions by the tensions in the bands. Use the fact that tension in each band is proportional to extension in order to verify the parallelogram law for the resultant of two forces acting at a point.

EXERCISE 11

Calculate the magnitude and direction of the resultant R of the two forces F_1 and F_2 acting at the point A given the magnitudes $F_1 = 1 \text{ N}$, $F_2 = 2 \text{ N}$ and directions $\theta_1 = 60^\circ$, $\theta_2 = 30^\circ$ (see Figure 1.10).

Problems 3 and 4.

1.9 Resultant of three coplanar forces acting at a point

Consider the three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ shown in Figure 1.11. The resultant $\mathbf{R}_1$ of $\mathbf{F}_1$ and $\mathbf{F}_2$ can be found by drawing the vector $\mathbf{F}_2$ on the end of $\mathbf{F}_1$ and joining the start of $\mathbf{F}_1$ to the end of $\mathbf{F}_2$. Then the final resultant $\mathbf{R}$ of $\mathbf{R}_1$ and $\mathbf{F}_3$, i.e. of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$, is found by drawing the vector $\mathbf{F}_3$ on the end of $\mathbf{R}_1$ and joining the start of $\mathbf{R}_1$ to the end of $\mathbf{F}_3$. Having done this, we see that the intermediate step of inserting $\mathbf{R}_1$ may be omitted. Hence, the construction shown in Figure 1.11 is replaced by that of Figure 1.12. The procedure is simply to join the vectors $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ end-on-end; then the resultant $\mathbf{R}$ corresponds to the vector joining the start of $\mathbf{F}_1$ to the end of $\mathbf{F}_3$.

The resultant vector $\mathbf{R}$ corresponds to the vector addition

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3.$$

In terms of Cartesian components:

$$R_x = F_{1x} + F_{2x} + F_{3x}$$

and

$$R_y = F_{1y} + F_{2y} + F_{3y}.$$

The three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ will be in equilibrium if their resultant $\mathbf{R}$ is zero. In this case, joining the vectors end-on-end, the end of $\mathbf{F}_3$ will coincide with the start of $\mathbf{F}_1$. Thus we have the triangle of forces, which states that three coplanar forces acting at a point are in equilibrium if their vectors joined end-on-end correspond to the sides of a triangle, as illustrated in Figure 1.13.

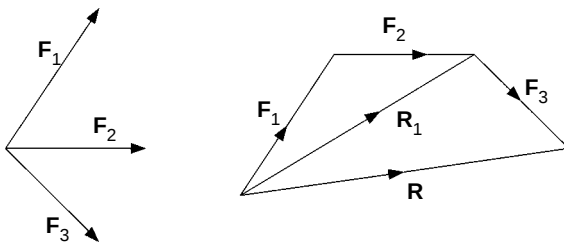


Figure 1.11. Constructing the resultant of three coplanar forces acting at a point.

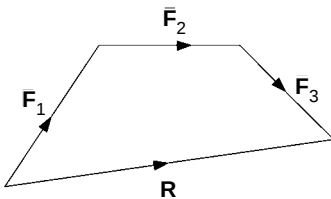


Figure 1.12. The final construction $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$.

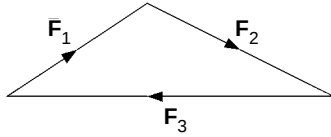


Figure 1.13. Triangle of forces.

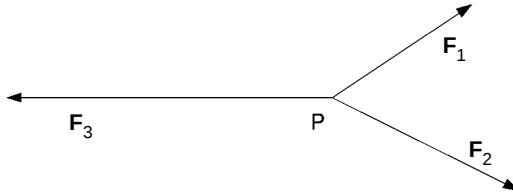


Figure 1.14. Three coplanar forces acting at a point.

Referring to Figures 1.13 and 1.14, we notice that the angle in the triangle between $\mathbf{F}_1$ and $\mathbf{F}_2$ is 180° minus the angle between the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at the point P, and similarly for the other angles. Now, the sine rule for a triangle states that the length of each side is proportional to the sine of the angle opposite. Since $\sin(180^\circ - \theta) = \sin \theta$, we have Lamy's theorem which states that 'three coplanar forces acting at a point are in equilibrium if the magnitude of each force is proportional to the sine of the angle between the other two forces'.

