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Introduction to tensors and dyadics

1.1 Introduction

Tensors play a fundamental role in theoretical physics. The reason for this is that physical laws written in tensor form are independent of the coordinate system used (Morse and Feshbach, 1953). Before elaborating on this point, consider a simple example, based on Segel (1977). Newton's second law is $\mathbf{f} = m\mathbf{a}$, where $\mathbf{f}$ and $\mathbf{a}$ are vectors representing the force and acceleration of an object of mass m. This basic law does not have a coordinate system attached to it. To apply the law in a particular situation it will be convenient to select a coordinate system that simplifies the mathematics, but there is no question that any other system will be equally acceptable. Now consider an example from elasticity, discussed in Chapter 3. The stress vector $\mathbf{T}$ (force/area) across a surface element in an elastic solid is related to the vector $\mathbf{n}$ normal to the same surface via the stress tensor. The derivation of this relation is carried out using a tetrahedron with faces along the three coordinate planes in a Cartesian coordinate system. Therefore, it is reasonable to ask whether the same result would have been obtained if a different Cartesian coordinate system had been used, or if a spherical, or cylindrical, or any other curvilinear system, had been used. Take another example. The elastic wave equation will be derived in a Cartesian coordinate system. As discussed in Chapter 4, two equations will be found, one in component form and one in vector form in terms of a combination of gradient, divergence, and curl. Again, here there are some pertinent questions regarding coordinate systems. For example, can either of the two equations be applied in non-Cartesian coordinate systems? The reader may already know that only the latter equation is generally applicable, but may not be aware that there is a mathematical justification for that fact, namely, that the gradient, divergence, and curl are independent of the coordinate system (Morse and Feshbach, 1953). These questions are generally not discussed in introductory texts, with the consequence that the reader fails to grasp the deeper meaning of the concepts of vectors and tensors. It is only when one realizes that physical entities (such as force, acceleration, stress tensor, and so on) and the relations among them have an existence independent of coordinate systems, that it is possible to appreciate that there is more to tensors than what is usually discussed. It is possible, however, to go through the basic principles of stress and strain without getting into the details of tensor analysis. Therefore, some parts of this chapter are not essential for the rest of the book.

Tensor analysis, in its broadest sense, is concerned with arbitrary curvilinear coordinates. A more restricted approach concentrates on *orthogonal curvilinear coordinates*, such as cylindrical and spherical coordinates. These coordinate systems have the property that the unit vectors at a given point in space are perpendicular (i.e. orthogonal) to each other. Finally, we have the rectangular Cartesian system, which is also orthogonal. The main difference between general orthogonal and Cartesian systems is that in the latter the unit vectors do not change as a function of position, while this is not true in the former. Unit vectors for the spherical system will be given in §9.9.1. The theory of tensors in non-Cartesian systems is exceedingly complicated, and for this reason we will limit our study to Cartesian tensors. However, some of the most important relations will be written using dyadics (see §1.6), which provide a symbolic representation of tensors independent of the coordinate system. It may be useful to note that there are oblique Cartesian coordinate systems of importance in crystallography, for example, but in the following we will consider the rectangular Cartesian systems only.

1.2 Summary of vector analysis

It is assumed that the reader is familiar with the material summarized in this section (see, e.g., Lass, 1950; Davis and Snider, 1991).

A vector is defined as a directed line segment, having both magnitude and direction. The magnitude, or length, of a vector $\mathbf{a}$ will be represented by $|\mathbf{a}|$. The sum and the difference of two vectors, and the multiplication of a vector by a scalar (real number) are defined using geometric rules. Given two vectors $\mathbf{a}$ and $\mathbf{b}$, two products between them have been defined.

Scalar, or dot, product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \alpha, \tag{1.2.1}$$

where α is the angle between the vectors.

