General Introduction and Outline

The foundations of the differential geometry of curves and surfaces were laid in the early part of the nineteenth century with the monumental works of Monge (1746–1818) and Gauss (1777–1855). Monge's major contributions were collected in his *Applications de l'Analyse à la Géometrie* published in 1807. The 1850 edition of that work is of particular value in that it includes an annotation by Liouville (1809–1882) detailing additional contributions to the subject by such luminaries as Frenet (1816–1888), Serret (1819–1885), Bertrand (1822–1900) and Saint-Venant (1796–1886), whose work in geometry was motivated by his interest in elasticity. Gauss' treatise on the geometry of surfaces, instigated by a geodetic study sponsored by the Elector of Hanover, was the *Disquisitiones Generales Circa Superficies Curvas* published in 1828. Therein, Gauss set down the system of equations that bears his name and which time has shown to be fundamental to the analysis of surfaces. Indeed, this Gauss system and the symmetries that it admits for privileged classes of surfaces underpin the remarkable connection between classical differential geometry and modern soliton theory to be the subject of this monograph.

The origins of soliton theory are likewise to be found in the early part of the nineteenth century. Thus, it was in 1834 that the Scottish engineer John Scott Russell recorded the first sighting, along a canal near Edinburgh, of the solitary hump-shaped wave to be rediscovered in 1965 in the context of the celebrated Fermi-Pasta-Ulam problem by Kruskal and Zabusky and termed a *soliton*. Scott Russell observed that his so-called *great wave of translation* proceeded with a speed proportional to its height. In a vivid account of water tank experiments set up to reproduce this large amplitude surface phenomenon, and described in a report to the British Association in 1844, there is also depicted the creation of two such waves. However, the limited duration of Scott Russell's experiments apparently did not allow him to observe the dramatic interaction properties of these waves in their entirety. Moreover, at that time, neither the nonlinear evolution equation descriptive of their propagation nor the analytic means to predict their interaction properties were to hand.

It was in 1895 that two Dutch mathematicians, Korteweg and de Vries, derived the nonlinear wave equation which now bears their name and adopts the canonical form

$$u_t + u_{xxx} + 6uu_x = 0. \tag{0.1}$$

This models long wave propagation in a rectangular channel and provides, through a simple travelling wave solution, a theoretical confirmation of the existence of the controversial solitary wave observed some sixty years earlier by Scott Russell on the Union canal. However, it is less well-known that what is now called the Korteweg–de Vries (KdV) equation had, in fact, been set down earlier by Boussinesq in his memoir of 1877 entitled *Essai sur la Théorie des Eaux Courantes*. Indeed, a pair of equations equivalent to the KdV equation (0.1) appeared as early as 1871 in two papers by Boussinesq devoted to wave propagation in rectangular channels.

The KdV equation was to be rediscovered in the mid-twentieth century by Gardner and Morikawa in 1960 in an analysis of the transmission of hydromagnetic waves. It has since been shown to be a canonical model for a rich diversity of large amplitude wave systems arising in the theory of solids, liquids and gases.

The advent of modern soliton theory was heralded in 1965 by the rediscovery of the KdV equation in the context of the celebrated Fermi-Pasta-Ulam problem. Thus, in a pioneering study by Kruskal and Zabusky, the KdV equation was obtained as a continuum limit of an anharmonic lattice model with cubic nonlinearity. The existence of solitary waves in this nonlinear model which possess the remarkable property that they preserve both their amplitude and speed subsequent upon interaction was revealed via a computational study. The term *soliton* was coined to describe such waves which had originally been observed in a hydrodynamic context by Scott Russell. However, the problem of obtaining an analytical expression descriptive of the interaction of solitons still remained.

It turns out that, remarkably, a generic method for the description of soliton interaction has its roots in a type of transformation originally introduced by Bäcklund in the nineteenth century to generate pseudospherical surfaces, that is, surfaces of constant negative Gaussian curvature $\mathcal{K} = -1/\rho^2$. The study of such surfaces goes back at least to Edmond Bour in 1862, who generated the celebrated sine-Gordon equation

$$\omega_{uv} = \frac{1}{\rho^2} \sin \omega \tag{0.2}$$

from the Gauss-Mainardi-Codazzi system for pseudospherical surfaces parametrised in terms of asymptotic coordinates. The sine-Gordon equation was subsequently rederived independently by both Bonnet in 1867 and Enneper in 1868 in a similar manner.

A purely geometric construction for pseudospherical surfaces was reformulated in mathematical terms as a transformation by Bianchi in 1879. In 1882, Bäcklund published details of his celebrated transformation $\mathbb{B}_\sigma$ which allows the iterative construction of pseudospherical surfaces. In 1883, Lie presented the decomposition $\mathbb{B}_\sigma = \mathbb{L}_\sigma^{-1}\mathbb{B}_{\pi/2}\mathbb{L}_\sigma$ which shows that the Bäcklund transformation $\mathbb{B}_\sigma$, in fact, represents a conjugation of Lie transformations $\mathbb{L}_\sigma, \mathbb{L}_\sigma^{-1}$ with the parameter-independent Bianchi transformation $\mathbb{B}_{\pi/2}$. Thus, the Lie transformations serve to intrude the key parameter σ into the original Bianchi transformation.