EXERCISE 12

Let three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ have magnitudes in newtons of $\sqrt{6}$, $1 + \sqrt{3}$ and 2, respectively. If the angles made with the positive x -direction are 45° for $\mathbf{F}_1$, 180° for $\mathbf{F}_2$ and -60° for $\mathbf{F}_3$, show that the three forces are in equilibrium by (a) calculating their resultant, (b) triangle of forces and (c) Lamy's theorem.

Problems 5 and 6.

1.10 Generalizations for forces acting at a point

Firstly, consider more than three coplanar forces acting at a point. The vector addition procedure can be continued. For instance, drawing the vectors end-on-end gives $\mathbf{R}_3$, say, for the resultant of the first three. Then, as shown in Figure 1.15, drawing the vector for $\mathbf{F}_4$ onto the end of $\mathbf{R}_3$, the resultant of $\mathbf{R}_3$ and $\mathbf{F}_4$ is given by the vector $\mathbf{R}_4$ joining the start of $\mathbf{R}_3$ to the end of $\mathbf{F}_4$. $\mathbf{R}_3$ can now be omitted.

This procedure is obviously valid for finding the resultant of any number of coplanar forces acting at a point. Furthermore, the forces will be in equilibrium if the final resultant is zero, i.e. when the end of the last vector coincides with the start of the first. Hence, we have the *polygon of forces*, which states that ' n coplanar forces acting

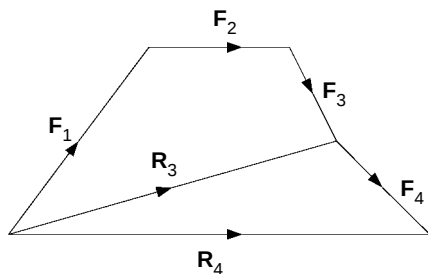


Figure 1.15. Constructing the resultant of four coplanar forces acting at a point.

at a point are in equilibrium if their vectors joined end-on-end complete an n -sided polygon'.

Although it is not convenient for two-dimensional drawing, the basic concept can be extended to finding the resultant of non-coplanar forces acting at a point. Any two of the forces are coplanar, so the resultant $\mathbf{R}_2$ of $\mathbf{F}_1$ and $\mathbf{F}_2$ is the vector sum $\mathbf{R}_2 = \mathbf{F}_1 + \mathbf{F}_2$. Then $\mathbf{R}_2$ and $\mathbf{F}_3$ must be coplanar with resultant $\mathbf{R}_3 = \mathbf{R}_2 + \mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$. Thus, if there are n forces, their resultant is $\mathbf{R}_n = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n$.

We have now moved from two-dimensional to three-dimensional space. Each vector has three *Cartesian components*, i.e. its x -, y - and z -components. The Cartesian components of $\mathbf{R}_n$ are :

$$R_{nx} = F_{1x} + F_{2x} + \cdots + F_{nx}$$

$$R_{ny} = F_{1y} + F_{2y} + \cdots + F_{ny}$$

$$R_{nz} = F_{1z} + F_{2z} + \cdots + F_{nz}$$

where x , y and z signify the corresponding component in each case.

As in the coplanar case, the forces will be in equilibrium if their resultant $\mathbf{R}_n$ is zero, i.e. $R_{nx} = R_{ny} = R_{nz} = 0$.

EXERCISE 13

Let four coplanar forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ acting at a point have magnitudes in newtons of 2, $\sqrt{6}$, 2 and $\sqrt{2}$, and directions relative to the positive x -axis of 150° , 45° , -60° and -135° , respectively. Show that the four forces are in equilibrium by (a) calculating their resultant and (b) using a polygon of forces.

EXERCISE 14

Find the resultant of three non-coplanar forces acting at a point, where each is given in newtons in terms of its Cartesian components: $\mathbf{F}_1 = (1, -2, -1)$, $\mathbf{F}_2 = (2, 1, -1)$ and $\mathbf{F}_3 = (-1, -1, 1)$. Besides giving the resultant in terms of its Cartesian components, find its magnitude and the angles which it makes with the x -, y - and z -axes.

Problems 7 and 8.

1.11 More exercises

EXERCISE 15

Figure 1.16 shows weights W_1 and W_2 attached to the ends of a string which passes over two smooth pegs A and B at the same height and 0.5 m apart. A third weight W_3 is suspended from a point C on the string between A and B.

(a) If $W_1 = 3 \text{ N}$, $W_2 = 4 \text{ N}$ and $W_3 = 5 \text{ N}$, find the horizontal and vertical distances a and d of C from A when the weights are in equilibrium.