Vector, or cross, product:

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin \alpha)\, \mathbf{n}, \tag{1.2.2}$$

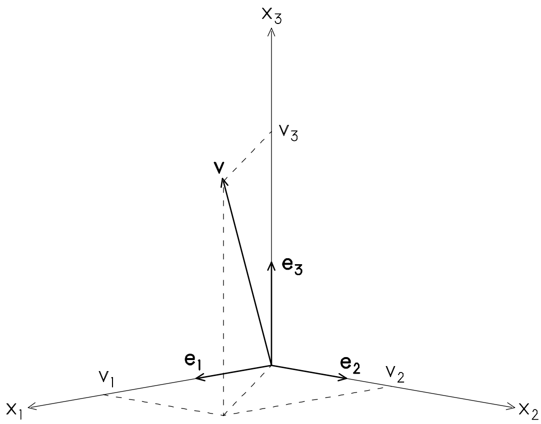

Fig. 1.1. Rectangular Cartesian coordinate system with unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, and decomposition of an arbitrary vector $\mathbf{v}$ into components v_1, v_2, v_3.

where α is as before, and $\mathbf{n}$ is a unit vector (its length is equal to 1) perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that the three vectors form a right-handed system.

An important property of the vector product, derived using geometric arguments, is the distributive law

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}. \tag{1.2.3}$$

By introducing a rectangular Cartesian coordinate system it is possible to write a vector in terms of three components. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ be the three unit vectors along the x_1, x_2, and x_3 axes of Fig. 1.1. Then any vector $\mathbf{v}$ can be written as

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = \sum_{i=1}^{3} v_i\mathbf{e}_i. \tag{1.2.4}$$

The components v_1, v_2, and v_3 are the orthogonal projections of $\mathbf{v}$ in the directions of the three axes (Fig. 1.1).

Before proceeding, a few words concerning the notation are necessary. A vector will be denoted by a bold-face letter, while its components will be denoted by the same letter in italics with subindices (literal or numerical). A bold-face letter with a subindex represents a vector, not a vector component. The three unit vectors defined above are examples of the latter. If we want to write the kth component of the unit vector $\mathbf{e}_j$ we will write $(\mathbf{e}_j)_k$. For example, $(\mathbf{e}_2)_1 = 0$, $(\mathbf{e}_2)_2 = 1$, and $(\mathbf{e}_2)_3 = 0$. In addition, although vectors will usually be written in row form (e.g., as in (1.2.4)), when they are involved in matrix operations they should be considered as column vectors, i.e., as matrices of one column and three rows. For example, the matrix form of the scalar product $\mathbf{a} \cdot \mathbf{b}$ is $\mathbf{a}^{\mathrm{T}}\mathbf{b}$, where T indicates transposition.

When the scalar product is applied to the unit vectors we find

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0 \tag{1.2.5}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{1.2.6}$$

Equations (1.2.5) and (1.2.6) can be summarized as follows:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases} \tag{1.2.7}$$

The symbol δ_{jk} is known as the *Kronecker delta*, which is an example of a second-order tensor, and will play an important role in this book. As an example of (1.2.7), $\mathbf{e}_2 \cdot \mathbf{e}_k$ is zero unless $k = 2$, in which case the scalar product is equal to 1.

Next we derive an alternative expression for a vector $\mathbf{v}$. Using (1.2.4), the scalar product of $\mathbf{v}$ and $\mathbf{e}_i$ is

$$\mathbf{v} \cdot \mathbf{e}_i = \left(\sum_{k=1}^{3} v_k \mathbf{e}_k \right) \cdot \mathbf{e}_i = \sum_{k=1}^{3} v_k \mathbf{e}_k \cdot \mathbf{e}_i = \sum_{k=1}^{3} v_k (\mathbf{e}_k \cdot \mathbf{e}_i) = v_i . \tag{1.2.8}$$

Note that when applying (1.2.4) the subindex in the summation must be different from i. To obtain (1.2.8) the following were used: the distributive law of the scalar product, the law of the product by a scalar, and (1.2.7). Equation (1.2.8) shows that the ith component of $\mathbf{v}$ can be written as

$$v_i = \mathbf{v} \cdot \mathbf{e}_i . \tag{1.2.9}$$

When (1.2.9) is introduced in (1.2.4) we find

$$\mathbf{v} = \sum_{i=1}^{3} (\mathbf{v} \cdot \mathbf{e}_i) \mathbf{e}_i . \tag{1.2.10}$$

This expression will be used in the discussion of dyadics (see §1.6).