In 1892, under the title *Sulla Trasformazione di Bäcklund per le Superficie Pseudosferiche*, in a masterly breakthrough, Bianchi demonstrated that the Bäcklund transformation $\mathbb{B}_\sigma$ admits a commutativity property $\mathbb{B}_{\sigma_2}\mathbb{B}_{\sigma_1} = \mathbb{B}_{\sigma_1}\mathbb{B}_{\sigma_2}$ a consequence of which is a nonlinear superposition principle embodied in what is termed a *permutability theorem*. The evidence that Bianchi's permutability theorem has important application in nonlinear physics had to await the work of Seeger et al. in 1953 on crystal dislocations. Therein, in the context of Frenkel and Kontorova's dislocation theory of 1938, the superposition of so-called *eigenmotions* was obtained via the classical permutability theorem. Indeed, the interaction of what today is called a breather with a kink-type dislocation was both described analytically by means of the permutability theorem and displayed graphically. The typical solitonic features to be later discovered numerically in 1965 for the KdV equation, namely, preservation of velocity and shape following interaction, as well as the concomitant phase shift, were all derived by means of the permutability theorem for the sine-Gordon equation in this remarkable paper.

In 1958, Skyrme derived a higher-dimensional sine-Gordon equation in a nonlinear theory of particle interaction, while in 1965 the same equation was set down by Josephson in his seminal study of the tunnelling phenomenon in superconductivity for which he was later to gain the Nobel Prize. In 1967, Lamb derived the classical sine-Gordon equation in an analysis of the propagation of ultrashort light pulses. Lamb, aware of the earlier work of Seeger et al., exploited the permutability theorem associated with the Bäcklund transformation to generate an analytic expression for pulse decomposition corresponding to the two-soliton solution. Later, in 1971, he used the permutability theorem to analyse the decomposition of $2N\pi$ light pulses into N stable 2π pulses. The experimental evidence for such a decomposition phenomenon had been provided by Gibbs and Slusher in 1970, who recorded the decomposition of a 6π pulse into three 2π pulses in a Rb vapour. In the same year, Scott had noted how the permutability theorem may also be exploited in the study of long Josephson junctions.

In 1973, Wahlquist and Estabrook demonstrated that the KdV equation, like the sine-Gordon equation, admits invariance under a Bäcklund-type transformation and moreover possesses an associated permutability theorem. The novel pulse interaction properties observed by Zabusky and Kruskal in their original numerical study of the KdV equation are captured analytically in the multi-soliton solutions generated by iterative application of this permutability theorem.

In 1974, a Bäcklund transformation for the nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + \nu q^2|q| = 0 \tag{0.3}$$

was constructed by Lamb via a classical method developed by Clairin in 1910. A nonlinear superposition principle may again be constructed by means of the Bäcklund transformation. The NLS equation has important applications in fibre optics. It seems to have been first set down independently by Kelley and Talanov in 1965 in studies of the self-focusing of optical beams in nonlinear Kerr media. Subsequently, in 1968, Zakharov derived the NLS equation in a study of deep water gravity waves. Hasimoto, in 1971, obtained the same equation in an approximation to the hydrodynamical motion of a thin isolated vortex filament. Implicit was a geometric derivation of the NLS equation wherein it is associated with a motion of an inextensible curve in $\mathbb{R}^3$. This association of an integrable equation with the spatial motion of an inextensible curve will arise naturally in our study of the geometry of solitons.

Thus, by 1974, the Bäcklund transformations for the canonical soliton equations (0.1)–(0.3) were all in place and in that year a National Science Foundation meeting was convened at Vanderbilt University in the USA to assess the status and potential role of Bäcklund transformations in soliton theory. In 1973, the celebrated generalised ZS-AKNS spectral system had been introduced by Ablowitz et al. A broad spectrum of 1+1-dimensional nonlinear evolution equations amenable to the Inverse Scattering Transform (IST) can be encapsulated as compatibility conditions for this ZS-AKNS system. The latter was exploited by Chen to derive auto-Bäcklund transformations for (0.1)–(0.3) in an elegant manner.

The linear structure of the ZS-AKNS system permits the application in soliton theory of another important class of transformations with their origin in the nineteenth century, namely, Darboux transformations. The latter arose in a study by Darboux in 1882 of Sturm-Liouville problems. However, they are but a special case of transformations due to Moutard and introduced earlier in 1878 in connection with the sequential reduction of linear hyperbolic equations to canonical form. Iterated Darboux transformations were constructed by Crum in 1955 in connection with related Sturm-Liouville problems. In 1975, the Crum transformation was taken up by Wadati et al. and used to generate multi-soliton solutions of integrable equations associated with the ZS-AKNS system. In geometric terms, these iterated versions of Darboux transformations occur in the classical theory of surfaces as Levy sequences as described in Eisenhart's *Transformations of Surfaces*.

