From the stripes of a zebra and the spots on a leopard’s back to the ripples on a sandy beach or desert dune, regular patterns arise in nature everywhere. The appearance and evolution of these phenomena have been a focus of recent research activity across several disciplines.

This book provides an introduction to the range of mathematical theory and methods used to analyse and explain these often intricate and beautiful patterns. Bringing together several different approaches, from group theoretic methods to envelope equations and the theory of patterns in large-aspect-ratio systems, the book provides insight behind the selection of one pattern over another.

Suitable as an upper-undergraduate textbook for mathematics students or as a fascinating, engaging, and fully illustrated resource for readers in physics and biology, Dr Hoyle’s book, using a nonpartisan approach, unifies a multiplicity of techniques used by active researchers in this growing field.

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Preface

Regular patterns are found in abundance in nature, from the spots on a leopard’s back to the ripples on a sandy beach or desert dune. There has been a flurry of recent research activity seeking to explain their appearance and evolution, and the selection of one pattern over another has turned out to be an inherently nonlinear phenomenon. My aim in writing this book has been to provide an introduction to the range of methods used to analyse natural patterns, at a level suitable for final year undergraduates and beginning graduate students in UK universities.

The book brings together several different approaches used in describing pattern formation, from group theoretic methods to envelope equations and the theory of patterns in large-aspect-ratio systems. The emphasis is on using symmetries to describe universal classes of pattern rather than restricting attention to physical systems with well-known governing equations, though connections with particular systems are also explored. I have taken a wholeheartedly nonpartisan approach, unifying for perhaps the first time in a textbook a multiplicity of methods used by active researchers in the field.

It was David Crighton who originally suggested I should write this book. I had been lecturing a Cambridge Part III course on pattern formation, and David mentioned in passing that it might be a nice idea to turn my lecture notes into a book. Of course I had no idea what I was letting myself in for, but David was always persuasive and inspirational so naturally I said yes. Several years of sweat and toil later I have finally produced the book, though it bears little resemblance to my Part III course, which is probably just as well. I am only sad that David is no longer here to see the result: he inspired and encouraged so many people, particularly those at the beginning of their careers, and he is sorely missed. Like so many others, I owe him a great debt.

Mike Proctor, my Ph.D. supervisor, first introduced me to pattern formation. Later, it was Mike who gave me the opportunity to lecture the course that led to this book. I am very grateful to him for all his support and encouragement over
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the years. I particularly enjoy and admire the great enthusiasm with which he approaches any problem, whether it be a tricky calculation or just a matter of finding the right pub.

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