PART I

Dynamic modelling

CHAPTER 1

Introduction

1.1 What this book is about

This is *not* a book on mathematics, nor is it a book on economics. It is true that the over-riding emphasis is on the economics, but the economics under review is specified very much in mathematical form. Our main concern is with *dynamics* and, most especially with phase diagrams, which have entered the economics literature in a major way since 1990. By their very nature, phase diagrams are a feature of dynamic systems.

But why have phase diagrams so dominated modern economics? Quite clearly it is because more emphasis is now placed on dynamics than in the past. Comparative statics dominated economics for a long time, and much of the teaching is still concerned with comparative statics. But the breakdown of many economies, especially under the pressure of high inflation, and the major influence of inflationary expectations, has directed attention to dynamics. By its very nature, dynamics involves time derivatives, dx/dt, where x is a continuous function of time, or difference equations, $x_t - x_{t-1}$ where time is considered in discrete units. This does not imply that these have not been considered or developed in the past. What has been the case is that they have been given only cursory treatment. The most distinguishing feature today is that dynamics is now taking a more central position.

In order to reveal this emphasis *and* to bring the material within the bounds of undergraduate (and postgraduate) courses, it has been necessary to consider dynamic modelling, in both its continuous and discrete forms. But in doing this the over-riding concern has been with the economic applications. It is easy to write a text on the formal mathematics, but what has always been demonstrated in teaching economics is the difficulty students have in relating the mathematics to the economics. This is as true at the postgraduate level as it is at the undergraduate level. This linking of the two disciplines is an art rather than a science. In addition, many books on dynamics are mathematical texts that often choose simple and brief examples from economics. Most often than not, these reduce down to a single differential equation or a single difference equation. Emphasis is on the mathematics. We do this too in part I. Even so, the concentration is on the mathematical concepts that have the widest use in the study of dynamic economics. In part II this emphasis is reversed. The mathematics is chosen in order to enhance the economics. The mathematics is applied to the economic problem rather than the (simple) economic problem being applied to the mathematics. We take a number of major economic areas and consider various aspects of their dynamics.

Because this book is intended to be self-contained, then it has been necessary to provide the mathematical *background*. By 'background' we, of course, mean that this must be mastered *before* the economic problem is reviewed. Accordingly, part I supplies this mathematical background. However, in order not to make part I totally mathematical we have discussed a number of economic applications. These are set out in part I for the first time, but the emphasis here is in illustrating the type of mathematics they involve so that we know what mathematical techniques are required in order to investigate them. Thus, the Malthusian population growth model is shown to be just a particular differential equation, if population growth is assumed to vary continuously over time. But equally, population growth can be considered in terms of a discrete time-period model. Hence, part I covers not only differential equations but also difference equations.

Mathematical specification can indicate that topics such as *A, B* and *C* should be covered. However, *A, B* and *C* are not always relevant to the economic problem under review. Our choice of material to include in part I, and the emphasis of this material, has been dictated by what mathematics is required to understand certain features of dynamic *economic* systems. It is quite clear when considering mathematical models of differential equations that the emphasis has been, and still is, with models from the physical sciences. This is not surprising given the development of science. In this text, however, we shall concentrate on economics as the *raison d'être* of the mathematics. In a nutshell, we have taken a number of economic dynamic models and asked: 'What mathematics is necessary to understand these?' This is the emphasis of part I. The content of part I has been dictated by the models developed in part II. Of course, if more economic models are considered then the mathematical background will inevitably expand. What we are attempting in this text is dynamic modelling that should be within the compass of an undergraduate with appropriate training in both economics *and* quantitative economics.

Not all dynamic questions are dealt with in this book. The over-riding concern has been to explain phase diagrams. Such phase diagrams have entered many academic research papers over the past decade, and the number is likely to increase. Azariades (1993) has gone as far as saying that

> Dynamical systems have spread so widely into macroeconomics that vector fields and phase diagrams are on the verge of displacing the familiar supply–demand schedules and Hicksian crosses of static macroeconomics. (p. xii)

The emphasis is therefore justified. Courses in quantitative economics generally provide inadequate training to master this material. They provide the basics in differentiation, integration and optimisation. But dynamic considerations get less emphasis – most usually because of a resource constraint. But this is a most unfortunate deficiency in undergraduate teaching that simply does not equip students to understand the articles dealing with dynamic systems. The present book is one attempt to bridge this gap.