In terms of its components the length of the vector is given by

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = (\mathbf{v} \cdot \mathbf{v})^{1/2}. \tag{1.2.11}$$

Using purely geometric arguments it is found that the scalar and vector products can be written in component form as follows:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{1.2.12}$$

and

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\mathbf{e}_1 + (u_3 v_1 - u_1 v_3)\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\mathbf{e}_3. \tag{1.2.13}$$

The last expression is based on the use of (1.2.3).

Vectors, and vector operations such as the scalar and vector products, among others, are defined independently of any coordinate system. Vector relations derived without recourse to vector components will be valid when written in component form regardless of the coordinate system used. Of course, the same vector may (and generally will) have different components in different coordinate systems, but they will represent the same geometric entity. This is true for Cartesian and more general coordinate systems, such as spherical and cylindrical ones, but in the following we will consider the former only.

Now suppose that we want to define new vector entities based on operations on the components of other vectors. In view of the comments in §1.1 it is reasonable to expect that not every arbitrary definition will represent a vector, i.e., an entity intrinsically independent of the coordinate system used to represent the space. To see this consider the following example, which for simplicity refers to vectors in two-dimensional (2-D) space. Given a vector $\mathbf{u} = (u_1, u_2)$, define a new vector $\mathbf{v} = (u_1 + \lambda, u_2 + \lambda)$, where λ is a nonzero scalar. Does this definition result in a vector? To answer this question draw the vectors $\mathbf{u}$ and $\mathbf{v}$ (Fig. 1.2a), rotate the original coordinate axes, decompose $\mathbf{u}$ into its new components u'_1 and u'_2, add λ to each of them, and draw the new vector $\mathbf{v}' = (u'_1 + \lambda, u'_2 + \lambda)$. Clearly, $\mathbf{v}$ and $\mathbf{v}'$ are not the same geometric object. Therefore, our definition does not represent a vector.

Now consider the following definition: $\mathbf{v} = (\lambda u_1, \lambda u_2)$. After a rotation similar to the previous one we see that $\mathbf{v} = \mathbf{v}'$ (Fig. 1.2b), which is not surprising, as this definition corresponds to the multiplication of a vector by a scalar.

Let us look now at a more complicated example. Suppose that given two vectors $\mathbf{u}$ and $\mathbf{v}$ we want to define a third vector $\mathbf{w}$ as follows:

$$\mathbf{w} = (u_2 v_3 + u_3 v_2)\mathbf{e}_1 + (u_3 v_1 + u_1 v_3)\mathbf{e}_2 + (u_1 v_2 + u_2 v_1)\mathbf{e}_3. \tag{1.2.14}$$

Note that the only difference with the vector product (see (1.2.13)) is the replacement of the minus signs by plus signs. As before, the question is whether this definition is independent of the coordinate system. In this case, however, finding an answer is not straightforward. What one should do is to compute the components w_1, w_2, w_3 in the original coordinate system, draw $\mathbf{w}$, perform a rotation of axes, find the new components of $\mathbf{u}$ and $\mathbf{v}$, compute w'_1, w'_2, and w'_3, draw $\mathbf{w}'$ and compare it with $\mathbf{w}$. If it is found that the two vectors are different, then it is obvious that (1.2.14) does not define a vector. If the two vectors are equal it might be tempting to say that (1.2.14) does indeed define a vector, but this conclusion would not be correct because there may be other rotations for which $\mathbf{w}$ and $\mathbf{w}'$ are not equal.

These examples should convince the reader that establishing the vectorial character of an entity defined by its components requires a definition of a vector that will take this question into account automatically. Only then will it be possible to

