Economic Dynamics
Phase Diagrams and Their Economic Application
Second Edition

This is the substantially revised and restructured second edition of Ron Shone’s successful undergraduate and graduate textbook Economic Dynamics.

The book provides detailed coverage of dynamics and phase diagrams including: quantitative and qualitative dynamic systems, continuous and discrete dynamics, linear and nonlinear systems and single equation and systems of equations. It illustrates dynamic systems using Mathematica, Maple and spreadsheets.

It provides a thorough introduction to phase diagrams and their economic application and explains the nature of saddle path solutions.

The second edition contains a new chapter on oligopoly and an extended treatment of stability of discrete dynamic systems and the solving of first-order difference equations. Detailed routines on the use of Mathematica and Maple are now contained in the body of the text, which now also includes advice on the use of Excel and additional examples and exercises throughout. The supporting website contains a solutions manual and learning tools.

Ronald Shone is Senior Lecturer in Economics at the University of Stirling. He is the author of eight books on economics covering the areas of microeconomics, macroeconomics and international economics at both undergraduate and postgraduate level. He has written a number of articles published in Oxford Economic Papers, the Economic Journal, Journal of Economic Surveys and Journal of Economic Studies.
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Contents

Preface to the second edition xi
Preface to the first edition xiii

PART I Dynamic modelling

1 Introduction 3
  1.1 What this book is about 3
  1.2 The rise in economic dynamics 5
  1.3 Stocks, flows and dimensionality 8
  1.4 Nonlinearities, multiple equilibria and local stability 12
  1.5 Nonlinearity and chaos 15
  1.6 Computer software and economic dynamics 17
  1.7 Mathematica and Maple 20
  1.8 Structure and features 24
  Additional reading 25

2 Continuous dynamic systems 26
  2.1 Some definitions 26
  2.2 Solutions to first-order linear differential equations 37
  2.3 Compound interest 39
  2.4 First-order equations and isoclines 41
  2.5 Separable functions 45
  2.6 Diffusion models 53
  2.7 Phase portrait of a single variable 54
  2.8 Second-order linear homogeneous equations 59
  2.9 Second-order linear nonhomogeneous equations 64
  2.10 Linear approximations to nonlinear differential equations 66
  2.11 Solving differential equations with Mathematica 70
  2.12 Solving differential equations with Maple 73
Appendix 2.1 Plotting direction fields for a single equation with Mathematica 77
Appendix 2.2 Plotting direction fields for a single equation with Maple 79
Exercises 80
Additional reading 84
## 3 Discrete dynamic systems  
3.1 Classifying discrete dynamic systems 85  
3.2 The initial value problem 86  
3.3 The cobweb model: an introduction 87  
3.4 Equilibrium and stability of discrete dynamic systems 88  
3.5 Solving first-order difference equations 99  
3.6 Compound interest 105  
3.7 Discounting, present value and internal rates of return 108  
3.8 Solving second-order difference equations 110  
3.9 The logistic equation: discrete version 118  
3.10 The multiplier–accelerator model 123  
3.11 Linear approximation to discrete nonlinear difference equations 127  
3.12 Solow growth model in discrete time 130  
3.13 Solving recursive equations with *Mathematica* and *Maple* 131  
Appendix 3.1 Two-cycle logistic equation using *Mathematica* 135  
Appendix 3.2 Two-cycle logistic equation using *Maple* 137  
Exercises 138  
Additional reading 141

## 4 Systems of first-order differential equations  
4.1 Definitions and autonomous systems 142  
4.2 The phase plane, fixed points and stability 145  
4.3 Vectors of forces in the phase plane 149  
4.4 Matrix specification of autonomous systems 156  
4.5 Solutions to the homogeneous differential equation system: real distinct roots 160  
4.6 Solutions with repeating roots 162  
4.7 Solutions with complex roots 164  
4.8 Nodes, spirals and saddles 166  
4.9 Stability/instability and its matrix specification 178  
4.10 Limit cycles 179  
4.11 Euler’s approximation and differential equations on a spreadsheet 183  
4.12 Solving systems of differential equations with *Mathematica* and *Maple* 186  
Appendix 4.1 Parametric plots in the phase plane: continuous variables 194  
Exercises 196  
Additional reading 200

