An Elementary Introduction to Mathematical Finance

Options and Other Topics

Second Edition

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1.1 Probabilities and Events

Consider an experiment and let S, called the *sample space*, be the set of all possible outcomes of the experiment. If there are m possible outcomes of the experiment then we will generally number them 1 through m, and so $S = \{1, 2, \ldots, m\}$. However, when dealing with specific examples, we will usually give more descriptive names to the outcomes.

Example 1.1a (i) Let the experiment consist of flipping a coin, and let the outcome be the side that lands face up. Thus, the sample space of this experiment is

$$S = \{h, t\},$$

where the outcome is h if the coin shows heads and t if it shows tails.

(ii) If the experiment consists of rolling a pair of dice – with the outcome being the pair (i, j), where i is the value that appears on the first die and j the value on the second – then the sample space consists of the following 36 outcomes:

$$\begin{gathered}
(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6),\\
(2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),\\
(3,1),\ (3,2),\ (3,3),\ (3,4),\ (3,5),\ (3,6),\\
(4,1),\ (4,2),\ (4,3),\ (4,4),\ (4,5),\ (4,6),\\
(5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,5),\ (5,6),\\
(6,1),\ (6,2),\ (6,3),\ (6,4),\ (6,5),\ (6,6).
\end{gathered}$$

(iii) If the experiment consists of a race of r horses numbered 1, 2, 3, $\ldots, r$, and the outcome is the order of finish of these horses, then the sample space is

$$S = \{\text{all orderings of the numbers } 1, 2, 3, \ldots, r\}.$$

For instance, if $r = 4$ then the outcome is $(1, 4, 2, 3)$ if the number 1 horse comes in first, number 4 comes in second, number 2 comes in third, and number 3 comes in fourth. $\square$

Consider once again an experiment with the sample space $S = \{1, 2, \ldots, m\}$. We will now suppose that there are numbers $p_1, \ldots, p_m$ with

$$p_i \geq 0, \ i = 1, \ldots, m, \quad \text{and} \quad \sum_{i=1}^{m} p_i = 1$$

and such that p_i is the *probability* that i is the outcome of the experiment.

Example 1.1b In Example 1.1a(i), the coin is said to be *fair* or *unbiased* if it is equally likely to land on heads as on tails. Thus, for a fair coin we would have that

$$p_h = p_t = 1/2.$$

If the coin were biased and heads were twice as likely to appear as tails, then we would have

$$p_h = 2/3, \qquad p_t = 1/3.$$

If an unbiased pair of dice were rolled in Example 1.1a(ii), then all possible outcomes would be equally likely and so

$$p_{(i,j)} = 1/36, \quad 1 \leq i \leq 6, \ 1 \leq j \leq 6.$$

If $r = 3$ in Example 1.1a(iii), then we suppose that we are given the six nonnegative numbers that sum to 1:

$$p_{1,2,3}, \ p_{1,3,2}, \ p_{2,1,3}, \ p_{2,3,1}, \ p_{3,1,2}, \ p_{3,2,1},$$

where $p_{i,j,k}$ represents the probability that horse i comes in first, horse j second, and horse k third. $\square$

Any set of possible outcomes of the experiment is called an *event*. That is, an event is a subset of S, the set of all possible outcomes. For any event A, we say that A *occurs* whenever the outcome of the experiment is a point in A. If we let $P(A)$ denote the probability that event A occurs, then we can determine it by using the equation

$$P(A) = \sum_{i \in A} p_i. \tag{1.1}$$

Note that this implies

$$P(S) = \sum_i p_i = 1. \tag{1.2}$$

In words, the probability that the outcome of the experiment is in the sample space is equal to 1 – which, since S consists of all possible outcomes of the experiment, is the desired result.