(b) If $W_1 = 10 \text{ N}$, the angle between CB and the vertical is 30° and $\angle ACB = 90^\circ$ when the weights are in equilibrium, find the weights W_2 and W_3 .

EXERCISE 16

Two smooth spheres, each of weight W and radius r , are placed in the bottom of a vertical cylinder of radius $3r/2$. Find the magnitudes of the forces R_a , R_b , R_c and R_p which act on the spheres as indicated in Figure 1.17.

EXERCISE 17

Four identical smooth spheres, each of weight W and radius r , are placed in the bottom of a hollow vertical circular cylinder of inner radius $2r$. Two spheres rest on the bottom and the other two settle as low as possible above the bottom two. Find all the forces acting on the spheres. Because of the symmetry, only one of the top and one of the bottom spheres need be considered.

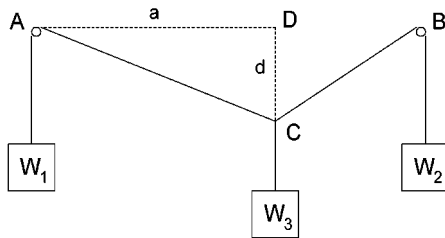


Figure 1.16. Three weights on a string.

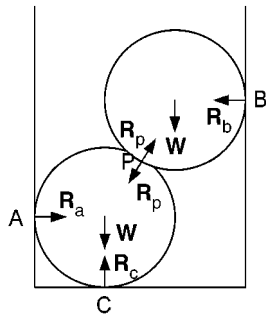


Figure 1.17. Two spheres in a hollow cylinder.

1.12 Answers to exercises

1. There are infinitely many possible answers to Exercise 1. The following are a couple of examples.
- If you hang a wet and heavy piece of clothing on a clothes-line, the weight of the clothing exerts a downward force on the line, which is opposed by an increased tension in the line. The latter is observed in a downward sag of the line from its unloaded position.

If you stand on weighing scales in your bathroom, your own weight exerts a downward force via your feet which is opposed by an upward force from the scales. The latter is observed both by the feeling of pressure on your feet and by the measurement of your weight as indicated by the scales.

2. There will be many different answers to Exercise 2. Here are a couple of examples from my own experience.

When I came out of the house this morning, I was carrying a brief-case. The weight of the case exerted a downward force via the handle on the fingers of my hand. The latter exerted an equal and opposite force on the handle of the brief-case.

Later, as I was driving my car at a steady speed, I had to exert a constant force on the accelerator pedal to keep it in a certain position. This force was transmitted to the pedal through the sole of my shoe. At the same time, the pedal was exerting an equal and opposite force on the sole of my shoe.

3. The distance in the inverse square law is measured from the centre of each body. Let W be the weight of a body and r be its distance from the centre of the earth. Also, let the Greek letter δ (delta) mean 'change in' so that δW is change in weight and δr is change in distance from the centre of the earth. Using k as a constant of proportionality, by the inverse square law, $W = k/r^2$.

If the weight reduces by δW when the body is raised from sea level to an altitude of δr , then $W - \delta W = k/(r + \delta r)^2$. Dividing this by W gives:

$$1 - \frac{\delta W}{W} = 1 - 0.01 = 0.99 = \frac{r^2}{(r + \delta r)^2} = \frac{1}{(1 + \frac{\delta r}{r})^2}.$$

Hence, $1 + \frac{\delta r}{r} = 1/\sqrt{0.99}$ and $\delta r = (\frac{1}{\sqrt{0.99}} - 1)r$. If $r = 6370$ km, altitude $\delta r = 6370(\frac{1}{\sqrt{0.99}} - 1) = 32.09$ km.

4. It follows from Exercise 3 that $\frac{\delta W}{W} = 1 - \frac{1}{(1 + \frac{\delta r}{r})^2}$. With $\delta r = 3$ and $r = 6370$, $\frac{\delta W}{W} = 0.00094 = 0.094\%$.
5. The weight of the aeroplane acts vertically downwards, so the force may be represented by an arrow pointing downwards (Figure 1.18a). The thrust of the engines is a force in the direction in which the aeroplane is travelling (Figure 1.18b). The force from the air on the aeroplane has two components: lift upwards and drag backwards, so the two together may be represented by an upward sloping backwards (Figure 1.18c).

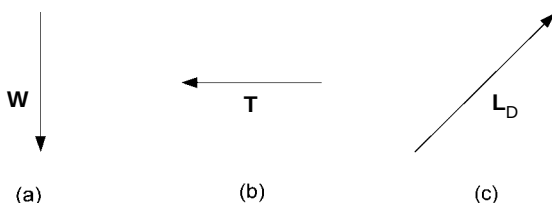


Figure 1.18. Forces on an aeroplane.