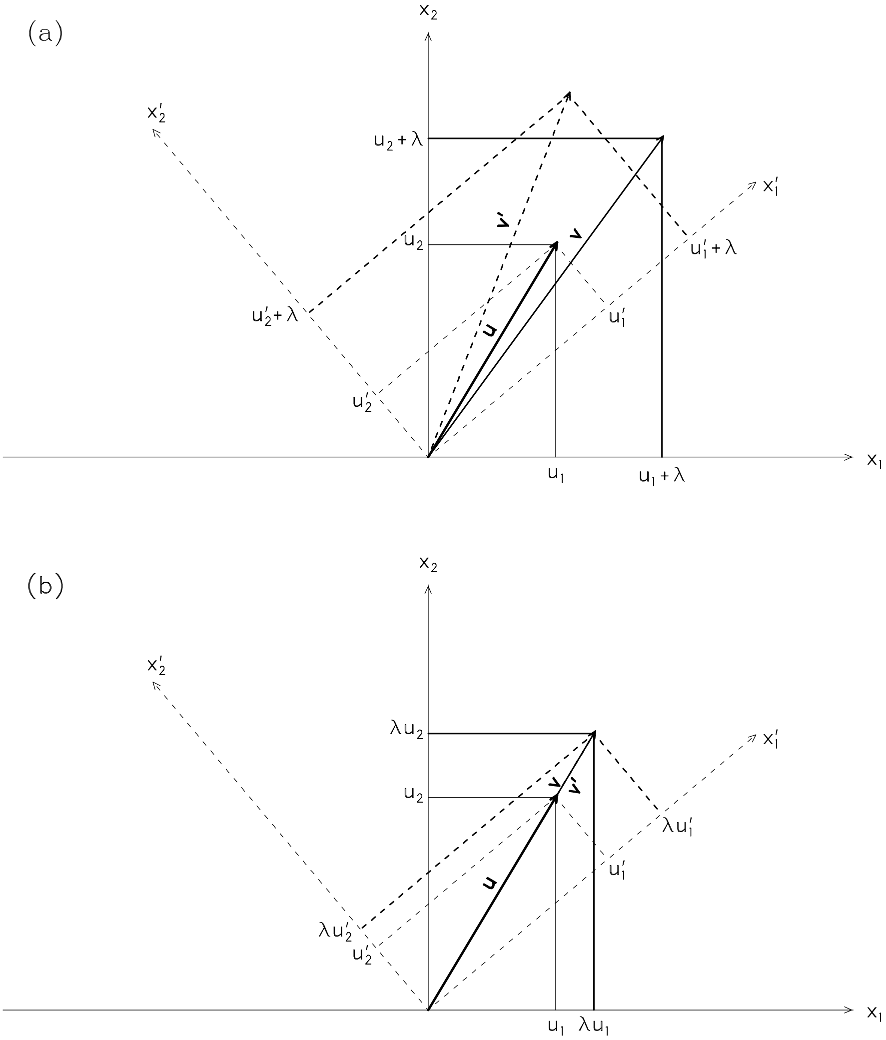

Fig. 1.2. (a) Vectors **v** and **v**′ obtained from a vector **u** as follows. For **v**, add a constant λ to the components u_1 and u_2. For **v**′, add a constant λ to the components u'_1 and u'_2 obtained by rotation of the axis. Because **v** and **v**′ are not the same vector, we can conclude that the entity obtained by adding a constant to the components of a vector does constitute a vector under a rotation of coordinates. (b) Similar to the construction above, but with the constant λ multiplying the vector components. In this case **v** and **v**′ coincide, which agrees with the fact that the operation defined is just the multiplication of a vector by a scalar. After Santalo (1969).

answer the previous question in a general way. However, before introducing the new definition it is necessary to study coordinate rotations in some more detail. This is done next.

1.3 Rotation of Cartesian coordinates. Definition of a vector

Let Ox_1, Ox_2, and Ox_3 represent a Cartesian coordinate system and Ox'_1, Ox'_2, Ox'_3 another system obtained from the previous one by a rotation about their common origin O (Fig. 1.3). Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ and $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3$ be the unit vectors along the three axes in the original and rotated systems. Finally, let a_{ij} denote the cosine of the angle between Ox'_i and Ox_j. The a_{ij}'s are known as direction cosines, and are related to $\mathbf{e}'_i$ and $\mathbf{e}_j$ by

$$\mathbf{e}'_i \cdot \mathbf{e}_j = a_{ij}. \tag{1.3.1}$$

Given an arbitrary vector $\mathbf{v}$ with components v_1, v_2, and v_3 in the original system, we are interested in finding the components v'_1, v'_2, and v'_3 in the rotated system. To find the relation between the two sets of components we will consider first the relation between the corresponding unit vectors. Using (1.3.1) $\mathbf{e}'_i$ can be written as

$$\mathbf{e}'_i = a_{i1}\mathbf{e}_1 + a_{i2}\mathbf{e}_2 + a_{i3}\mathbf{e}_3 = \sum_{j=1}^{3} a_{ij}\mathbf{e}_j \tag{1.3.2}$$

(Problem 1.3a).