I have assumed some basic knowledge of differentiation and integration, along with some basic knowledge of difference equations. However, I have made great pains to spell out the modelling specifications and procedures. This should enable a student to follow how the mathematics and economics interrelate. Such knowledge can be imparted only by demonstration. I have always been disheartened by the idea that you can teach the mathematics and statistics in quantitative courses, and you can teach the economics in economics courses, and by some unspecified osmosis the two areas are supposed to fuse together in the minds of the student. For some, this is true. But I suspect that for the bulk of students this is simply *not* true. Students require knowledge and experience in how to relate the mathematics and the economics.

As I said earlier, this is more of an art than a science. But more importantly, it shows how a problem excites the economist, how to then specify the problem in a formal (usually mathematical) way, and how to solve it. At each stage ingenuity is required. Economics at the moment is very much in the mould of problem solving. It appears that the procedure the investigator goes through[1] is:

(1) Specify the problem
(2) Mathematise the problem
(3) See if the problem's solution conforms to standard mathematical solutions
(4) Investigate the properties of the solution.

It is not always possible to mathematise a problem and so steps (2)–(4) cannot be undertaken. However, in many such cases a verbal discussion is carried out in which a 'story' is told about the situation. This is no more than a heuristic model, but a model just the same. In such models the dynamics are part of the 'story' – about how adjustment takes place over time. It has long been argued by some economists that only those problems that can be mathematised get investigated. There are advantages to formal modelling, of going beyond heuristics. In this book we concentrate only on the formal modelling process.

1.2 The rise in economic dynamics

Economic dynamics has recently become more prominent in mainstream economics. This influence has been quite pervasive and has influenced both microeconomics and macroeconomics. Its influence in macroeconomics, however, has been much greater. In this section we outline some of the main areas where economic dynamics has become more prominent and the possible reasons for this rise in the subject.

1.2.1 Macroeconomic dynamics

Economists have always known that the world is a dynamic one, and yet a scan of the books and articles over the past twenty years or so would make one wonder if they really believed it. With a few exceptions, dynamics has been notably absent from published works. This began to change in the 1970s. The 1970s became a watershed in both economic analysis and economic policy. It was a turbulent time.

[1] For an extended discussion of the modelling process, see Mooney and Swift (1999, chapter 0).

Economic relationships broke down, stagflation became typical of many Western economies, and Conservative policies became prominent. Theories, especially macroeconomic theories, were breaking down, or at best becoming poor predictors of economic changes. The most conspicuous change was the rapid (and accelerating) rise in inflation that occurred with rising unemployment. This became a feature of most Western economies. Individuals began to expect price rises and to build this into their decision-making. If such behaviour was to be modelled, and it was essential to do so, then it inevitably involved a dynamic model of the macroeconomy. More and more, therefore, articles postulated dynamic models that often involved inflationary expectations.

Inflation, however, was not the only issue. As inflation increased, as OPEC changed its oil price and as countries discovered major resource deposits, so there were major changes to countries' balance of payments situations. Macroeconomists had for a long time considered their models in the context of a closed economy. But with such changes, the fixed exchange rate system that operated from 1945 until 1973 had to give way to floating. Generalised floating began in 1973. This would not have been a problem if economies had been substantially closed. But trade in goods and services was growing for most countries. Even more significant was the increase in capital flows between countries. Earlier trade theories concentrated on the current account. But with the growth of capital flows, such models became quite unrealistic. The combination of major structural changes and the increased flows of capital meant that exchange rates had substantial impacts on many economies. It was no longer possible to model the macroeconomy as a closed economy. But with the advent of generalised floating changes in the exchange rate needed to be modelled. Also, like inflation, market participants began to formulate expectations about exchange rate movements and act accordingly. It became essential, then, to model exchange rate expectations. This modelling was inevitably dynamic. More and more articles considered dynamic models, and are still doing so.

One feature of significance that grew out of *both* the closed economy modelling and the open economy modelling was the stock-flow aspects of the models. Keynesian economics had emphasised a flow theory. This was because Keynes himself was very much interested in the short run – as he aptly put it: 'In the long run we are all dead.' Even growth theories allowed investment to take place (a flow) but assumed the stock of capital constant, even though such investment added to the capital stock! If considering only one or two periods, this may be a reasonable approximation. However, economists were being asked to predict over a period of five or more years. More importantly, the change in the bond issue (a flow) altered the National Debt (a stock), and also the interest payment on this debt. It is one thing to consider a change in government spending and the impact this has on the budget balance; but the budget, or more significantly the National Debt, gives a stock dimension to the long-run forces. Governments are not unconcerned with the size of the National Debt.

