Part I

Statics

1 Forces

1.1 Force

In the study of statics we are concerned with two fundamental quantities: length or distance, which requires no explanation, and force. The quantity length can be seen with the eye but with force, the only thing that is ever seen is its effect. We can see a spring being stretched or a rubber ball being squashed but what is seen is only the effect of a force being applied and not the force itself. With a rigid body there is no distortion due to the force and in statics it does not move either. Hence, there is no visual indication of forces being applied.

We detect a force being applied to our human body by our sense of touch or feel. Again, it is not the force itself but its effect which is felt – we feel the movement of our stomachs when we go over a humpback bridge in a fast car; we feel that the soles of our feet are squashed slightly when we stand.

We have now encountered one of the fundamental conceptual difficulties in the study of mechanics. Force cannot be seen or measured directly but must always be imagined. Generally the existence of some force requires little imagination but to imagine all the different forces which exist in a given situation may not be too easy. Furthermore, in order to perform any analysis, the forces must be defined precisely in mathematical terms.

For the moment we shall content ourselves with a qualitative definition of force. 'A force is that quantity which tries to move the object on which it acts.' This qualitative definition will suffice for statical problems in which the object does not move but we shall have to give it further consideration when we study the subject of dynamics. If the object does not move, the force must be opposed and balanced by another force. If we push with our hand against a wall, we know that we are exerting a force; we also know that the wall would be pushed over if it were not so strong. By saying that the wall is strong we mean that the wall itself can produce a force to balance the one applied by us.

EXERCISE 1

Note down a few different forces and state whether, and if so how, they might be observed.

1.2 Forces of contact

Before giving a precise mathematical description of force, we shall discuss two general categories. We shall start with the type which is more easily imagined; this is that due to contact between one object and another.

In the example of pushing against a wall with one's hand, the wall and hand are in contact, a force is exerted by the hand on the wall and this is opposed by another force from the wall to the hand. In the same way, when we are standing on the ground we can feel the force of the ground on our feet in opposition to the force due to our weight transmitted through our feet to the ground. Sometimes we think of a force being a pull but if we analyse the situation, the force of contact from one object to another is still a push. For instance, suppose a rope is tied around an object so that the latter may be pulled along. When this happens, the force from the rope which moves the object is a push on the rear of the object.

Another form of contact force is that which occurs when a moving object strikes another one. Any player of ball games will be familiar with this type of force. It only acts for a short time and is called an impulsive force. It is given special consideration in dynamics but it also occurs in statics in the following sense. When the surface of an object is in contact with a gas, the gas exerts a pressure, that is a force spread over the surface. The pressure is caused by the individual particles of the gas bouncing against the surface and exerting impulsive forces. The magnitude of each force is so small but the frequency of occurrence is so high that the effect is that of a force continuously distributed over the whole surface.

Forces of contact need not be exerted normal to the surface of contact. It is also possible to exert what is called a tangential or frictional component of force. In this case the force is applied obliquely to the surface; we can think of part being applied normally, i.e. perpendicular to the surface, and part tangentially. The maximum proportion of the tangential part which may be applied depends upon the nature of the surfaces in contact. Ice skaters know how small the tangential component can be and manufacturers of motor car tyres know how high.

A fact which must be emphasized concerning contact forces is that the forces each way are always equal and opposite, i.e. action and reaction are equal and opposite. When you push against the wall with your hand, the force from your hand on the wall is equal and opposite to the force from the wall against your hand. The rule is true for any pair of contact forces.

EXERCISE 2

Note down some of the contact forces which you have experienced or which have been applied to objects with which you have been concerned during the day. For each contact force, note the equal and opposite force which opposed it.

1.3 Mysterious forces

It is not too difficult to imagine the contact forces already described from our everyday experience but what is it that prevents a solid object from bending, squashing or just falling apart under the action of such forces? Mysterious forces of attraction act between the separate molecules of the material binding them together in a particular way and resisting outside forces which try to disturb the pattern. These *intermolecular forces* constitute the strength of the material. Although we shall not be concerned with it here, knowledge of the strength of materials is of great importance to engineers when designing buildings, machinery, etc.

Another mysterious force which will concern us deeply is the *force of gravity*. The magnitude of the force of gravity acting on a particular object depends on the size and physical nature of the object. In our study this force will remain constant and it will always act vertically downwards, this being referred to as the *weight* of the object. This is sufficient for most earthbound problems but when studying artificial satellites and space-craft it is necessary to consider the full properties of gravity.

Gravity is a force of attraction between any two bodies. It needs no material for its transmission nor is it impeded or changed in any way by material placed in between the bodies in question. The magnitude of the force was given mathematical form by Sir Isaac Newton and published in his *Philosophiae Naturalis Principia Mathematica* in 1687. The force is proportional to the product of the masses of the two bodies divided by the square of the distance between them. We shall say more about mass when studying dynamics but it is a constant property of any body. The law of gravitation, i.e. the inverse square law, was deduced by correlating it with the elliptical motion of planets about the sun as focus. Newton proved that such motion would be produced by the inverse square law of attraction to the sun acting on each planet.