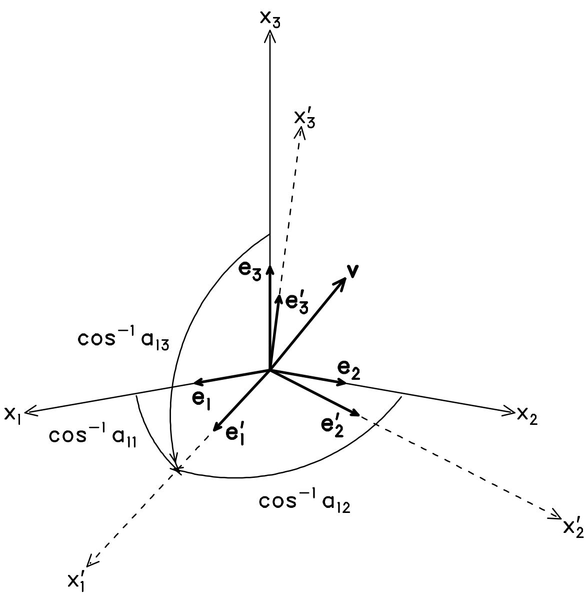

Fig. 1.3. Rotation of axes. Primed and unprimed quantities refer to the original and rotated coordinate systems, respectively. Both systems are rectangular Cartesian. The quantities a_{ij} indicate the scalar product $\mathbf{e}'_i \cdot \mathbf{e}_j$. The vector $\mathbf{v}$ exists independent of the coordinate system. Three relevant angles are shown.

Furthermore, in the original and rotated systems $\mathbf{v}$ can be written as

$$\mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e}_j \tag{1.3.3}$$

and

$$\mathbf{v} = \sum_{i=1}^{3} v'_i \mathbf{e}'_i. \tag{1.3.4}$$

Now introduce (1.3.2) in (1.3.4)

$$\mathbf{v} = \sum_{i=1}^{3} v'_i \sum_{j=1}^{3} a_{ij} \mathbf{e}_j \equiv \sum_{j=1}^{3} \left(\sum_{i=1}^{3} a_{ij} v'_i \right) \mathbf{e}_j. \tag{1.3.5}$$

Since (1.3.3) and (1.3.5) represent the same vector, and the three unit vectors $\mathbf{e}_1$, $\mathbf{e}_1$, and $\mathbf{e}_3$ are independent of each other, we conclude that

$$v_j = \sum_{i=1}^{3} a_{ij} v'_i. \tag{1.3.6}$$

If we write the $\mathbf{e}_j$s in terms of the $\mathbf{e}'_i$s and replace them in (1.3.3) we find that

$$v'_i = \sum_{j=1}^{3} a_{ij} v_j \tag{1.3.7}$$

(Problem 1.3b).

Note that in (1.3.6) the sum is over the first subindex of a_{ij}, while in (1.3.7) the sum is over the second subindex of a_{ij}. This distinction is critical and must be respected.

Now we are ready to introduce the following definition of a vector:

three scalars are the components of a vector if under a rotation of coordinates they transform according to (1.3.7).

What this definition means is that if we want to define a vector by some set of rules, we have to verify that the vector components satisfy the transformation equations.

Before proceeding we will introduce a *summation convention* (due to Einstein) that will simplify the mathematical manipulations significantly. The convention applies to monomial expressions (such as a single term in an equation) and consists of dropping the sum symbol and summing over repeated indices.[1] This convention requires that the same index should appear no more than twice in the same term.

