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Part I

The SEMTSA approach

1 Time series analysis and simultaneous equation econometric models (1974)

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1 Introduction

In this chapter we take up the analysis of dynamic simultaneous equation models (SEMs) within the context of general linear multiple time series processes such as studied by Quenouille (1957). As noted by Quenouille, if a set of variables is generated by a multiple time series process, it is often possible to solve for the processes generating individual variables, namely the "final equations" of Tinbergen (1940), and these are in the autoregressive-moving average (ARMA) form. ARMA processes have been studied intensively by Box and Jenkins (1970). Further, if a general multiple time series process is appropriately specialized, we obtain a usual dynamic SEM in structural form. By algebraic manipulations, the associated reduced form and transfer function equation systems can be derived. In what follows, these equation systems are presented and their properties and uses are indicated.

It will be shown that assumptions about variables being exogenous, about lags in structural equations of SEMs, and about serial correlation properties of structural disturbance terms have strong implications for the properties of transfer functions and final equations that can be tested. Further, we show how large sample posterior odds and likelihood ratios can be used to appraise alternative hypotheses. In agreement with Pierce and Mason (1971), we believe that testing the implications of structural assumptions for transfer functions and, we add, final equations is an important element in the process of iterating in on a model that is reasonably in accord with the information in a sample of data. To illustrate these general points and to provide applications of the above methods,

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a dynamic version of a SEM due to Haavelmo (1947) is analyzed using US post-Second World War quarterly data.

The plan of the chapter is as follows. In section 2, a general multiple time series model is specified, its final equations are obtained, and their properties set forth. Then the implications of assumptions needed to specialize the multiple time series model to become a dynamic SEM for transfer functions and final equations are presented. In section 3, the algebraic analysis is applied to a small dynamic SEM. Quarterly US data are employed in sections 4 and 5 to analyze the final and transfer equations of the dynamic SEM. Section 6 provides a discussion of the empirical results, their implications for the specification and estimation of the structural equations of the model, and some concluding remarks.

2 General formulation and analysis of a system of dynamic equations

As indicated by Quenouille (1957), a linear multiple time series process can be represented as follows:[1]

$$\underset{p\times p}{H(L)}\ \underset{p\times 1}{\boldsymbol{z}_t} = \underset{p\times p}{F(L)}\ \underset{p\times 1}{\boldsymbol{e}_t}, \quad t = 1, 2, \ldots, T, \tag{2.1}$$

where $\boldsymbol{z}_t' = (z_{1t}, z_{2t}, \ldots, z_{pt})$ is a vector of random variables, $\boldsymbol{e}_t' = (e_{1t}, e_{2t}, \ldots, e_{pt})$ is a vector of random errors, and $H(L)$ and $F(L)$ are each $p \times p$ matrices, assumed of full rank, whose elements are finite polynomials in the lag operator L, defined as $L^n z_t = z_{t-n}$. Typical elements of $H(L)$ and $F(L)$ are given by $h_{ij} = \sum_{l=0}^{r_{ij}} h_{ijl} L^l$ and $f_{ij} = \sum_{l=0}^{q_{ij}} f_{ijl} L^l$. Further, we assume that the error process has a zero mean, an identity covariance matrix and no serial correlation, that is:

$$E\boldsymbol{e}_t = 0, \tag{2.2}$$

for all t and t',

$$E\boldsymbol{e}_t\boldsymbol{e}_{t'}' = \delta_{tt'} I, \tag{2.3}$$

where I is a unit matrix and $\delta_{tt'}$ is the Kronecker delta. The assumption in (2.3) does not involve a loss of generality since correlation of errors can be introduced through the matrix $F(L)$.

The model in (2.1) is a multivariate autoregressive-moving average (ARMA) process. If $H(L) = H_0$, a matrix of degree zero in L, (2.1) is a

[1] In (2.1), $\boldsymbol{z}_t$ is assumed to be mean-corrected, that is $\boldsymbol{z}_t$ is a deviation from a population mean vector. Below, we relax this assumption.

moving average (MA) process; if $F(L) = F_0$, a matrix of degree zero in L, it is an autoregressive (AR) process. In general, (2.1) can be expressed as:

$$\sum_{l=0}^{r} H_l L^l \boldsymbol{z}_t = \sum_{l=0}^{q} F_l L^l \boldsymbol{e}_t, \tag{2.4}$$

where H_l and F_l are matrices with all elements not depending on L, $r = \max_{i,j} r_{ij}$ and $q = \max_{i,j} q_{ij}$.