Given that a body generates an attractive force proportional to its mass, it is reasonable that an inverse square law with respect to distance should apply. The force acts in towards the body from all directions around. However, the force acts over a larger area as the distance from the body increases. Since the effort is spread out over a larger area we can expect the strength at any particular point in the area to decrease accordingly. Thus we expect the magnitude of the force to be inversely proportional to the area of the sphere with the body at the centre and the point at which the force acts being on the surface of the sphere. The area is proportional to the square of the radius of the sphere; hence the inverse square law follows.

Magnetic and electrostatic forces are also important mysterious forces. However, they will not be dicussed here since we shall not be concerned with them in this text.

EXERCISE 3

Find the altitude at which the weight of a body is one per cent less than its weight at sea level. (Assume that the radius of the earth from sea level is 6370 km.)

EXERCISE 4

Find the percentage reduction in weight when the body is lifted from sea level to a height of 3 km.

1.4 Quantitative definition of force

In statics, force is that quantity which tries to move the object on which it acts. The magnitude of a force is the measure of its strength. It is then necessary to define basic units of measurement.

In lifting different objects we are very familiar with the concept of weight, which is the downward gravitational force on an object. It is tempting to use the weight of a particular object as the unit of force. However, weight varies with altitude (see Exercises 3 and 4) and also with latitude. To avoid this, a dynamical unit of force has been adopted.

The basic SI unit (Système International d'Unités) is the newton (symbol N). It is the force which would give a mass of one kilogramme (1 kg) an acceleration of one metre per second per second (1 m/s^2 or 1 m s^{-2}). The kilogramme is the mass of a particular piece of platinum–iridium. Of course, once a standard has been set, other masses can easily be evaluated by comparing relative weights. Incidentally, the mass of 1 kg is approximately the mass of one cubic decimetre of distilled water at the temperature (3.98°C) at which its density is maximum.

If you are more familiar with the pound-force (lbf) as the unit of force, then

$1 \text{ lbf} = 4.449 \text{ N}$ or $1 \text{ N} = 0.2248 \text{ lbf}$.

In quantifying a force, not only must its magnitude be given but also its direction of application, i.e. the direction in which it tries to move the object on which it acts. Having both magnitude and direction, force is a *vector* quantity. Sometimes it is convenient to represent a force graphically by an arrow (see Figure 1.1) which points in a direction corresponding to the direction of the force and has a length proportional to the magnitude of the force.

EXERCISE 5

Consider an aeroplane (see Figure 1.2) flying along at constant speed and height. Since there is no acceleration, forces should balance out in the same way that they do in statics. Draw vectors which might correspond to (a) the weight of the aeroplane, (b) the thrust from its engines and (c) the force from the surrounding air on the aeroplane which is a combination of lift and drag (lift/drag).

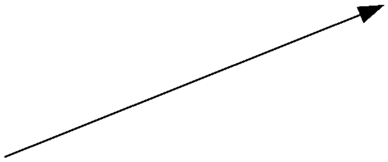

Figure 1.1. Force vector.

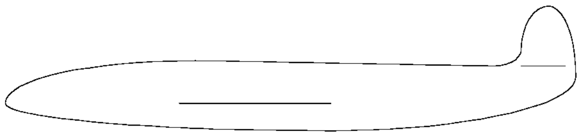

Figure 1.2. Simple sketch of an aeroplane.

1.5 Point of application

In studying forces acting on a rigid body, it is necessary to know the points of the body to which the forces are applied. For instance, consider a horizontal force applied to a stone which is resting on horizontal ground. If the force is strong enough the stone will move, but whether it moves by toppling or slipping depends on where the force is applied.

Forces rarely act at a single point of a body. Usually the force is spread out over a surface or volume. If the stone mentioned above is pushed with your hand, then the force from your hand is spread out over the surface of contact between your hand and the stone. The force from the ground which is acting on the stone is spread out over the surface of contact with the ground. The gravitational force acting on the stone is spread out over the whole volume of the stone. In order to perform the analysis in minute detail it would be necessary to consider each small force acting on each small element of area and on each small element of volume. However, since we are only considering rigid bodies, we are not concerned with internal stress. Thus we can replace many small forces by one large force. In our example, the small forces from the small elements of area of contact of your hand are represented by a single large force acting on the stone. Similarly, we have a single large force acting from the ground. Also, for the small gravitational forces acting on all the small elements of volume of the stone, we have instead a single force equal to the weight of the stone acting at a point in the stone which is called the *centre of gravity*.