Example 1.1c Suppose the experiment consists of rolling a pair of fair dice. If A is the event that the sum of the dice is equal to 7, then

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

and

$$P(A) = 6/36 = 1/6.$$

If we let B be the event that the sum is 8, then

$$P(B) = p_{(2,6)} + p_{(3,5)} + p_{(4,4)} + p_{(5,3)} + p_{(6,2)} = 5/36.$$

If, in a horse race between three horses, we let A denote the event that horse number 1 wins, then $A = \{(1, 2, 3), (1, 3, 2)\}$ and

$$P(A) = p_{1,2,3} + p_{1,3,2}. \qquad \square$$

For any event A, we let A^c, called the *complement* of A, be the event containing all those outcomes in S that are not in A. That is, A^c occurs if and only if A does not. Since

$$\begin{aligned} 1 &= \sum_i p_i \\ &= \sum_{i \in A} p_i + \sum_{i \in A^c} p_i \\ &= P(A) + P(A^c), \end{aligned}$$

we see that

$$P(A^c) = 1 - P(A). \tag{1.3}$$

That is, the probability that the outcome is not in A is 1 minus the probability that it is in A. The complement of the sample space S is the null event $\emptyset$, which contains no outcomes. Since $\emptyset = S^c$, we obtain from Equations (1.2) and (1.3) that

$$P(\emptyset) = 0.$$

For any events A and B we define $A \cup B$, called the *union* of A and B, as the event consisting of all outcomes that are in A, or in B, or in both A and B. Also, we define their *intersection* AB (sometimes written $A \cap B$) as the event consisting of all outcomes that are both in A and in B.

Example 1.1d Let the experiment consist of rolling a pair of dice. If A is the event that the sum is 10 and B is the event that both dice land on even numbers greater than 3, then

$$A = \{(4,6),(5,5),(6,4)\}, \qquad B = \{(4,4),(4,6),(6,4),(6,6)\}.$$

Therefore,

$$\begin{aligned} A \cup B &= \{(4,4),(4,6),(5,5),(6,4),(6,6)\}, \\ AB &= \{(4,6),(6,4)\}. \end{aligned} \qquad \square$$

For any events A and B, we can write

$$\begin{aligned} P(A \cup B) &= \sum_{i \in A \cup B} p_i, \\ P(A) &= \sum_{i \in A} p_i, \\ P(B) &= \sum_{i \in B} p_i. \end{aligned}$$

Since every outcome in both A and B is counted twice in $P(A) + P(B)$ and only once in $P(A \cup B)$, we obtain the following result, often called the *addition theorem of probability.*

Proposition 1.1.1

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

Thus, the probability that the outcome of the experiment is either in A or in B equals the probability that it is in A, plus the probability that it is in B, minus the probability that it is in both A and B.

Example 1.1e Suppose the probabilities that the Dow-Jones stock index increases today is .54, that it increases tomorrow is .54, and that it increases both days is .28. What is the probability that it does not increase on either day?

Solution. Let A be the event that the index increases today, and let B be the event that it increases tomorrow. Then the probability that it increases on at least one of these days is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(AB) \\ &= .54 + .54 - .28 = .80. \end{aligned}$$

Therefore, the probability that it increases on neither day is $1 - .80 = .20$. $\square$

If $AB = \emptyset$, we say that A and B are *mutually exclusive* or *disjoint*. That is, events are mutually exclusive if they cannot both occur. Since $P(\emptyset) = 0$, it follows from Proposition 1.1.1 that, when A and B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B).$$

1.2 Conditional Probability

Suppose that each of two teams is to produce an item, and that the two items produced will be rated as either acceptable or unacceptable. The sample space of this experiment will then consist of the following four outcomes:

$$S = \{(a, a), (a, u), (u, a), (u, u)\},$$

where (a, u) means, for instance, that the first team produced an acceptable item and the second team an unacceptable one. Suppose that the probabilities of these outcomes are as follows:

$$\begin{aligned} P(a, a) &= .54, \\ P(a, u) &= .28, \\ P(u, a) &= .14, \\ P(u, u) &= .04. \end{aligned}$$

If we are given the information that exactly one of the items produced was acceptable, what is the probability that it was the one produced by the first team? To determine this probability, consider the following reasoning. Given that there was exactly one acceptable item produced, it follows that the outcome of the experiment was either (a, u) or (u, a). Since the outcome (a, u) was initially twice as likely as the outcome (u, a), it should remain twice as likely given the information that one of them occurred. Therefore, the probability that the outcome was (a, u) is $2/3$, whereas the probability that it was (u, a) is $1/3$.