## 5 Discrete systems of equations  
5.1 Introduction 201  
5.2 Basic matrices with *Mathematica* and *Maple* 204  
5.3 Eigenvalues and eigenvectors 208
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.4</td>
<td>Mathematica and Maple for solving discrete systems</td>
<td>214</td>
</tr>
<tr>
<td>5.5</td>
<td>Graphing trajectories of discrete systems</td>
<td>220</td>
</tr>
<tr>
<td>5.6</td>
<td>The stability of discrete systems</td>
<td>223</td>
</tr>
<tr>
<td>5.7</td>
<td>The phase plane analysis of discrete systems</td>
<td>235</td>
</tr>
<tr>
<td>5.8</td>
<td>Internal and external balance</td>
<td>239</td>
</tr>
<tr>
<td>5.9</td>
<td>Nonlinear discrete systems</td>
<td>245</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>247</td>
</tr>
<tr>
<td></td>
<td>Additional reading</td>
<td>250</td>
</tr>
<tr>
<td>6</td>
<td>Optimal control theory</td>
<td>251</td>
</tr>
<tr>
<td>6.1</td>
<td>The optimal control problem</td>
<td>251</td>
</tr>
<tr>
<td>6.2</td>
<td>The Pontryagin maximum principle: continuous model</td>
<td>252</td>
</tr>
<tr>
<td>6.3</td>
<td>The Pontryagin maximum principle: discrete model</td>
<td>259</td>
</tr>
<tr>
<td>6.4</td>
<td>Optimal control with discounting</td>
<td>265</td>
</tr>
<tr>
<td>6.5</td>
<td>The phase diagram approach to continuous time control models</td>
<td>270</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>283</td>
</tr>
<tr>
<td></td>
<td>Additional reading</td>
<td>285</td>
</tr>
<tr>
<td>7</td>
<td>Chaos theory</td>
<td>286</td>
</tr>
<tr>
<td>7.1</td>
<td>Introduction</td>
<td>286</td>
</tr>
<tr>
<td>7.2</td>
<td>Bifurcations: single-variable case</td>
<td>287</td>
</tr>
<tr>
<td>7.3</td>
<td>The logistic equation, periodic-doubling bifurcations and chaos</td>
<td>293</td>
</tr>
<tr>
<td>7.4</td>
<td>Feigenbaum’s universal constant</td>
<td>301</td>
</tr>
<tr>
<td>7.5</td>
<td>Sarkovskii theorem</td>
<td>302</td>
</tr>
<tr>
<td>7.6</td>
<td>Van der Pol equation and Hopf bifurcations</td>
<td>304</td>
</tr>
<tr>
<td>7.7</td>
<td>Strange attractors</td>
<td>307</td>
</tr>
<tr>
<td>7.8</td>
<td>Rational choice and erratic behaviour</td>
<td>312</td>
</tr>
<tr>
<td>7.9</td>
<td>Inventory dynamics under rational expectations</td>
<td>315</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>319</td>
</tr>
<tr>
<td></td>
<td>Additional reading</td>
<td>321</td>
</tr>
<tr>
<td>PART II</td>
<td>Applied economic dynamics</td>
<td>325</td>
</tr>
<tr>
<td>8</td>
<td>Demand and supply models</td>
<td>325</td>
</tr>
<tr>
<td>8.1</td>
<td>Introduction</td>
<td>325</td>
</tr>
<tr>
<td>8.2</td>
<td>A simple demand and supply model in continuous time</td>
<td>326</td>
</tr>
<tr>
<td>8.3</td>
<td>The cobweb model</td>
<td>332</td>
</tr>
<tr>
<td>8.4</td>
<td>Cobwebs with Mathematica and Maple</td>
<td>338</td>
</tr>
<tr>
<td>8.5</td>
<td>Cobwebs in the phase plane</td>
<td>339</td>
</tr>
<tr>
<td>8.6</td>
<td>Cobwebs in two interrelated markets</td>
<td>346</td>
</tr>
<tr>
<td>8.7</td>
<td>Demand and supply with stocks</td>
<td>349</td>
</tr>
<tr>
<td>8.8</td>
<td>Stability of the competitive equilibrium</td>
<td>353</td>
</tr>
<tr>
<td>8.9</td>
<td>The housing market and demographic changes</td>
<td>358</td>
</tr>
<tr>
<td>8.10</td>
<td>Chaotic demand and supply</td>
<td>363</td>
</tr>
</tbody>
</table>
Appendix 8.1 Obtaining cobwebs using Mathematica and Maple 367
Exercises 371
Additional reading 374