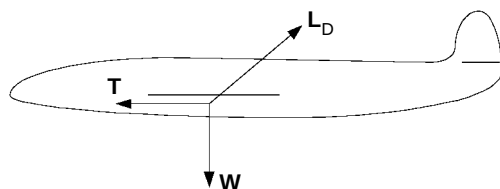


Figure 1.19. Forces shown acting on an aeroplane.

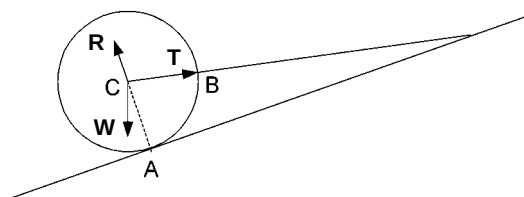


Figure 1.20. A smooth sphere held by a string on an inclined plane.

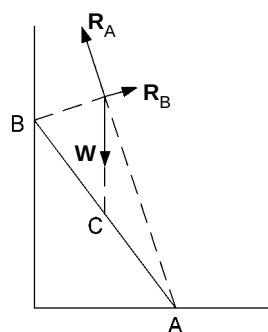


Figure 1.21. Forces acting on a ladder.

6. The weight vector $\mathbf{W}$ must act downwards through the centre of gravity somewhere in the middle of the aeroplane. The thrust vector $\mathbf{T}$ is level with the engines which are assumed in Figure 1.19 to be in pods under the wings. The lift/drag vector $\mathbf{L}_D$ must act through the point of intersection of $\mathbf{T}$ and $\mathbf{W}$ in order to avoid any turning effect.
7. The weight $\mathbf{W}$ acts through the centre of the sphere and so does the reaction $\mathbf{R}$ from the inclined plane, since the sphere is smooth and therefore the reaction is perpendicular to the plane. The tension $\mathbf{T}$ in the string must also apply a force acting through the centre of the sphere and hence may be represented as shown in Figure 1.20. Notice that, by the principle of transmissibility, we can draw in $\mathbf{R}$ and $\mathbf{T}$ as forces acting at the centre C even though the points of application are actually A and B , respectively.
8. The weight $\mathbf{W}$ acts vertically downwards through the centre C of the ladder. Assuming that there are frictional components of reaction, the lines of action of the corresponding total reactions $\mathbf{R}_A$ and $\mathbf{R}_B$ will be inclined in the manner shown in Figure 1.21. Notice, this time, that the forces act through a point outside the body, i.e. outside the ladder. We shall see later that this does not affect the usefulness of this procedure even though we can no longer appeal to the principle of transmissibility.

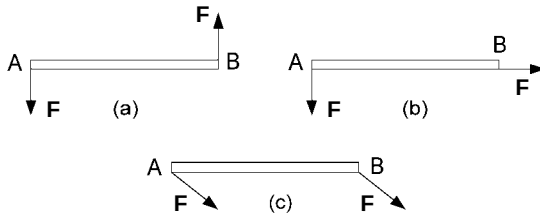


Figure 1.22. A rod acted on by two coplanar forces.



Figure 1.23. A rod in equilibrium under the action of two forces.

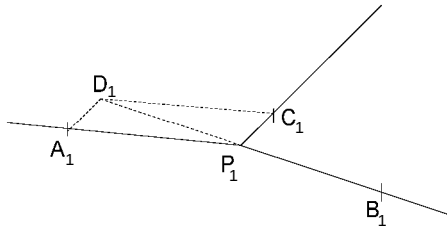


Figure 1.24. Testing the parallelogram law.

9. If the forces are equal and opposite and perpendicular to the rod (Figure 1.22a), the forces would start to rotate the rod. If the force at end A is as before, but that at end B is along its length (Figure 1.22b), the rod would start to both rotate and translate. If the forces both act in the same direction (Figure 1.22c), the rod would start to translate in that direction.

Obviously, there are many more possible examples but moving on to those which result in equilibrium, we remember that for this to exist, the two forces must not only be equal in magnitude but also opposite in direction and collinear. For the latter to be true, both forces must act along the length of the rod. Then to be opposite in direction as well, they must either both pull outwards (Figure 1.23a) or both push inwards (Figure 1.23b).