Let $A = \{(a, u), (a, a)\}$ denote the event that the item produced by the first team is acceptable, and let $B = \{(a, u), (u, a)\}$ be the event that exactly one of the produced items is acceptable. The probability that the item produced by the first team was acceptable given that exactly one of the produced items was acceptable is called the *conditional probability* of A given that B has occurred; this is denoted as

$$P(A|B).$$

A general formula for $P(A|B)$ is obtained by an argument similar to the one given in the preceding. Namely, if the event B occurs then, in order for the event A to occur, it is necessary that the occurrence be a point in both A and B; that is, it must be in AB. Now, since we know that B has occurred, it follows that B can be thought of as the new sample space, and hence the probability that the event AB occurs will equal the probability of AB relative to the probability of B. That is,

$$P(A|B) = \frac{P(AB)}{P(B)}. \tag{1.4}$$

Example 1.2a A coin is flipped twice. Assuming that all four points in the sample space $S = \{(h, h), (h, t), (t, h), (t, t)\}$ are equally likely, what is the conditional probability that both flips land on heads, given that

(a) the first flip lands on heads, and
(b) at least one of the flips lands on heads?

Solution. Let $A = \{(h, h)\}$ be the event that both flips land on heads; let $B = \{(h, h), (h, t)\}$ be the event that the first flip lands on heads; and let $C = \{(h, h), (h, t), (t, h)\}$ be the event that at least one of the flips lands on heads. We have the following solutions:

$$\begin{aligned} P(A|B) &= \frac{P(AB)}{P(B)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t)\})} \\ &= \frac{1/4}{2/4} \\ &= 1/2 \end{aligned}$$

and

$$\begin{aligned} P(A|C) &= \frac{P(AC)}{P(C)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t),(t,h)\})} \\ &= \frac{1/4}{3/4} \\ &= 1/3. \end{aligned}$$

Many people are initially surprised that the answers to parts (a) and (b) are not identical. To understand why the answers are different, note first that – conditional on the first flip landing on heads – the second one is still equally likely to land on either heads or tails, and so the probability in part (a) is $1/2$. On the other hand, knowing that at least one of the flips lands on heads is equivalent to knowing that the outcome is not (t, t). Thus, given that at least one of the flips lands on heads, there remain three equally likely possibilities, namely (h, h), (h, t), (t, h), showing that the answer to part (b) is $1/3$. □

It follows from Equation (1.4) that

$$P(AB) = P(B)P(A|B). \tag{1.5}$$

That is, the probability that both A and B occur is the probability that B occurs multiplied by the conditional probability that A occurs given that B occurred; this result is often called the *multiplication theorem of probability*.

Example 1.2b Suppose that two balls are to be withdrawn, without replacement, from an urn that contains 9 blue and 7 yellow balls. If each ball drawn is equally likely to be any of the balls in the urn at the time, what is the probability that both balls are blue?

Solution. Let B_1 and B_2 denote, respectively, the events that the first and second balls withdrawn are blue. Now, given that the first ball withdrawn is blue, the second ball is equally likely to be any of the remaining 15 balls, of which 8 are blue. Therefore, $P(B_2|B_1) = 8/15$. As $P(B_1) = 9/16$, we see that

$$P(B_1B_2) = \frac{9}{16}\frac{8}{15} = \frac{3}{10}. \qquad \square$$

The conditional probability of A given that B has occurred is not generally equal to the unconditional probability of A. In other words, knowing that the outcome of the experment is an element of B generally changes the probability that it is an element of A. (What if A and B are mutually exclusive?) In the special case where $P(A|B)$ is equal to $P(A)$, we say that A is *independent* of B. Since

$$P(A|B) = \frac{P(AB)}{P(B)},$$

we see that A is independent of B if

$$P(AB) = P(A)P(B). \tag{1.6}$$

The relation in (1.6) is symmetric in A and B. Thus it follows that, whenever A is independent of B, B is also independent of A – that is, A and B are *independent events.*

Example 1.2c Suppose that, with probability .52, the closing price of a stock is at least as high as the close on the previous day, and that the results for succesive days are independent. Find the probability that the closing price goes down in each of the next four days, but not on the following day.