9 Dynamic theory of oligopoly 375
  9.1 Static model of duopoly 375
  9.2 Discrete oligopoly models with output adjusting completely and instantaneously 377
  9.3 Discrete oligopoly models with output not adjusting completely and instantaneously 389
  9.4 Continuous modelling of oligopoly 398
  9.5 A nonlinear model of duopolistic competition (R&D) 405
  9.6 Schumpeterian dynamics 414
  Exercises 419
  Additional reading 423

10 Closed economy dynamics 424
  10.1 Goods market dynamics 425
  10.2 Goods and money market dynamics 429
  10.3 IS-LM continuous model: version 1 431
  10.4 Trajectories with Mathematica, Maple and Excel 437
  10.5 Some important propositions 442
  10.6 IS-LM continuous model: version 2 447
  10.7 Nonlinear IS-LM model 453
  10.8 Tobin–Blanchard model 455
  10.9 Conclusion 465
  Exercises 467
  Additional reading 469

11 The dynamics of inflation and unemployment 470
  11.1 The Phillips curve 470
  11.2 Two simple models of inflation 472
  11.3 Deflationary ‘death spirals’ 484
  11.4 A Lucas model with rational expectations 490
  11.5 Policy rules 493
  11.6 Money, growth and inflation 494
  11.7 Cagan model of hyperinflation 500
  11.8 Unemployment and job turnover 506
  11.9 Wage determination models and the profit function 509
  11.10 Labour market dynamics 513
  Exercises 516
  Additional reading 518

12 Open economy dynamics: sticky price models 519
  12.1 The dynamics of a simple expenditure model 519
  12.2 The balance of payments and the money supply 524
12.3 Fiscal and monetary expansion under fixed exchange rates 532
12.4 Fiscal and monetary expansion under flexible exchange rates 539
12.5 Open economy dynamics under fixed prices and floating 545
Exercises 551
Additional reading 552

13 Open economy dynamics: flexible price models 553
13.1 A simplified Dornbusch model 554
13.2 The Dornbusch model 559
13.3 The Dornbusch model: capital immobility 564
13.4 The Dornbusch model under perfect foresight 567
13.5 Announcement effects 573
13.6 Resource discovery and the exchange rate 581
13.7 The monetarist model 586
Exercises 589
Additional reading 592

14 Population models 593
14.1 Malthusian population growth 593
14.2 The logistic curve 596
14.3 An alternative interpretation 601
14.4 Multispecies population models: geometric analysis 603
14.5 Multispecies population models: mathematical analysis 619
14.6 Age classes and projection matrices 626
Appendix 14.1 Computing $a$ and $b$ for the logistic equation using Mathematica 630
Appendix 14.2 Using Maple to compute $a$ and $b$ for the logistic equation 631
Appendix 14.3 Multispecies modelling with Mathematica and Maple 632
Exercises 634
Additional reading 637

15 The dynamics of fisheries 638
15.1 Biological growth curve of a fishery 638
15.2 Harvesting function 644
15.3 Industry profits and free access 647
15.4 The dynamics of open access fishery 650
15.5 The dynamics of open access fishery: a numerical example 654
15.6 The fisheries control problem 658
15.7 Schooling fishery 661
15.8 Harvesting and age classes 669
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exercises</td>
<td>673</td>
</tr>
<tr>
<td>Additional reading</td>
<td>676</td>
</tr>
<tr>
<td><strong>Answers to selected exercises</strong></td>
<td>677</td>
</tr>
<tr>
<td>Bibliography</td>
<td>688</td>
</tr>
<tr>
<td>Author index</td>
<td>697</td>
</tr>
<tr>
<td>Subject index</td>
<td>700</td>
</tr>
</tbody>
</table>
Preface to the second edition

I was very encouraged with the reception of the first edition, from both staff and students. Correspondence eliminated a number of errors and helped me to improve clarity. Some of the new sections are in response to communications I received.