Since $H(L)$ in (2.1) is assumed to have full rank, (2.1) can be solved for $\boldsymbol{z}_t$ as follows:

$$\boldsymbol{z}_t = H^{-1}(L)F(L)\boldsymbol{e}_t, \tag{2.5a}$$

or

$$\boldsymbol{z}_t = [H^*(L)/|H(L)|]F(L)\boldsymbol{e}_t, \tag{2.5b}$$

where $H^*(L)$ is the adjoint matrix associated with $H(L)$ and $|H(L)|$ is the determinant which is a scalar, finite polynomial in L. If the process is to be invertible, the roots of $|H(L)| = 0$ have to lie outside the unit circle. Then (2.5) expresses $\boldsymbol{z}_t$ as an infinite MA process that can be equivalently expressed as the following system of finite order ARMA equations:

$$|H(L)|\boldsymbol{z}_t = H^*(L)F(L)\boldsymbol{e}_t. \tag{2.6}$$

The ith equation of (2.6) is given by:

$$|H(L)|z_{it} = \boldsymbol{\alpha}_i' \boldsymbol{e}_t, \quad i = 1, 2, \ldots, p, \tag{2.7}$$

where $\boldsymbol{\alpha}_i'$ is the ith row of $H^*(L)F(L)$.

The following points regarding the set of final equations in (2.7) are of interest:

(i) Each equation is in ARMA form, as pointed out by Quenouille (1957, p. 20). Thus the ARMA processes for individual variables are compatible with some, perhaps unknown, joint process for a set of random variables and are thus not necessarily "naive," "ad hoc" alternative models.

(ii) The order and parameters of the autoregressive part of each equation, $|H(L)|\ z_{it}, i = 1, 2, \ldots, p$, will usually be the same.[2]

(iii) Statistical methods can be employed to investigate the form and properties of the ARMA equations in (2.7). Given that their forms, that is the degree of $|H(L)|$ and the order of the moving average

[2] In some cases in which $|H(L)|$ contains factors in common with those appearing in all elements of the vectors α_i', e.g. when H is triangular, diagonal or block diagonal, some cancelling will take place. In such cases the statement in (ii) has to be qualified.

errors, have been determined, they can be estimated and used for prediction.

(iv) The equations of (2.7) are in the form of a restricted "seemingly unrelated" autoregressive model with correlated moving average error processes.[3]

The general multiple time series model in (2.1) can be specialized to a usual dynamic simultaneous equation model (SEM) if some prior information about H and F is available. That is, prior information may indicate that it is appropriate to regard some of the variables in $\boldsymbol{z}_t$ as being endogenous and the remaining variables as being exogenous, that is, generated by an independent process. To represent this situation, we partition (2.1) as follows:

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{x}_t \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_{1t} \\ \boldsymbol{e}_{2t} \end{pmatrix}. \tag{2.8}$$

If the $p_1 \times 1$ vector $\boldsymbol{y}_t$ is endogenous and the $p_2 \times 1$ vector $\boldsymbol{x}_t$ is exogenous, this implies the following restrictions on the submatrices of H and F:

$$H_{21} \equiv 0, \quad F_{21} \equiv 0, \quad \text{and} \quad F_{12} \equiv 0. \tag{2.9}$$

With the assumptions in (2.9), the elements of $\boldsymbol{e}_{1t}$ do not affect the elements of $\boldsymbol{x}_t$ and the elements of $\boldsymbol{e}_{2t}$ affect the elements of $\boldsymbol{y}_t$ only through the elements of $\boldsymbol{x}_t$. Under the hypotheses in (2.9), (2.8) is in the form of a dynamic SEM with endogenous variable vector $\boldsymbol{y}_t$ and exogenous variable vector $\boldsymbol{x}_t$ generated by an ARMA process. The usual structural equations, from (2.8) subject to (2.9), are:[4]

$$\underset{p_1\times p_1}{H_{11}(L)}\ \underset{p_1\times 1}{\boldsymbol{y}_t} + \underset{p_1\times p_2}{H_{12}(L)}\ \underset{p_2\times 1}{\boldsymbol{x}_t} = \underset{p_1\times p_1}{F_{11}(L)}\ \underset{p_1\times 1}{\boldsymbol{e}_{1t}}, \tag{2.10}$$

while the process generating the exogenous variables is:

$$\underset{p_2\times p_2}{H_{22}(L)}\ \underset{p_2\times 1}{\boldsymbol{x}_t} = \underset{p_2\times p_2}{F_{22}(L)}\ \underset{p_2\times 1}{\boldsymbol{e}_{2t}}, \tag{2.11}$$

with $p_1 + p_2 = p$.