[1] In this book the convention will not be applied to uppercase indices

Repeated indices are known as *dummy indices*, while those that are not repeated are called *free indices*. Using this convention, we will write, for example,

$$\mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e}_j = v_j \mathbf{e}_j \tag{1.3.8}$$

$$v_j = \sum_{i=1}^{3} a_{ij} v'_i = a_{ij} v'_i \tag{1.3.9}$$

$$v'_i = \sum_{j=1}^{3} a_{ij} v_j = a_{ij} v_j. \tag{1.3.10}$$

It is important to have a clear idea of the difference between free and dummy indices. A particular dummy index can be changed at will as long as it is replaced (in its two occurrences) by some other index not equal to any other existing indices in the same term. Free indices, on the other hand, are fixed and cannot be changed inside a single term. However, a free index can be replaced by another as long as the change is effected in all the terms in an equation, and the new index is different from all the other indices in the equation. In (1.3.9) i is a dummy index and j is a free index, while in (1.3.10) their role is reversed. The examples below show legal and illegal index manipulations.

The following relations, derived from (1.3.9), are true

$$v_j = a_{ij} v'_i = a_{kj} v'_k = a_{lj} v'_l \tag{1.3.11}$$

because the repeated index i was replaced by a different repeated index (equal to k or l). However, it would not be correct to replace i by j because j is already present in the equation. If i were replaced by j we would have

$$v_j = a_{jj} v'_j, \tag{1.3.12}$$

which would not be correct because the index j appears more than twice in the right-hand term, which is not allowed. Neither would it be correct to write

$$v_j = a_{ik} v'_i \tag{1.3.13}$$

because the free index j has been changed to k only in the right-hand term. On the other hand, (1.3.9) can be written as

$$v_k = a_{ik} v'_i \tag{1.3.14}$$

because the free index j has been replaced by k on both sides of the equation.

As (1.3.9) and (1.3.10) are of fundamental importance, it is necessary to pay attention to the fact that in the former the sum is over the first index of a_{ij} while in the latter the sum is over the second index of a_{ij}. Also note that (1.3.10) can be written as the product of a matrix and a vector:

$$\mathbf{v}' = \begin{pmatrix} v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \equiv \mathsf{A}\,\mathbf{v}, \tag{1.3.15}$$

where A is the matrix with elements a_{ij}.

It is clear that (1.3.9) can be written as

$$\mathbf{v} = \mathsf{A}^{\mathrm{T}}\mathbf{v}', \tag{1.3.16}$$

where the superscript T indicates transposition.

Now we will derive an important property of A. By introducing (1.3.10) in (1.3.9) we obtain

$$v_j = a_{ij}a_{ik}v_k. \tag{1.3.17}$$

Note that it was necessary to change the dummy index in (1.3.10) to satisfy the summation convention. Equation (1.3.17) implies that any of the three components of $\mathbf{v}$ is a combination of all three components. However, this cannot be generally true because $\mathbf{v}$ is an arbitrary vector. Therefore, the right-hand side of (1.3.17) must be equal to v_j, which in turn implies that the product $a_{ij}a_{ik}$ must be equal to unity when $j = k$, and equal to zero when $j \neq k$. This happens to be the definition of the Kronecker delta δ_{jk} introduced in (1.2.7), so that

$$a_{ij}a_{ik} = \delta_{jk}. \tag{1.3.18}$$

If (1.3.9) is introduced in (1.3.10) we obtain

$$a_{ij}a_{kj} = \delta_{ik}. \tag{1.3.19}$$

Setting $i = k$ in (1.3.19) and writing in full gives

$$1 = a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = |\mathbf{e}'_i|^2; \qquad i = 1, 2, 3, \tag{1.3.20}$$

where the equality on the right-hand side follows from (1.3.2).

When $i \neq k$, (1.3.19) gives

$$0 = a_{i1}a_{k1} + a_{i2}a_{k2} + a_{i3}a_{k3} = \mathbf{e}'_i \cdot \mathbf{e}'_k, \tag{1.3.21}$$

where the equality on the right-hand side also follows from (1.3.2). Therefore, (1.3.19) summarizes the fact that the $\mathbf{e}'_j$s are unit vectors orthogonal to each other, while (1.3.18) does the same thing for the $\mathbf{e}_i$s. Any set of vectors having these properties is known as an orthonormal set.