The derivation of the points of action of these equivalent resultant forces will be discussed later. For the time being we shall assume that the representation is valid so that we can study the example of the stone as though there were only three forces acting on it, one from your hand, one from the ground and one from gravity.

EXERCISE 6

Continue Exercise 5 by drawing in the three force vectors on a rough sketch of the aeroplane.

1.6 Line of action

In the answer to Exercise 6, it appears that the three resultant forces of weight, thrust and lift/drag all act at the same point. Of course this may not be so but it does not matter provided the lines of action of the three resultant forces intersect at one point. For instance, this point need not coincide with the centre of gravity but it must be in the same vertical line as the centre of gravity.

Thus, with a rigid body the effect of a force is the same for any point of application along its line of action. This property is referred to as the *principle of transmissibility*. If two non-parallel but coplanar forces act on a body, it is convenient to imagine them to be acting at the point of intersection of their lines of action.

EXERCISE 7

Suppose that a smooth sphere is held on an inclined plane by a string which is fastened to a point on the surface of the sphere at one end and to a point on the plane at the other end. Sketch the side view and draw in the force vectors at the points of intersection of their lines of action.

EXERCISE 8

Do the same as in Exercise 7 for a ladder leaning against a wall, assuming that the lines of action of the three forces (weight and reactions from wall and ground) are concurrent.

Problems 1 and 2.

1.7 Equilibrium of two forces

A force tries to move its point of application and it will move it unless there is an equal and opposite counterbalancing force. When you push a wall with your hand, the wall will move unless it is strong enough to produce an equal and opposite force on your hand. If you are holding a dog with a lead, you will only remain stationary if you pull on the lead with the same amount of force as that exerted by the dog. By considering such physical examples we can see that for two forces to balance each other, they must be equal in magnitude and opposite in direction.

Yet another property is also required for the balance to exist. Suppose we have a large wheel mounted on a vertical axle. If one person pushes the wheel tangentially along the rim on one side and another person pushes on the other side, the wheel will start to move if the two pushes are equal in magnitude and opposite in direction. In fact two forces only balance each other if not only are they equal in magnitude and opposite in direction but also have the same line of action. When you are holding the dog, the line of the lead is the line of action of both the force from your hand and the force from the dog.

When the three conditions hold, we say that the two forces are in *equilibrium*. If a rigid body is acted on by only two such forces, the body will not move and we say that the body is in equilibrium. When a stone rests in equilibrium on the ground, the resultant contact force from the ground is equal, opposite and collinear to the resultant gravitational force acting on the stone.

EXERCISE 9

Suppose that a rigid straight rod rests on its side on a smooth horizontal surface. Let two horizontal forces of equal magnitude be applied to the rod simultaneously, one at either end. Consider what will happen to the rod immediately after the forces have been applied for a few different situations regarding the directions in which the separate forces are applied. Show that there will be only two possible situations in which the rod will remain in equilibrium.

1.8 Parallelogram of forces (vector addition)

If two non-parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act at a point A, they have a combined effect equivalent to a single force $\mathbf{R}$ acting at A. The single force $\mathbf{R}$ is called the *resultant* and it may be found as follows. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be represented in magnitude and direction by two sides of a parallelogram meeting at A. Then $\mathbf{R}$ is represented in magnitude and direction by the diagonal of the parallelogram from A, as shown in Figure 1.3. This is an empirical result referred to as the parallelogram law.

The parallelogram law may be illustrated by the following experiment. Take three different known weights of magnitudes W_1, W_2 and W_3, and attach W_1 and W_2 to either end of a length of string. Drape the string over two smooth pegs set a distance apart at about the same height. Then attach W_3 with a small piece of string to a point A of the other string between the two pegs. Finally, allow W_3 to drop gently and possibly move sideways until an equilibrium position is established (see Figure 1.4).

Now measure the angles to the horizontal made by the sections of string between the two pegs and A. Make an accurate drawing of the strings which meet at A and mark off distances proportional to W_1 and W_2 as shown in Figure 1.5. Since the pegs are smooth,

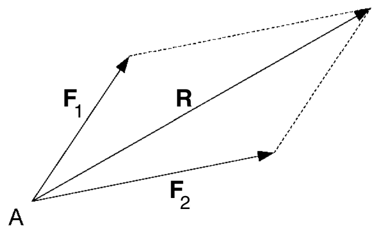

Figure 1.3. Parallelogram of forces.

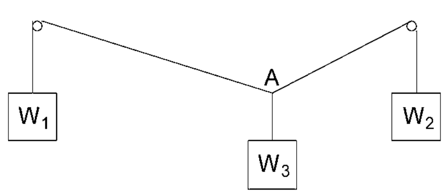

Figure 1.4. String over two smooth pegs.