Analogous to (2.4), the system (2.10) can be expressed as:

$$\sum_{l=0}^{r} H_{11l} L^l \boldsymbol{y}_t + \sum_{l=0}^{r} H_{12l} L^l \boldsymbol{x}_t = \sum_{l=0}^{q} F_{11l} L^l \boldsymbol{e}_{1t}, \tag{2.12}$$

where H_{11l}, H_{12l} and F_{11l} are matrices the elements of which are coefficients of L^l. Under the assumption that $H_{11\,0}$ is of full rank, the reduced

[3] See Nelson (1970) and Akaike (1973) for estimation results for systems similar to (2.7).
[4] Hannan (1969, 1971) has analysed the identification problem for systems in the form of (2.10).

form equations, which express the current values of endogenous variables as functions of the lagged endogenous and current and lagged exogenous variables, are:

$$\boldsymbol{y}_t = -\sum_{l=1}^{r} H_{110}^{-1} H_{11l} L^l \boldsymbol{y}_t - \sum_{l=0}^{r} H_{110}^{-1} H_{12l} L^l \boldsymbol{x}_t + \sum_{l=0}^{q} H_{110}^{-1} F_{11l} L^l \boldsymbol{e}_{1t}. \tag{2.13}$$

The reduced form system in (2.13) is a system of p_1 stochastic difference equations of maximal order r.

The "final form" of (2.13), Theil and Boot (1962), or "set of fundamental dynamic equations" associated with (2.13), Kmenta (1971), which expresses the current values of endogenous variables as functions of only the exogenous variables, is given by:

$$\boldsymbol{y}_t = -H_{11}^{-1}(L) H_{12}(L) \boldsymbol{x}_t + H_{11}^{-1}(L) F_{11}(L) \boldsymbol{e}_{1t}. \tag{2.14}$$

If the process is invertible, i.e. if the roots of $|H_{11}(L)| = 0$ lie outside the unit circle, (2.14) is an infinite MA process in $\boldsymbol{x}_t$ and $\boldsymbol{e}_{1t}$. Note that (2.14) is a set of "rational distributed lag" equations, Jorgenson (1966), or a system of "transfer function" equations, Box and Jenkins (1970). Also, the system in (2.14) can be brought into the following form:

$$|H_{11}(L)| \boldsymbol{y}_t = -H_{11}^{*}(L) H_{12}(L) \boldsymbol{x}_t + H_{11}^{*}(L) F_{11}(L) \boldsymbol{e}_{1t}, \tag{2.15}$$

where $H_{11}^{*}(L)$ is the adjoint matrix associated with $H_{11}(L)$ and $|H_{11}(L)|$ is the determinant of $H_{11}(L)$. The equation system in (2.15), where each endogenous variable depends only on its own lagged values and on the exogenous variables, with or without lags, has been called the "separated form," Marschak (1950), "autoregressive final form," Dhrymes (1970), "transfer function form," Box and Jenkins (1970), or "fundamental dynamic equations," Pierce and Mason (1971).[5] As in (2.7), the p_1 endogenous variables in $\boldsymbol{y}_t$ have autoregressive parts with identical order and parameters, a point emphasized by Pierce and Mason (1971).

Having presented several equation systems above, it is useful to consider their possible uses and some requirements that must be met for these uses. As noted above, the final equations in (2.7) can be used to predict the future values of some or all variables in $\boldsymbol{z}_t$, given that the forms of the ARMA processes for these variables have been determined and that

[5] If some of the variables in $\boldsymbol{x}_t$ are non-stochastic, say time trends, they will appear the final equations of the system.

parameters have been estimated. However, these final equations cannot be used for control and structural analysis. On the other hand, the reduced form equations (2.13) and transfer equations (2.15) can be employed for both prediction and control but not generally for structural analysis except when structural equations are in reduced form ($H_{110} \equiv I$ in (2.12)) or in final form [$H_{11} \equiv I$ in (2.10)]. Note that use of reduced form and transfer function equations implies that we have enough prior information to distinguish endogenous and exogenous variables. Further, if data on some of the endogenous variables are unavailable, it may be impossible to use the reduced form equations whereas it will be possible to use the transfer equations relating to those endogenous variables for which data are available. When the structural equation system in (2.10) is available, it can be employed for structural analysis and the associated "restricted" reduced form or transfer equations can be employed for prediction and control. Use of the structural system (2.10) implies not only that endogenous and exogenous variables have been distinguished, but also that prior information is available to identify structural parameters and that the dynamic properties of the structural equations have been determined. Also, structural analysis of the complete system in (2.10) will usually require that data be available on all variables.[6] For the reader's convenience, some of these considerations are summarized in table 1.1.