10. Draw three straight lines from a point P_1 in exactly the directions of the bands PA, PB and PC shown in Figure 1.9. Mark off distances from P_1 proportional to the band extensions and therefore to their tensions. Denote these distance marks A_1 , B_1 and C_1 , respectively, as shown in Figure 1.24. Complete the parallelogram on the sides P_1A_1 and P_1C_1 , and denote the fourth corner D_1 . If the parallelogram law for the resultant of two forces holds, the diagonal P_1D_1 should be collinear with and of equal length to P_1B_1 .

Note that the parallelogram law may also be tested by completing a parallelogram on the sides P_1A_1 and P_1B_1 or on P_1B_1 and P_1C_1 .

11. Referring to Figure 1.25, $F_{1x} = F_1 \cos \theta_1 = 1/2$. $F_{1y} = F_1 \sin \theta_1 = \sqrt{3}/2$. $F_{2x} = F_2 \cos \theta_2 = 2\sqrt{3}/2 = \sqrt{3}$. $F_{2y} = F_2 \sin \theta_2 = 2/2 = 1$.

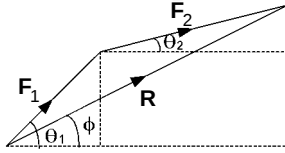


Figure 1.25. Resultant $\mathbf{R}$ of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at a point.

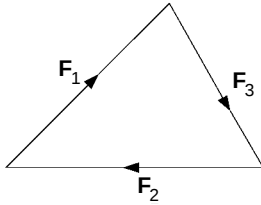


Figure 1.26. Triangle of forces.

$$\begin{aligned} \text{Thus } R_x &= F_{1x} + F_{2x} = \frac{1}{2} + \sqrt{3} = \frac{1+2\sqrt{3}}{2} \quad \text{and} \quad R_y = F_{1y} + F_{2y} = \frac{\sqrt{3}}{2} + 1 = \frac{\sqrt{3}+2}{2} \\ R^2 &= R_x^2 + R_y^2 = 5 + 2\sqrt{3}, \quad R = 2.91 \text{ N} \\ \tan \phi &= R_y/R_x = \frac{\sqrt{3}+2}{1+2\sqrt{3}}, \quad \phi = 39.9^\circ. \end{aligned}$$

12. (a) Calculate the Cartesian components of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ as follows. $F_{1x} = \sqrt{6} \cos 45^\circ = \sqrt{6}/\sqrt{2} = \sqrt{3}$, $F_{2x} = -(1 + \sqrt{3})$, $F_{3x} = 2 \cos(-60^\circ) = 1$, $F_{1y} = \sqrt{6} \sin 45^\circ = \sqrt{6}/\sqrt{2} = \sqrt{3}$, $F_{2y} = 0$, $F_{3y} = 2 \sin(-60^\circ) = 2(-\sqrt{3}/2) = -\sqrt{3}$. Then the x - and y -components of the resultant are:

$$\begin{aligned} R_x &= F_{1x} + F_{2x} + F_{3x} = \sqrt{3} - (1 + \sqrt{3}) + 1 = 0 \quad \text{and} \\ R_y &= F_{1y} + F_{2y} + F_{3y} = \sqrt{3} + 0 - \sqrt{3} = 0. \end{aligned}$$

Hence, the resultant $\mathbf{R} = 0$ and the forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are in equilibrium.

(b) Draw the vectors corresponding to $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ end-on-end as shown in Figure 1.26. Since they form the sides of a triangle, the forces must be in equilibrium. Note that the order in which the vectors are joined does not matter provided that all point the same way around the triangle, i.e. all clockwise or all anti-clockwise.

(c) Referring to Figure 1.27: $\frac{F_1}{\sin 120^\circ} = \frac{\sqrt{6}}{\sin 120^\circ} = 2.828$, $\frac{F_2}{\sin 105^\circ} = \frac{1+\sqrt{3}}{\sin 105^\circ} = 2.828$ and $\frac{F_3}{\sin 135^\circ} = \frac{2}{\sin 135^\circ} = 2.828$. The forces are in equilibrium since the magnitude of each is proportional to the sine of the angle between the other two.