The same was true of the open economy. The balance of payments is a flow. The early models, especially those ignoring the capital account, were concerned only with the impact of the difference between the exports and imports of goods and services. In other words, the inflow and outflow of goods and services to and from an economy. This was the emphasis of modelling under fixed exchange rates. But a deficit leads to a reduction in the level of a country's *stock* of reserves. A surplus does the opposite. Repeated deficits lead to a repeated decline in a country's level of reserves and to the money stock. Printing more money could, of course, offset the latter (sterilisation), but this simply complicates the adjustment process. At best it delays the adjustment that is necessary. Even so, the adjustment requires *both* a change in the flows and a change in stocks.

What has all this to do with dynamics? Flows usually (although not always) take place in the same time period, say over a year. Stocks are at points in time. To change stock levels, however, to some desired amount would often take a number of periods to achieve. There would be stock-adjustment flows. These are inherently dynamic. Such stock-adjustment flows became highly significant in the 1970s and needed to be included in the modelling process. Models had to become more dynamic if they were to become more realistic or better predictors.

These general remarks about why economists need to consider dynamics, however, hide an important distinction in the way dynamics enters economics. It enters in two quite different and fundamental ways (Farmer 1999). The first, which has its counterpart in the natural sciences, is from the fact that the present depends upon the past. Such models typically are of the form

$$y_t = f(y_{t-1}) \tag{1.1}$$

where we consider just a one-period lag. The second way dynamics enters macroeconomics, *which has no counterpart in the natural sciences*, arises from the fact that economic agents in the present have expectations (or beliefs) about the future. Again taking a one-period analysis, and denoting the present expectation about the variable y one period from now by Ey_{t+1}, then

$$y_t = g(Ey_{t+1}) \tag{1.2}$$

Let us refer to the first lag as a *past lag* and the second a *future lag*. There is certainly no reason to suppose modelling past lags is the same as modelling future lags. Furthermore, a given model can incorporate both past lags and future lags.

The natural sciences provide the mathematics for handling past lags but has nothing to say about how to handle future lags. It is the future lag that gained most attention in the 1970s, most especially with the rise in rational expectations. Once a future lag enters a model it becomes absolutely essential to model expectations, and at the moment there is no generally accepted way of doing this. This does not mean that we should not model expectations, rather it means that at the present time there are a variety of ways of modelling expectations, each with its strengths and weaknesses. This is an area for future research.

1.2.2 Environmental issues

Another change was taking place in the 1970s. Environmental issues were becoming, and are becoming, more prominent. Environmental economics as a subject began to have a clear delineation from other areas of economics. It is true that environmental economics already had a body of literature. What happened in the 1970s and 1980s was that it became a recognised sub-discipline.

Economists who had considered questions in the area had largely confined themselves to the static questions, most especially the questions of welfare and cost–benefit analysis. But environmental issues are about resources. Resources have a stock and there is a rate of depletion and replenishment. In other words, there is the inevitable stock-flow dimension to the issue. Environmentalists have always known this, but economists have only recently considered such issues. Why? Because the issues are dynamic. Biological species, such as fish, grow and decline, and decline most especially when harvested by humans. Forests decline and take a long time to replace. Fossil fuels simply get used up. These aspects have led to a number of dynamic models – some discrete and some continuous. Such modelling has been influenced most particularly by control theory. We shall briefly cover some of this material in chapters 6 and 15.

1.2.3 The implication for economics

All the changes highlighted have meant a significant move towards economic dynamics. But the quantitative courses have in large part not kept abreast of these developments. The bulk of the mathematical analysis is still concerned with equilibrium and comparative statics. Little consideration is given to dynamics – with the exception of the cobweb in microeconomics and the multiplier–accelerator model in macroeconomics.

Now that more attention has been paid to economic dynamics, more and more articles are highlighting the problems that arise from nonlinearity which typify many of the dynamic models we shall be considering in this book. It is the presence of nonlinearity that often leads to more than one equilibrium; and given more than one equilibrium then only local stability properties can be considered. We discuss these issues briefly in section 1.4.

1.3 Stocks, flows and dimensionality

Nearly all variables and parameters – whether they occur in physics, biology, sociology or economics – have units in which they are defined and measured. Typical units in physics are weight and length. Weight can be measured in pounds or kilograms, while length can be measured in inches or centimetres. We can add together length and we can add together weight, but what we cannot do is add length to weight. This makes no sense. Put simply, we can add only things that have the same dimension.