Solution. Let A_i be the event that the closing price goes down on day i. Then, by independence, we have

$$\begin{aligned} P(A_1A_2A_3A_4A_5^c) &= P(A_1)P(A_2)P(A_3)P(A_4)P(A_5^c) \\ &= (.48)^4(.52) = .0276. \end{aligned} \qquad \square$$

1.3 Random Variables and Expected Values

Numerical quantities whose values are determined by the outcome of the experiment are known as *random variables*. For instance, the sum obtained when rolling dice, or the number of heads that result in a series of coin flips, are random variables. Since the value of a random variable is determined by the outcome of the experiment, we can assign probabilities to each of its possible values.

Example 1.3a Let the random variable X denote the sum when a pair of fair dice are rolled. The possible values of X are $2, 3, \ldots, 12$, and they have the following probabilities:

$$
\begin{aligned}
P\{X=2\} &= P\{(1,1)\} = 1/36,\\
P\{X=3\} &= P\{(1,2),(2,1)\} = 2/36,\\
P\{X=4\} &= P\{(1,3),(2,2),(3,1)\} = 3/36,\\
P\{X=5\} &= P\{(1,4),(2,3),(3,2),(4,1)\} = 4/36,\\
P\{X=6\} &= P\{(1,5),(2,4),(3,3),(4,2),(5,1)\} = 5/36,\\
P\{X=7\} &= P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = 6/36,\\
P\{X=8\} &= P\{(2,6),(3,5),(4,4),(5,3),(6,2)\} = 5/36,\\
P\{X=9\} &= P\{(3,6),(4,5),(5,4),(6,3)\} = 4/36,\\
P\{X=10\} &= P\{(4,6),(5,5),(6,4)\} = 3/36,\\
P\{X=11\} &= P\{(5,6),(6,5)\} = 2/36,\\
P\{X=12\} &= P\{(6,6)\} = 1/36.
\end{aligned}
\qquad \square
$$

If X is a random variable whose possible values are $x_1, x_2, \ldots, x_n$, then the set of probabilities $P\{X = x_j\}$ $(j = 1, \ldots, n)$ is called the *probability distribution* of the random variable. Since X must assume one of these values, it follows that

$$\sum_{j=1}^{n} P\{X = x_j\} = 1.$$

Definition If X is a random variable whose possible values are $x_1, x_2, \ldots, x_n$, then the *expected value* of X, denoted by $E[X]$, is defined by

$$E[X] = \sum_{j=1}^{n} x_j P\{X = x_j\}.$$

Alternative names for $E[X]$ are the *expectation* or the *mean* of X.

In words, $E[X]$ is a weighted average of the possible values of X, where the weight given to a value is equal to the probability that X assumes that value.

Example 1.3b Let the random variable X denote the amount that we win when we make a certain bet. Find $E[X]$ if there is a 60% chance that we lose 1, a 20% chance that we win 1, and a 20% chance that we win 2.

Solution.

$$E[X] = -1(.6) + 1(.2) + 2(.2) = 0.$$

Thus, the expected amount that is won on this bet is equal to 0. A bet whose expected winnings is equal to 0 is called a *fair* bet. □

Example 1.3c A random variable X, which is equal to 1 with probability p and to 0 with probability $1 - p$, is said to be a *Bernoulli* random variable with parameter p. Its expected value is

$$E[X] = 1(p) + 0(1 - p) = p. \qquad \square$$

A useful and easily established result is that, for constants a and b,

$$E[aX + b] = aE[X] + b. \tag{1.7}$$

To verify Equation (1.7), let $Y = aX + b$. Since Y will equal $ax_j + b$ when $X = x_j$, it follows that

$$\begin{aligned}
E[Y] &= \sum_{j=1}^{n} (ax_j + b) P\{X = x_j\} \\
&= \sum_{j=1}^{n} ax_j P\{X = x_j\} + \sum_{j=1}^{n} bP\{X = x_j\} \\
&= a \sum_{j=1}^{n} x_j P\{X = x_j\} + b \sum_{j=1}^{n} P\{X = x_j\} \\
&= aE[X] + b.
\end{aligned}$$

An important result is that the expected value of a sum of random variables is equal to the sum of their expected values.