13. (a) $F_{1x} = 2 \cos 150^\circ = 2(-\sqrt{3}/2) = -\sqrt{3}$, $F_{2x} = \sqrt{6} \cos 45^\circ = \sqrt{6}/\sqrt{2} = \sqrt{3}$, $F_{3x} = 2 \cos(-60^\circ) = 2/2 = 1$, $F_{4x} = \sqrt{2} \cos(-135^\circ) = \sqrt{2}(-1/\sqrt{2}) = -1$. Therefore, $R_x = F_{1x} + F_{2x} + F_{3x} + F_{4x} = -\sqrt{3} + \sqrt{3} + 1 - 1 = 0$.

$$\begin{aligned} F_{1y} &= 2 \sin 150^\circ = 2/2 = 1, \quad F_{2y} = \sqrt{6} \sin 45^\circ = \sqrt{6}/\sqrt{2} = \sqrt{3}, \quad F_{3y} = 2 \sin(-60^\circ) = 2(-\sqrt{3}/2) = -\sqrt{3}, \\ F_{4y} &= \sqrt{2} \sin(-135^\circ) = \sqrt{2}(-1/\sqrt{2}) = -1. \quad \text{Therefore, } R_y = F_{1y} + F_{2y} + F_{3y} + F_{4y} = 1 + \sqrt{3} - \sqrt{3} - 1 = 0. \end{aligned}$$

Since both R_x and R_y are zero, the resultant $\mathbf{R}$ is zero and therefore the four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ are in equilibrium.

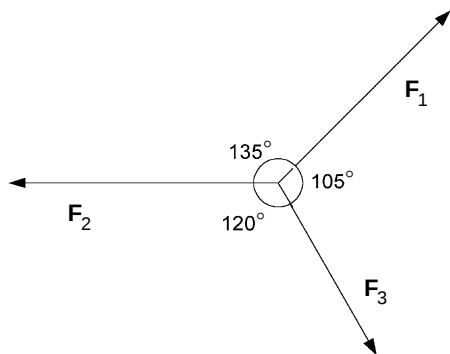


Figure 1.27. Three forces acting at a point.

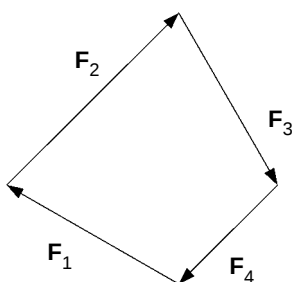


Figure 1.28. Tetragon of forces.

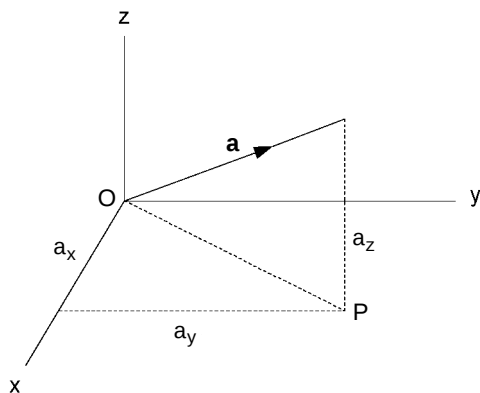


Figure 1.29. Vector **a** in three-dimensional space.

(b) Draw the vectors corresponding to F_1 , F_2 , F_3 and F_4 end-on-end as shown in Figure 1.28. Since they complete the sides of a tetragon (four-sided polygon), by the polygon of forces, the forces must be in equilibrium.

14. Before answering the specific problem, let us consider the properties of a vector **a** as shown supposedly in three-dimensional space in Figure 1.29. This has been simplified by localizing the vector **a** at the

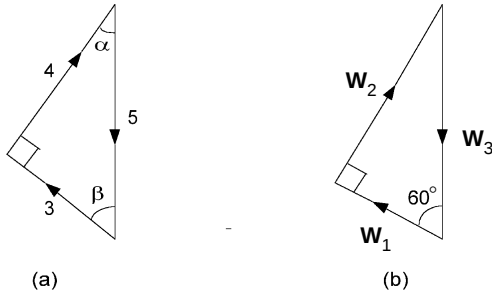


Figure 1.30. Triangles of tension forces acting at C.

origin of the three-dimensional Cartesian coordinate system. If P is the projection of the end of $\mathbf{a}$ onto the x, y plane, by Pythagoras' theorem, $a^2 = OP^2 + a_z^2 = a_x^2 + a_y^2 + a_z^2$. In other words, the square of the length of a vector equals the sum of the squares of its x -, y - and z -components. Also, $a_x = a \cos \theta_x$, $a_y = a \cos \theta_y$ and $a_z = a \cos \theta_z$, where θ_x , θ_y and θ_z are the angles which the vector $\mathbf{a}$ makes with the positive x -, y - and z -axes, respectively.