Aside from the differing data requirements for use of the various equation systems considered in table 1.1, it should be appreciated that before each of the equation systems can be employed, the form of its equations must be ascertained. For example, in the case of the structural equation system (2.10), not only must endogenous and exogenous variables be distinguished, but also lag distributions, serial correlation properties of error terms, and identifying restrictions must be specified. Since these are often difficult requirements, it may be that some of the simpler equation systems will often be used although their uses are more limited than those of structural equation systems. Furthermore, even when the objective of an analysis is to obtain a structural equation system, the other equation systems, particularly the final equations and transfer equations, will be found useful. That is, structural assumptions regarding lag structures, etc. have implications for the forms and properties of final and transfer equations that can be checked with data. Such checks on structural assumptions can reveal weaknesses in them and possibly suggest alternative structural assumptions more in accord with the information in the data. In the following sections we illustrate these points in the analysis of a small dynamic structural equation system.

[6] This requirement will not be as stringent for partial analyses and for fully recursive models.

Table 1.1 *Uses and requirements for various equation systems*

Equation system	Uses of equation systems: Prediction	Control	Structural analysis	Requirements for use of equation systems
1. Final equations[a] (2.7)	yes	no	no	Forms of ARMA processes and parameter estimates
2. Reduced form equations (2.13)	yes	yes	no	Endogenous–exogenous classification of variables, forms of equations, and parameter estimates
3. Transfer equations[b] (2.15)	yes	yes	no	Endogenous–exogenous classification of variables, forms of equations, and parameter estimates
4. Final form equations[c] (2.14)	yes	yes	no	Endogenous–exogenous classification of variables, forms of equations, and parameter estimates
5. Structural equations (2.10)	yes	yes	yes	Endogenous–exogenous variable classification, identifying information,[d] forms of equations, and parameter estimates

Notes:
[a] This is Tinbergen's (1940) term.
[b] These equations are also referred to as "separated form" or "autoregressive final form" equations.
[c] As noted in the text, these equations are also referred to as "transfer function," "fundamental dynamic," and "rational distributed lag" equations.
[d] That is, information in the form of restrictions to identify structural parameters.

3 Algebraic analysis of a dynamic version of Haavelmo's model

Haavelmo (1947) formulated and analyzed the following static model with annual data for the United States, 1929–41:

$$c_t = \alpha y_t + \beta + u_t, \tag{3.1a}$$
$$r_t = \mu(c_t + x_t) + v + w_t \tag{3.1b}$$
$$y_t = c_t + x_t - r_t \tag{3.1c}$$

where c_t, y_t and r_t are endogenous variables, x_t is exogenous, u_t and w_t are disturbance terms, and α, β, μ and v are scalar parameters. The

definitions of the variables, all on a price-deflated, *per capita* basis, are:
c_t = personal consumption expenditures,
y_t = personal disposable income,
r_t = gross business saving, and
x_t = gross investment.[7]

Equation (3.1a) is a consumption relation, (3.1b) a gross business saving equation, and (3.1c) an accounting identity.

In Chetty's (1966, 1968) analyzes of the system (3.1) employing Haavelmo's annual data, he found the disturbance terms highly autocorrelated, perhaps indicating that the static nature of the model is not appropriate. In view of this possibility, (3.1) is made dynamic in the following way:

$$c_t = \alpha(L)y_t + \beta + u_t, \tag{3.2a}$$
$$r_t = \mu(L)(c_t + x_t) + v + w_t \tag{3.2b}$$
$$y_t = c_t + x_t - r_t \tag{3.2c}$$

In (3.2a), $\alpha(L)$ is a polynomial lag operator that serves to make c_t a function of current and lagged values of income. Similarly, $\mu(L)$ in (3.2b) is a polynomial lag operator that makes r_t depend on current and lagged values of $c_t + x_t$, a variable that Haavelmo refers to as "gross disposable income." On substituting for r_t in (3.2b) from (3.2c), the equations for c_t and y_t are:

$$c_t = \alpha(L)y_t + \beta + u_t, \tag{3.3a}$$
$$y_t = [1 - \mu(L)](c_t + x_t) - v - w_t. \tag{3.3b}$$

With respect to the disturbance terms in (3.3), we assume:

$$\begin{pmatrix} u_t \\ -w_t \end{pmatrix} = \begin{pmatrix} f_{11}(L) & f_{12}(L) \\ f_{21}(L) & f_{22}(L) \end{pmatrix} \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}, \tag{3.4}$$

where the $f_{ij}(L)$ are polynomials in L, e_{1t} and e_{2t} have zero means, unit variances, and are contemporaneously and serially uncorrelated.