DEFINITION
Any set of additive quantities is a **dimension**. A *primary dimension* is not expressible in terms of any other dimension; a *secondary dimension* is defined in terms of primary dimensions.[2]

[2] An elementary discussion of dimensionality in economics can be found in Neal and Shone (1976, chapter 3). The definitive source remains De Jong (1967).

To clarify these ideas, and other to follow, we list the following set of primary dimensions used in economics:

(1) Money $[M]$
(2) Resources or quantity $[Q]$
(3) Time $[T]$
(4) Utility or satisfaction $[S]$

Apples has, say, dimension $[Q1]$ and bananas $[Q2]$. We cannot add an apple to a banana (we can of course add the number of objects, but that is not the same thing). The *value* of an apple has dimension $[M]$ and the value of a banana has dimension $[M]$, so we can add the value of an apple to the value of a banana. They have the same dimension. Our reference to $[Q1]$ and $[Q2]$ immediately highlights a problem, especially for macroeconomics. Since we cannot add apples and bananas, it is sometimes assumed in macroeconomics that there is a *single* aggregate good, which then involves dimension $[Q]$.

For any set of primary dimensions, and we shall use money $[M]$ and time $[T]$ to illustrate, we have the following three propositions:

(1) If $a \in [M]$ and $b \in [M]$ then $a \pm b \in [M]$
(2) If $a \in [M]$ and $b \in [T]$ then $ab \in [MT]$ and $a/b \in [MT^{-1}]$
(3) If $y = f(x)$ and $y \in [M]$ then $f(x) \in [M]$.

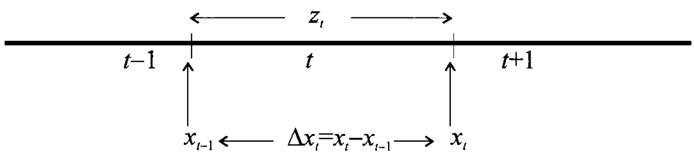
Proposition (1) says that we can add or subtract only things that have the same dimension. Proposition (2) illustrates what is meant by secondary dimensions, e.g., $[MT^{-1}]$ is a secondary or derived dimension. Proposition (3) refers to equations and states that an equation must be dimensionally consistent. Not only must the two sides of an equation have the same value, but it must also have the same dimension, i.e., the equation must be *dimensionally homogeneous.*

The use of time as a primary dimension helps us to clarify most particularly the difference between stocks and flows. A stock is something that occurs at a *point* in time. Thus, the money supply, Ms, has a certain value on 31 December 2001. Ms is a stock with dimension $[M]$, i.e., $Ms \in [M]$. A stock variable is independent of the dimension $[T]$. A flow, on the other hand, is something that occurs over a *period* of time. A flow variable must involve the dimension $[T^{-1}]$. In demand and supply analysis we usually consider demand and supply per period of time. Thus, q^d and q^s are the quantities demanded and supplied per period of time. More specifically, $q^d \in [QT^{-1}]$ and $q^s \in [QT^{-1}]$. In fact, all flow variables involve dimension $[T^{-1}]$. The nominal rate of interest, i, for example, is a per cent per period, so $i \in [T^{-1}]$ and is a flow variable. Inflation, π, is the percentage change in prices per period, say a year. Thus, $\pi \in [T^{-1}]$. The real rate of interest, defined as $r = i - \pi$, is dimensionally consistent since $r \in [T^{-1}]$, being the difference of two variables each with dimension $[T^{-1}]$.

Continuous variables, such as $x(t)$, can be a stock or a flow but are still defined for a point in time. In dealing with discrete variables we need to be a little more careful. Let x_t denote a stock variable. We define this as the value at the *end of period t*.[3] Figure 1.1 uses three time periods to clarify our discussion: $t - 1$, t and

[3] We use this convention throughout this book.