Returning to the specific problem, the resultant $\mathbf{R}$ of the three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ will have x -, y - and z -components as follows:

$$R_x = F_{1x} + F_{2x} + F_{3x} = 1 + 2 - 2 = 2$$

$$R_y = F_{1y} + F_{2y} + F_{3y} = -2 + 1 - 1 = -2$$

$$R_z = F_{1z} + F_{2z} + F_{3z} = -1 - 1 + 1 = -1.$$

Therefore, $\mathbf{R} = (2, -2, -1)$.

$$R^2 = R_x^2 + R_y^2 + R_z^2 = 4 + 4 + 1 = 9. \text{ Hence, } R = 3 \text{ N.}$$

$$R_x = R \cos \theta_x, \theta_x = \cos^{-1}(2/3) = 48.2^\circ$$

$$R_y = R \cos \theta_y, \theta_y = \cos^{-1}(-2/3) = 131.8^\circ$$

$$R_z = R \cos \theta_z, \theta_z = \cos^{-1}(-1/3) = 109.5^\circ.$$

15. (a) The three tensions acting at C are in equilibrium so they must obey the triangle of forces as shown in Figure 1.30a. W_3 is vertically downwards and since $3^2 + 4^2 = 5^2$, the angle between the W_1 and W_2 tensions is a right-angle. Comparing Figure 1.16 with Figure 1.30a, $\angle BAC = \alpha$ and $\angle ABC = \beta$. Therefore, the triangle of forces is similar to $\triangle ABC$ which in turn is similar to $\triangle ACD$. Since $AB = 0.5 \text{ m}$, $AC = 0.5(4/5) = 0.4 \text{ m}$. Then, $a = (4/5)AC = 0.32 \text{ m}$ and $d = (3/5)AC = 0.24 \text{ m}$.

(b) In this case, the triangle of forces is as shown in Figure 1.30b. Remembering that a 30° right-angled triangle has sides of length proportional to $1 : \sqrt{3} : 2$, we see that $W_3 = (2/1)W_1 = 20 \text{ N}$ and $W_2 = (\sqrt{3}/1)W_1 = 10\sqrt{3} \text{ N}$.

16. The diameter of the cylinder is $3r$ and the diameter of each sphere is $2r$. Thus the length of the line joining the centres of the spheres is $2r$ and the horizontal displacement between the centres is r , as shown in Figure 1.31a. Consequently, the line of centres makes an angle of 30° with the vertical, since $\sin 30^\circ = 1/2$.

We can now draw the triangle of forces for the three forces acting on the upper sphere, as in Figure 1.31b. From this we see that $R_b = W \tan 30^\circ = W/\sqrt{3}$. Also, $R_p = W \sec 30^\circ = 2W/\sqrt{3}$.

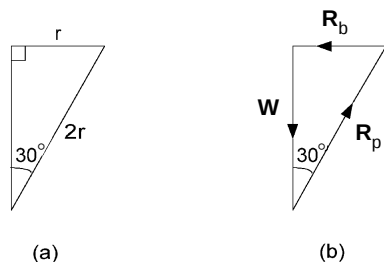


Figure 1.31. Triangle of forces acting on upper sphere.

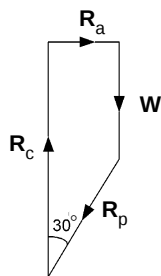


Figure 1.32. Tetragon of forces acting on lower sphere.

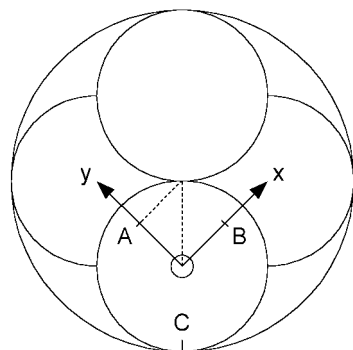


Figure 1.33. Top view of the four spheres in the cylinder.

Next we draw the polygon of forces for the four forces acting on the lower sphere (see Figure 1.32). From this, we see that $R_a = R_p \sin 30^\circ = 2W/2\sqrt{3} = W/\sqrt{3}$. Also, $R_c = W + R_p \cos 30^\circ = W + \frac{2}{\sqrt{3}}W \frac{\sqrt{3}}{2} = 2W$.

17. Figure 1.33 shows the top view of the four spheres in the cylinder. Consider one of the top spheres and draw in x - and y -axes as shown with the origin at the centre of the sphere. The z -axis will be at right-angles vertically upwards. The points of contact with the bottom spheres will be on the lines of centres below the points A and B. The top view diagram (Figure 1.33) shows that $AO = BO = r/\sqrt{2}$.