The book has retained its basic structure, but there have been extensive revisions to the text. Part I, containing the mathematical background, has been considerably enhanced in all chapters. All chapters contain new material. This new material is largely in terms of the mathematical content, but there are some new economic examples to illustrate the mathematics. Chapter 1 contains a new section on dimensionality in economics, a much-neglected topic in my view. Chapter 3 on discrete systems has been extensively revised, with a more thorough discussion of the stability of discrete dynamical systems and an extended discussion of solving second-order difference equations. Chapter 5 also contains a more extensive discussion of discrete systems of equations, including a more thorough discussion of solving such systems. Direct solution methods using Mathematica and Maple are now provided in the main body of the text. Indirect solution methods using the Jordan form are new to this edition. There is also a more thorough treatment of the stability of discrete systems.

The two topics covered in chapter 6 of the first edition have now been given a chapter each. This has allowed topics to be covered in more depth. Chapter 6 on control theory now includes the use of Excel’s Solver for solving discrete control problems. Chapter 7 on chaos theory has also been extended, with a discussion of Sarkovskii’s theorem. It also contains a much more extended discussion of bifurcations and strange attractors.

Changes to part II, although less extensive, are quite significant. The mathematical treatment of cobwebs in chapter 8 has been extended and there is now a new section on stock models and another on chaotic demand and supply. Chapter 9 on dynamic oligopoly is totally new to this edition. It deals with both discrete and continuous dynamic oligopoly and goes beyond the typical duopoly model. There is also a discussion of an R&D dynamic model of duopoly and a brief introduction to Schumpeterian dynamics. Chapter 11 now includes a discussion of deflationary ‘death spirals’ which have been prominent in discussions of Japan’s downturn. Cagan’s model of hyperinflations is also a new introduction to this chapter.

The open economy was covered quite extensively in the first edition, so these chapters contain only minor changes. Population models now include a consideration of age classes and Leslie projection matrices. This material is employed...
Preface to the second edition

in chapter 15 to discuss culling policy. The chapter on overlapping generations modelling has been dropped in this edition to make way for the new material. Part of the reason for this is that, as presented, it contained little by the way of dynamics. It had much more to say about nonlinearity.

Two additional changes have been made throughout. Mathematica and Maple routines are now generally introduced into the main body of the text rather than as appendices. The purpose of doing this is to show that these programmes are ‘natural’ tools for the economist. Finally, there has been an increase in the number of questions attached to almost all chapters. As in the first edition, the full solution to all these questions is provided on the Cambridge University website, which is attached to this book: one set of solutions provided in Mathematica notebooks and an alternative set of solutions provided in Maple worksheets.

Writing a book of this nature, involving as it does a number of software packages, has become problematic with constant upgrades. This is especially true with Mathematica and Maple. Some of the routines provided in the first edition no longer work in the upgrade versions. Even in the final stages of preparing this edition, new upgrades were occurring. I had to make a decision, therefore, at which upgrade I would conclude. All routines and all solutions on the web site are carried out with Mathematica 4 and Maple 6.

I would like to thank all those individuals who wrote or emailed me on material in the first edition. I would especially like to thank Mary E. Edwards, Yee-Tien Fu, Christian Groth, Cars Hommes, Alkis Karabalis, Julio Lopez-Gallardo, Johannes Ludsteck and Yanghoon Song. I would also like to thank Simon Whitby for information and clarification on new material in chapter 9. I would like to thank Ashwin Rattan for his continued support of this project and Barbara Docherty for an excellent job of copy-editing, which not only eliminated a number of errors but improved the final project considerably.