Proposition 1.3.1 *For random variables $X_1, \ldots, X_k$,*

$$E\left[\sum_{j=1}^{k} X_j\right] = \sum_{j=1}^{k} E[X_j].$$

Example 1.3d Consider n independent trials, each of which is a success with probability p. The random variable X, equal to the total number of successes that occur, is called a *binomial* random variable with parameters n and p. We can determine its expectation by using the representation

$$X = \sum_{j=1}^{n} X_j,$$

where X_j is defined to equal 1 if trial j is a success and to equal 0 otherwise. Using Proposition 1.3.1, we obtain that

$$E[X] = \sum_{j=1}^{n} E[X_j] = np,$$

where the final equality used the result of Example 1.3c. □

The random variables $X_1, \ldots, X_n$ are said to be *independent* if probabilities concerning any subset of them are unchanged by information as to the values of the others.

Example 1.3e Suppose that k balls are to be randomly chosen from a set of N balls, of which n are red. If we let X_i equal 1 if the ith ball chosen is red and 0 if it is black, then $X_1, \ldots, X_n$ would be independent if each selected ball is replaced before the next selection is made, but they would not be independent if each selection is made without replacing previously selected balls. (Why not?) □

Whereas the average of the possible values of X is indicated by its expected value, its spread is measured by its variance.

Definition The *variance* of X, denoted by $\text{Var}(X)$, is defined by

$$\text{Var}(X) = E[(X - E[X])^2].$$

In other words, the variance measures the average square of the difference between X and its expected value.

Example 1.3f Find $\text{Var}(X)$ when X is a Bernoulli random variable with parameter p.

Solution. Because $E[X] = p$ (as shown in Example 1.3c), we see that

$$(X - E[X])^2 = \begin{cases} (1-p)^2 & \text{with probability } p \\ p^2 & \text{with probability } 1-p. \end{cases}$$

Hence,

$$\begin{aligned} \text{Var}(X) &= E[(X - E[X])^2] \\ &= (1-p)^2 p + p^2(1-p) \\ &= p - p^2. \end{aligned}$$

□

If a and b are constants, then

$$\begin{aligned} \text{Var}(aX+b) &= E[(aX+b-E[aX+b])^2] \\ &= E[(aX - aE[X])^2] \qquad \text{(by Equation (1.7))} \\ &= E[a^2(X - E[X])^2] \\ &= a^2\,\text{Var}(X). \end{aligned} \tag{1.8}$$

Although it is not generally true that the variance of the sum of random variables is equal to the sum of their variances, this *is* the case when the random variables are independent.

Proposition 1.3.2 *If $X_1, \ldots, X_k$ are independent random variables, then*

$$\text{Var}\left(\sum_{j=1}^{k} X_j\right) = \sum_{j=1}^{k} \text{Var}(X_j).$$

Example 1.3g Find the variance of X, a binomial random variable with parameters n and p.

Solution. Recalling that X represents the number of successes in n *independent* trials (each of which is a success with probability p), we can represent it as

$$X = \sum_{j=1}^{n} X_j,$$

where X_j is defined to equal 1 if trial j is a success and 0 otherwise. Hence,

$$\begin{aligned}\mathrm{Var}(X) &= \sum_{j=1}^{n} \mathrm{Var}(X_j) \qquad \text{(by Proposition 1.3.2)} \\ &= \sum_{j=1}^{n} p(1-p) \qquad \text{(by Example 1.3f)} \\ &= np(1-p).\end{aligned}$$ □

The square root of the variance is called the *standard deviation*. As we shall see, a random variable tends to lie within a few standard deviations of its expected value.

1.4 Covariance and Correlation

The covariance of any two random variables X and Y, denoted by $\mathrm{Cov}(X, Y)$, is defined by

$$\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])].$$

Upon multiplying the terms within the expectation, and then taking expectation term by term, it can be shown that

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

A positive value of the covariance indicates that X and Y both tend to be large at the same time, whereas a negative value indicates that when one is large the other tends to be small. (Independent random variables have covariance equal to 0.)