If P is the point of contact below A, we see from Figure 1.34 of the APO triangle in the y, z plane that PO and therefore the direction of the reaction force at P is at 45° to the vertical, PO being the sphere radius r .

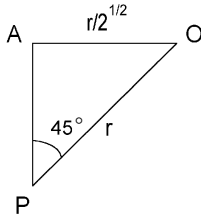


Figure 1.34. Triangle APO in the vertical y, z plane.

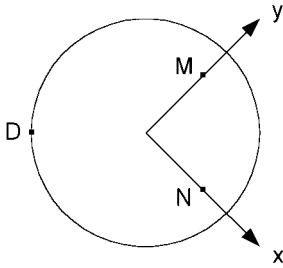


Figure 1.35. Horizontal x - and y -axes with origin at centre of left lower sphere.

We now deduce that the reaction force from P in the direction of O can be written in terms of its Cartesian components as $\mathbf{R}_p = (0, -R/\sqrt{2}, R/\sqrt{2})$, where R is its magnitude. We now have another reaction force at the point of contact Q below B given by $\mathbf{R}_q = (-R/\sqrt{2}, 0, R/\sqrt{2})$.

Since the top two spheres try to move down and out, we can assume that there will be no reaction force between the two. However, there will be one from the wall of the cylinder directed towards O which can be written as $\mathbf{R}_c = (R_c/\sqrt{2}, R_c/\sqrt{2}, 0)$, where R_c is the magnitude and its x - and y -components are at 45° to the direction of $\mathbf{R}_c$. Finally, the weight of the sphere can be written as the force $\mathbf{W} = (0, 0, -W)$.

Hence, the sphere is kept in equilibrium by the four forces $\mathbf{R}_p$, $\mathbf{R}_q$, $\mathbf{R}_c$ and $\mathbf{W}$ all acting through its centre O. For equilibrium, the sum of the x -components must be zero, the sum of the y -components must be zero and the sum of the z -components must be zero. Thus, $0 - R/\sqrt{2} + R_c/\sqrt{2} + 0 = 0$ and $-R/\sqrt{2} + 0 + R_c/\sqrt{2} + 0 = 0$, each of which implies that $R = R_c$, and $R/\sqrt{2} + R/\sqrt{2} + 0 - W = 0$, i.e. $R = W/\sqrt{2}$.

Now consider the bottom two spheres. The top two try to push them apart, so we can assume that there is no reactive force between the bottom two spheres. Again, because of symmetry, we only need to study one of the spheres. Let us take the one on the left and draw in x - and y -axes with origin at the centre as shown in Figure 1.35. The z -axis will again be vertically upwards. The point D is the point of contact with the cylinder and a reactive force will act on the sphere at D towards its centre. This force may be written as $\mathbf{R}_d = (R_d/\sqrt{2}, R_d/\sqrt{2}, 0)$. The points of contact with the upper spheres are above M and N in Figure 1.35 and the corresponding downward sloping forces acting on our bottom sphere can be written as $\mathbf{R}_m = (0, -R/\sqrt{2}, -R/\sqrt{2})$ and $\mathbf{R}_n = (-R/\sqrt{2}, 0, -R/\sqrt{2})$. We have already shown that $R = W/\sqrt{2}$, so $R/\sqrt{2} = W/2$.

Besides these three forces, we have the weight of the sphere $\mathbf{W} = (0, 0, -W)$ and an upward reaction force through the base of the sphere given by $\mathbf{R}_b = (0, 0, R_b)$.

There are thus five forces acting on the sphere through its centre. For equilibrium we can in turn equate to zero the sum of the x -components, the sum of the y -components and the sum of the

z-components. Hence, $R_d/\sqrt{2} + 0 - W/2 + 0 + 0 = 0$ and $R_d/\sqrt{2} - W/2 + 0 + 0 + 0 = 0$, each of which gives $R_d = W/\sqrt{2}$, and $0 - W/2 - W/2 - W + R_b = 0$, i.e. $R_b = 2W$.

To summarize the results: (1) the force between each top sphere and the cylinder is $W/\sqrt{2}$; (2) the forces at the points of contact between upper and lower spheres are each equal to $W/\sqrt{2}$; (3) the force between each bottom sphere and the cylinder is $W/\sqrt{2}$; (4) the force between each bottom sphere and the base of the cylinder is $2W$.