Figure 1.1.



period $t+1$. Thus x_{t-1} is the stock at the end of period $t-1$ and x_t is the stock at the end of period t. Now let z_t be a flow variable *over* period t, and involving dimension $[T^{-1}]$. Of course, there is also z_{t-1} and z_{t+1}. Now return to variable x. It is possible to consider the change in x over period t, which we write as

$$\Delta x_t = x_t - x_{t-1}$$

This immediately shows up a problem. Let x_t have dimension $[Q]$, then by proposition (1) so would Δx_t. But this cannot be correct! Δx_t is the change *over* period t and must involve dimension $[T^{-1}]$. So how can this be? The correct formulation is, in fact,

(1.3)
$$\frac{\Delta x_t}{\Delta t} = \frac{x_t - x_{t-1}}{t-(t-1)} \in [QT^{-1}]$$

Implicit is that $\Delta t = 1$ and so $\Delta x_t = x_t - x_{t-1}$. But this 'hides' the dimension $[T^{-1}]$. This is because $\Delta t \in [T]$, even though it has a value of unity, $\Delta x_t/\Delta t \in [QT^{-1}]$.

Keeping with the convention $\Delta x_t = x_t - x_{t-1}$, then $\Delta x_t \in [QT^{-1}]$ is referred to as a **stock-flow variable**. Δx_t must be kept quite distinct from z_t. The variable z_t is a flow variable and has no stock dimension. Δx_t, on the other hand, is a difference of two stocks defined over period t.

Example 1.1

Consider the quantity equation $MV = Py$. M is the stock of money, with dimension $[M]$. The variable y is the level of real output. To make dimensional sense of this equation, we need to assume a single-good economy. It is usual to consider y as real GDP over a period of time, say one year. So, with a single-good economy with goods having dimension $[Q]$, then $y \in [QT^{-1}]$. If we have a single-good economy, then P is the money per unit of the good and has dimension $[MQ^{-1}]$. V is the income velocity of circulation of money, and indicates the average number of times a unit of money circulates over a period of time. Hence $V \in [T^{-1}]$. Having considered the dimensions of the variables separately, do we have dimensional consistency?

$$MV \in [M][T^{-1}] = [MT^{-1}]$$
$$Py \in [MQ^{-1}][QT^{-1}] = [MT^{-1}]$$

and so we do have dimensional consistency. Notice in saying this that we have utilised the feature that dimensions 'act like algebra' and so dimensions cancel, as with $[QQ^{-1}]$. Thus

$$Py \in [MQ^{-1}][QT^{-1}] = [MQ^{-1}QT^{-1}] = [MT^{-1}]$$

Example 1.2

Consider again the nominal rate of interest, denoted *i*. This can more accurately be defined as the amount of money received over some interval of time divided by the capital outlay. Hence,

$$i \in \frac{[MT^{-1}]}{[M]} = [T^{-1}]$$

Example 1.3

Consider the linear static model of demand and supply, given by the following equations.

$$\begin{aligned} q^d &= a - bp \qquad a, b > 0 \\ q^s &= c + dp \qquad d > 0 \\ q^d &= q^s = q \end{aligned} \tag{1.4}$$

with equilibrium price and quantity

$$p^* = \frac{a-c}{b+d}, \qquad q^* = \frac{ad+bc}{b+d}$$

and with dimensions

$$q^d, q^s \in [QT^{-1}], \qquad p \in [MQ^{-1}]$$

The model is a flow model since q^d and q^s are defined as quantities per period of time.[4] It is still, however, a static model because all variables refer to time period *t*. Because of this we conventionally do not include a time subscript.

Now turn to the parameters of the model. If the demand and supply equations are to be dimensionally consistent, then

$$a, c \in [QT^{-1}] \qquad \text{and} \qquad b, d \in [Q^2T^{-1}M^{-1}]$$

Then

$$\begin{aligned} &a - c \in [QT^{-1}] \\ &b + d \in [Q^2T^{-1}M^{-1}] \\ &p^* \in \frac{[QT^{-1}]}{[Q^2T^{-1}M^{-1}]} = [MQ^{-1}] \end{aligned}$$

Also

$$\begin{aligned} &ad \in [QT^{-1}][Q^2T^{-1}M^{-1}] = [Q^3T^{-2}M^{-1}] \\ &bc \in [Q^2T^{-1}M^{-1}][QT^{-1}] = [Q^3T^{-2}M^{-1}] \\ &q^* \in \frac{[Q^3T^{-2}M^{-1}]}{[Q^2T^{-1}M^{-1}]} = [QT^{-1}] \end{aligned}$$

Where a problem sometimes occurs in writing formulas is when parameters have values of unity. Consider just the demand equation and suppose it takes the

[4] We could have considered a stock demand and supply model, in which case q^d and q^s would have dimension $[Q]$. Such a model would apply to a particular point in time.