The author and publishers wish to thank the following for permission to use copyright material: Springer-Verlag for the programme listing on p. 192 of A First Course in Discrete Dynamic Systems and the use of the Visual D Solve software package from Visual D Solve; Cambridge University Press for table 3 from British Economic Growth 1688–1959, p. 8.

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March 2002
The conception of this book began in the autumn semester of 1990 when I undertook a course in Advanced Economic Theory for undergraduates at the University of Stirling. In this course we attempted to introduce students to dynamics and some of the more recent advances in economic theory. In looking at this material it was quite clear that phase diagrams, and what mathematicians would call qualitative differential equations, were becoming widespread in the economics literature. There is little doubt that in large part this was a result of the rational expectations revolution going on in economics. With a more explicit introduction of expectations into economic modelling, adjustment processes became the mainstay of many economic models. As such, there was a movement away from models just depicting comparative statics. The result was a more explicit statement of a model’s dynamics, along with its comparative statics. A model’s dynamics were explicitly spelled out, and in particular, vectors of forces indicating movements when the system was not in equilibrium. This led the way to solving dynamic systems by employing the theory of differential equations. Saddle paths soon entered many papers in economic theory. However, students found this material hard to follow, and it did not often use the type of mathematics they were taught in their quantitative courses. Furthermore, the material that was available was very scattered indeed.

But there was another change taking place in Universities which has a bearing on the way the present book took shape. As the academic audit was about to be imposed on Universities, there was a strong incentive to make course work assessment quite different from examination assessment. Stirling has always had a long tradition of course work assessment. In the earlier period there was a tendency to make course work assessment the same as examination assessment: the only real difference being that examinations could set questions which required greater links between material since the course was by then complete. In undertaking this new course, I decided from the very outset that the course work assessment would be quite different from the examination assessment. In particular, I conceived the course work to be very ‘problem oriented’. It was my belief that students come to a better understanding of the economics, and its relation to mathematics, if they carry out problems which require them to explicitly solve models, and to go on to discuss the implications of their analysis.

This provided me with a challenge. There was no material available of this type. Furthermore, many economics textbooks of an advanced nature, and certainly the
published articles, involved setting up models in general form and carrying out very
tedious algebraic manipulations. This is quite understandable. But such algebraic
manipulation does not give students the same insight it may provide the research
academic. A compromise is to set out models with specific numerical coefficients.
This has at least four advantages.

It allowed the models to be solved explicitly. This means that students can
get to grips with the models themselves fairly quickly and easily.

Generalisation can always be achieved by replacing the numerical coef-
ficients by unspecified parameters. Or alternatively, the models can be
solved for different values, and students can be alerted to the fact that
a model’s solution is quite dependent on the value (sign) of a particular
parameter.

The dynamic nature of the models can more readily be illustrated. Ac-
cordingly concentration can be centred on the economics and not on the
mathematics.

Explicit solutions to saddle paths can be obtained and so students can
explicitly graph these solutions. Since it was the nature of saddle paths
which gave students the greatest conceptual difficulty, this approach soon
provided students with the insight into their nature that was lacking from
a much more formal approach. Furthermore, they acquired this insight
by explicitly dealing with an economic model.

I was much encouraged by the students’ attitude to this ‘problem oriented’
approach. The course work assignments that I set were far too long and required far
more preparation than could possibly be available under examination conditions.
However, the students approached them with vigour during their course work
period. Furthermore, it led to greater exchanges between students and a positive
externality resulted.

This book is an attempt to bring this material together, to extend it, and make
it more widely available. It is suitable for core courses in economic theory, and
reading for students undertaking postgraduate courses and to researchers who
require to acquaint themselves with the phase diagram technique. In addition, it
can also be part of courses in quantitative economics. Outside of economics, it is
also applicable to courses in mathematical modelling.

Finally, I would like to thank Cambridge University Press and the department of
economics at Stirling for supplying the two mathematical software programmes;
the copy editor, Anne Rix, for an excellent job on a complex manuscript; and my
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