Letting $\boldsymbol{z}_t' = (c_t, y_t, x_t)$, the general multiple time series model for $\boldsymbol{z}_t$, in the matrix form (2.1), is:

$$\underset{3\times 3}{H(L)}\ \underset{3\times 1}{\boldsymbol{z}_t} = \underset{3\times 1}{\boldsymbol{\theta}} + \underset{3\times 3}{F(L)}\ \underset{3\times 1}{\boldsymbol{e}_t}, \tag{3.5}$$

[7] In Haavelmo's paper, gross investment, x_t, is defined equal to "government expenditures + transfers − all taxes + gross private capital formation," while gross business saving, r_t, is defined equal to "depreciation and depletion charges + capital outlay charged to current expense + income credited to other business reserves − revaluation of business inventories + corporate savings".

where $\boldsymbol{e}_t' = (e_{1t}, e_{2t}, e_{3t})$ satisfies (2.2)–(2.3) and $\boldsymbol{\theta}' = (\theta_1, \theta_2, \theta_3)$ is a vector of constants. In explicit form, (3.5) is:

$$\begin{pmatrix} h_{11}(L) & h_{12}(L) & h_{13}(L) \\ h_{21}(L) & h_{22}(L) & h_{23}(L) \\ h_{31}(L) & h_{32}(L) & h_{33}(L) \end{pmatrix} \begin{pmatrix} c_t \\ y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} + \begin{pmatrix} f_{11}(L) & f_{12}(L) & f_{13}(L) \\ f_{21}(L) & f_{22}(L) & f_{23}(L) \\ f_{31}(L) & f_{32}(L) & f_{33}(L) \end{pmatrix} \begin{pmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{pmatrix}. \tag{3.6}$$

To specialize (3.6) to represent the dynamic version of Haavelmo's model in (3.3) with x_t exogenous, we must have $\theta_1 = \beta, \theta_2 = v$,

$$\begin{array}{lll} h_{11}(L) \equiv 1 & h_{12}(L) \equiv -\alpha(L) & h_{13}(L) \equiv 0 \\ h_{21}(L) \equiv -[1-\mu(L)] & h_{22}(L) \equiv 1 & h_{23}(L) \equiv -[1-\mu(L)] \\ h_{31}(L) \equiv 0 & h_{32}(L) \equiv 0 & h_{33}(L) \end{array} \tag{3.7a}$$

and

$$f_{13}(L) \equiv f_{23}(L) \equiv f_{31}(L) \equiv f_{32}(L) \equiv 0. \tag{3.7b}$$

Utilizing the conditions in (3.7), (3.6) becomes:

$$\begin{bmatrix} 1 & h_{12}(L) & 0 \\ h_{21}(L) & 1 & h_{23}(L) \\ 0 & 0 & h_{33}(L) \end{bmatrix} \begin{bmatrix} c_t \\ y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} f_{11}(L) & f_{12}(L) & 0 \\ f_{21}(L) & f_{22}(L) & 0 \\ 0 & 0 & f_{33}(L) \end{bmatrix} \begin{bmatrix} e_{1t} \\ e_{2t} \\ e_{3t} \end{bmatrix}. \tag{3.8}$$

Note that the process on the exogenous variable is $h_{33}(L)x_t = f_{33}(L)e_{3t} + \theta_3$ and the fact that x_t is assumed exogenous requires that $h_{31}(L) \equiv h_{32}(L) \equiv 0$ and that $F(L)$ be block diagonal as shown in (3.8).

In what follows, we shall denote the degree of $h_{ij}(L)$ by r_{ij} and the degree of $f_{ij}(L)$ by q_{ij}.

From (3.8), the final equations for c_t and y_t are given by:

$$\begin{aligned}(1 - h_{12}h_{21})h_{33}c_t = \theta_1' &+ (f_{11} - f_{21}h_{12})h_{33}e_{1t} \\ &+ (f_{12} - f_{22}h_{12})h_{33}e_{2t} + f_{33}h_{12}h_{23}e_{3t}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}(1 - h_{12}h_{21})h_{33}y_t = \theta_2' &+ (f_{21} - f_{11}h_{21})h_{33}e_{1t} \\ &+ (f_{22} - f_{12}h_{21})h_{33}e_{2t} - f_{33}h_{23}e_{3t},\end{aligned} \tag{3.10}$$