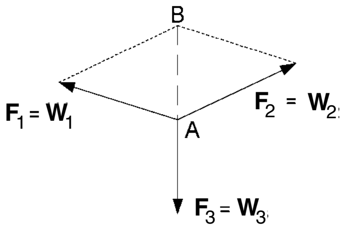

Figure 1.5. Three forces acting at A.

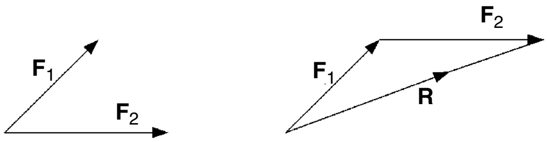

Figure 1.6. Vector addition.

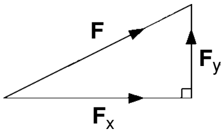

Figure 1.7. Cartesian components $\mathbf{F}_x$ and $\mathbf{F}_y$ of vector $\mathbf{F}$.

the tensions in the string of magnitudes F_1 and F_2 must be equal to the weights W_1 and W_2, respectively. Complete the parallelogram on the sides F_1 and F_2 and let B be the corner opposite A.

Since the point A is in equilibrium, the resultant of F_1 and F_2 should be equal, opposite and collinear to F_3 which is the tension in the string supporting W_3 with $F_3 = W_3$. If the parallelogram law holds, then AB should be collinear with the line corresponding to the vertical string and the length AB should correspond to the weight W_3.

The parallelogram law also applies to the *vector sum* of two vectors. Hence, the resultant of two forces acting at a point is their vector sum. Thus, if we use boldface letters to indicate vector quantities, the resultant $\mathbf{R}$ of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at a point may be written as $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$.

Also, since opposite sides of a parallelogram are equal, $\mathbf{R}$ may be found by drawing $\mathbf{F}_2$ onto the end of $\mathbf{F}_1$ and joining the start of $\mathbf{F}_1$ to the end of $\mathbf{F}_2$ as shown in Figure 1.6.

A force vector $\mathbf{F}$ may also be written in terms of its *Cartesian components* $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$ as shown in Figure 1.7.

Similarly, if we want the resultant $\mathbf{R}$ of two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at a point, then

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_{1x} + \mathbf{F}_{1y} + \mathbf{F}_{2x} + \mathbf{F}_{2y} = (\mathbf{F}_{1x} + \mathbf{F}_{2x}) + (\mathbf{F}_{1y} + \mathbf{F}_{2y}) = \mathbf{R}_x + \mathbf{R}_y.$$

In other words, the x-component of $\mathbf{R}$ is the sum of the x-components of $\mathbf{F}_1$ and $\mathbf{F}_2$ and the y-component of $\mathbf{R}$ is the sum of the y-components of $\mathbf{F}_1$ and $\mathbf{F}_2$. This

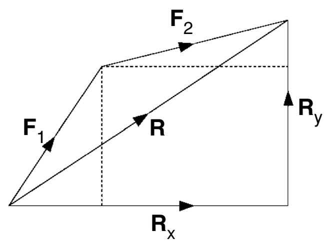

Figure 1.8. Addition of Cartesian components in vector addition.

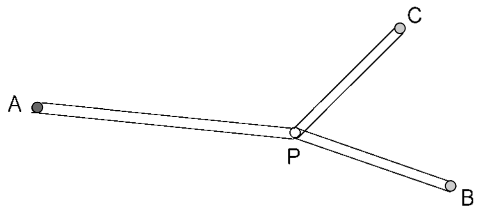

Figure 1.9. Three elastic bands used to demonstrate the parallelogram law.

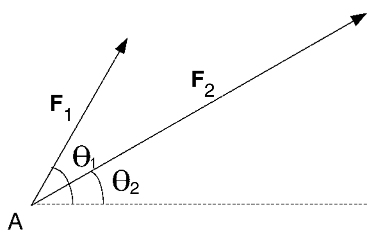

Figure 1.10. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at a point A.

can be seen diagramatically by drawing in the Cartesian components as illustrated in Figure 1.8.

EXERCISE 10

Use a piece of cotton thread to tie together three identical elastic bands. Having measured the unstretched length of the bands, peg them out as indicated in Figure 1.9, so that each band is in a stretched state but not beyond the elastic limit. The points A, B and C represent the fixed positions of the pegs but the point P takes up its equilibrium position pulled in three directions by the tensions in the bands. Use the fact that tension in each band is proportional to extension in order to verify the parallelogram law for the resultant of two forces acting at a point.

EXERCISE 11

Calculate the magnitude and direction of the resultant $\mathbf{R}$ of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting at the point A given the magnitudes $F_1 = 1\,\mathrm{N}$, $F_2 = 2\,\mathrm{N}$ and directions $\theta_1 = 60^\circ$, $\theta_2 = 30^\circ$ (see Figure 1.10).

Problems 3 and 4.