Example 1.4a Let X and Y both be Bernoulli random variables. That is, each takes on either the value 0 or 1. Using the identity

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

and noting that XY will equal 1 or 0 depending upon whether both X and Y are equal to 1, we obtain that

$$\text{Cov}(X, Y) = P\{X=1,\ Y=1\} - P\{X=1\}P\{Y=1\}.$$

From this, we see that

$$\begin{aligned}
\text{Cov}(X, Y) > 0 \iff & P\{X=1,\ Y=1\} > P\{X=1\}P\{Y=1\} \\
\iff & \frac{P\{X=1,\ Y=1\}}{P\{X=1\}} > P\{Y=1\} \\
\iff & P\{Y=1 \mid X=1\} > P\{Y=1\}.
\end{aligned}$$

That is, the covariance of X and Y is positive if the outcome that $X = 1$ makes it more likely that $Y = 1$ (which, as is easily seen, also implies the reverse). □

The following properties of covariance are easily established. For random variables X and Y, and constant c:

$$\begin{aligned}
\text{Cov}(X, Y) &= \text{Cov}(Y, X), \\
\text{Cov}(X, X) &= \text{Var}(X), \\
\text{Cov}(cX, Y) &= c\,\text{Cov}(X, Y), \\
\text{Cov}(c, Y) &= 0.
\end{aligned}$$

Covariance, like expected value, satisfies a linearity property – namely,

$$\text{Cov}(X_1 + X_2, Y) = \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y). \tag{1.9}$$

Equation (1.9) is proven as follows:

$$\begin{aligned}
\text{Cov}(X_1 + X_2, Y) &= E[(X_1 + X_2)Y] - E[X_1 + X_2]E[Y] \\
&= E[X_1Y + X_2Y] - (E[X_1] + E[X_2])E[Y] \\
&= E[X_1Y] - E[X_1]E[Y] + E[X_2Y] - E[X_2]E[Y] \\
&= \text{Cov}(X_1, Y) + \text{Cov}(X_2, Y).
\end{aligned}$$

Equation (1.9) is easily generalized to yield the following useful identity:

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j). \tag{1.10}$$

Equation (1.10) yields a useful formula for the variance of the sum of random variables:

$$\begin{aligned} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) &= \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} \mathrm{Cov}(X_i, X_i) + \sum_{i=1}^{n} \sum_{j \neq i} \mathrm{Cov}(X_i, X_j) \\ &= \sum_{i=1}^{n} \mathrm{Var}(X_i) + \sum_{i=1}^{n} \sum_{j \neq i} \mathrm{Cov}(X_i, X_j). \end{aligned} \tag{1.11}$$

The degree to which large values of X tend to be associated with large values of Y is measured by the *correlation* between X and Y, denoted as $\rho(X, Y)$ and defined by

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

It can be shown that

$$-1 \leq \rho(X, Y) \leq 1.$$

If X and Y are linearly related by the equation

$$Y = a + bX,$$

then $\rho(X, Y)$ will equal 1 when b is positive and -1 when b is negative.

1.5 Exercises

Exercise 1.1 When typing a report, a certain typist makes i errors with probability p_i $(i \geq 0)$, where

$$p_0 = .20, \quad p_1 = .35, \quad p_2 = .25, \quad p_3 = .15.$$

What is the probability that the typist makes

(a) at least four errors;
(b) at most two errors?

Exercise 1.2 A family picnic scheduled for tomorrow will be postponed if it is either cloudy or rainy. If the probability that it will be cloudy is .40, the probability that it will be rainy is .30, and the probability that it will be both rainy and cloudy is .20, what is the probabilty that the picnic will not be postponed?

Exercise 1.3 If two people are randomly chosen from a group of eight women and six men, what is the probability that

(a) both are women;
(b) both are men;
(c) one is a man and the other a woman?

Exercise 1.4 A club has 120 members, of whom 35 play chess, 58 play bridge, and 27 play both chess and bridge. If a member of the club is randomly chosen, what is the conditional probability that she

(a) plays chess given that she plays bridge;
(b) plays bridge given that she plays chess?

Exercise 1.5 Cystic fibrosis (CF) is a genetically caused disease. A child that receives a CF gene from each of its parents will develop the disease either as a teenager or before, and will not live to adulthood. A child that receives either zero or one CF gene will not develop the disease. If an individual has a CF gene, then each of his or her children will independently receive that gene with probability $1/2$.

(a) If both parents possess the CF gene, what is the probability that their child will develop cystic fibrosis?
(b) What is the probability that a 30-year old who does not have cystic fibrosis, but whose sibling died of that disease, possesses a CF gene?

Exercise 1.6 Two cards are randomly selected from a deck of 52 playing cards. What is the conditional probability they are both aces, given that they are of different suits?

Exercise 1.7 If A and B are independent, show that so are

(a) A and B^c;
(b) A^c and B^c.

Exercise 1.8 A gambling book recommends the following strategy for the game of roulette. It recommends that the gambler bet 1 on red. If red appears (which has probability $18/38$ of occurring) then the gambler should take his profit of 1 and quit. If the gambler loses this bet, he should then make a second bet of size 2 and then quit. Let X denote the gambler's winnings.

(a) Find $P\{X > 0\}$.
(b) Find $E[X]$.

Exercise 1.9 Four buses carrying 152 students from the same school arrive at a football stadium. The buses carry (respectively) 39, 33, 46, and 34 students. One of the 152 students is randomly chosen. Let X denote the number of students who were on the bus of the selected student. One of the four bus drivers is also randomly chosen. Let Y be the number of students who were on that driver's bus.

(a) Which do you think is larger, $E[X]$ or $E[Y]$?
(b) Find $E[X]$ and $E[Y]$.

Exercise 1.10 Two players play a tennis match, which ends when one of the players has won two sets. Suppose that each set is equally likely to be won by either player, and that the results from different sets are independent. Find (a) the expected value and (b) the variance of the number of sets played.

Exercise 1.11 Verify that

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

Hint: Starting with the definition

$$\mathrm{Var}(X) = E[(X - E[X])^2],$$

square the expression on the right side; then use the fact that the expected value of a sum of random variables is equal to the sum of their expectations.

Exercise 1.12 A lawyer must decide whether to charge a fixed fee of $5,000 or take a contingency fee of $25,000 if she wins the case (and 0 if she loses). She estimates that her probability of winning is .30. Determine the mean and standard deviation of her fee if

(a) she takes the fixed fee;
(b) she takes the contingency fee.

Exercise 1.13 Let $X_1, \ldots, X_n$ be independent random variables, all having the same distribution with expected value μ and variance σ^2. The random variable $\bar{X}$, defined as the arithmetic average of these variables, is called the *sample mean*. That is, the sample mean is given by

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}.$$

(a) Show that $E[\bar{X}] = \mu$.
(b) Show that $\mathrm{Var}(\bar{X}) = \sigma^2/n$.

The random variable S^2, defined by

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n - 1},$$

is called the *sample variance.*

(c) Show that $\sum_{i=1}^{n} (X_i - \bar{X})^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$.
(d) Show that $E[S^2] = \sigma^2$.

Exercise 1.14 Verify that

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y].$$

Exercise 1.15 Prove:

(a) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$;
(b) $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$;
(c) $\mathrm{Cov}(cX, Y) = c\,\mathrm{Cov}(X, Y)$;
(d) $\mathrm{Cov}(c, Y) = 0$.

Exercise 1.16 If U and V are independent random variables, both having variance 1, find $\mathrm{Cov}(X, Y)$ when

$$X = aU + bV, \qquad Y = cU + dV.$$

Exercise 1.17 If $\mathrm{Cov}(X_i, X_j) = ij$, find

(a) $\mathrm{Cov}(X_1 + X_2,\ X_3 + X_4)$;
(b) $\mathrm{Cov}(X_1 + X_2 + X_3,\ X_2 + X_3 + X_4)$.

Exercise 1.18 Suppose that – in any given time period – a certain stock is equally likely to go up 1 unit or down 1 unit, and that the outcomes of different periods are independent. Let X be the amount the stock goes up (either 1 or -1) in the first period, and let Y be the cumulative amount it goes up in the first three periods. Find the correlation between X and Y.

Exercise 1.19 Can you construct a pair of random variables such that $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$ and $\mathrm{Cov}(X, Y) = 2$?

REFERENCE

[1] Ross, S. M. (2002). *A First Course in Probability*, 6th ed. Englewood Cliffs, NJ: Prentice-Hall.