In 1976, Lund and Regge, en route to the celebrated solitonic system which bears their name, made the crucial observation that the ZS-AKNS system for the sine-Gordon equation is nothing but a 2×2 representation of the classical Gauss-Weingarten system for pseudospherical surfaces. This connection was made independently in the same year by Pohlmeyer.

Thus, by 1976, it was clear that Bäcklund and Darboux transformations, with their origins in the classical differential geometry of surfaces, have deep connections with soliton theory. The aim of the present monograph is to bring together these strands and to give an account not only of their historical connections, but also of modern advances. It builds upon the complementary earlier monograph by Rogers and Shadwick (1982), which presented a non-geometric account of Bäcklund transformations and their applications in soliton theory and continuum mechanics. The geometric viewpoint in this monograph is inspired in many respects by the work of Antoni Sym published in 1981 under the title *Soliton Theory is Surface Theory*. It is the exploration of this theme that, in part, motivated the present work.

Chapter 1 presents an account of the connection between the classical Bäcklund transformation and its variants and modern soliton theory. It opens with the derivation of a classical nonlinear system due to Bianchi which embodies the Gauss-Mainardi-Codazzi equations for hyperbolic surfaces described in asymptotic coordinates. Specialisation to pseudospherical surfaces produces the celebrated sine-Gordon equation. There follows, in Section 1.2, a description of the geometric procedure for the construction of pseudospherical surfaces along with the derivation of the induced auto-Bäcklund transformation for the sine-Gordon equation. In Section 1.3, Bianchi's permutability theorem is derived via this Bäcklund transformation, and a lattice is introduced whereby multi-soliton solutions may be generated in a purely algebraic manner. Pseudospherical surfaces corresponding to one- and two-soliton solutions of the sine-Gordon equation are constructed in Section 1.4. Thus, the stationary single soliton solution is seen to correspond to the pseudosphere, while the non-stationary soliton leads to the Dini surface, namely the helicoid generated by simultaneous rotation and translation of Huygen's tractrix. The two-soliton solution is obtained via the permutability theorem, and pseudospherical surfaces corresponding to entrapped periodic solutions known as breathers are presented. In Section 1.5, it is shown that the Bäcklund transformation for surfaces parallel to pseudospherical surfaces may be induced in a straightforward manner. This extends the action of the classical Bäcklund transformation to a class of Weingarten surfaces. The chapter concludes with a treatment of another important class of surfaces which have a solitonic connection, namely that which bears the name of Bianchi. This class is determined by the system of equations

$$\begin{gathered}
a_v + \frac{1}{2}\frac{\rho_v}{\rho}a - \frac{1}{2}\frac{\rho_u}{\rho}b\cos\omega = 0,\\
b_u + \frac{1}{2}\frac{\rho_u}{\rho}b - \frac{1}{2}\frac{\rho_v}{\rho}a\cos\omega = 0,\\
\omega_{uv} + \frac{1}{2}\left(\frac{\rho_u}{\rho}\frac{b}{a}\sin\omega\right)_u + \frac{1}{2}\left(\frac{\rho_v}{\rho}\frac{a}{b}\sin\omega\right)_v - ab\sin\omega = 0,\\
\rho_{uv} = 0,
\end{gathered} \tag{0.4}$$

where $\mathcal{K} = -1/\rho^2$ is the Gaussian curvature and u, v are asymptotic coordinates. In 1890, Bianchi presented a purely geometric construction for such hyperbolic surfaces. The determining constraint $\rho_{uv} = 0$ was retrieved one hundred years later by Levi and Sym (1990) in their search for the subclass of hyperbolic surfaces which possess an associated integrable Gauss-Mainardi-Codazzi system. Their procedure was based on the intrusion by Lie group methods of a spectral parameter into a 2×2 linear representation of the Gauss-Weingarten system for hyperbolic surfaces. In Section 1.6, a spherical representation is used to show that the Bianchi system (0.4) is, in fact, equivalent to the nonlinear sigma-type model

$$\begin{gathered}
(\rho N N_u)_v + (\rho N N_v)_u = 0, \quad N^2 = \mathbb{1}, \quad N^\dagger = N\\
\rho_{uv} = 0.
\end{gathered} \tag{0.5}$$

Thus, this important system of modern soliton theory has its origin in classical differential geometry. Indeed, a vector version of (0.5) is implicit in the work of Bianchi.

An elliptic variant of the Bianchi system is shown to deliver the well-known Ernst equation of general relativity, namely

$$\mathcal{E}_{z\bar{z}} + \frac{1}{2}\frac{\rho_{\bar{z}}}{\rho}\mathcal{E}_z + \frac{1}{2}\frac{\rho_z}{\rho}\mathcal{E}_{\bar{z}} = \frac{\mathcal{E}_z\mathcal{E}_{\bar{z}}}{\Re(\mathcal{E})}, \quad \rho_{z\bar{z}} = 0. \tag{0.6}$$

To conclude, a Bäcklund transformation that connects hyperbolic surfaces is constructed in a geometric manner. This is then specialised to provide an invariance which admits the constraint associated with the Bianchi system. The resulting Bäcklund transformation is then applied to a degenerate seed Bianchi surface to generate a one-soliton Bianchi surface.

Chapter 2 is concerned with how certain motions of privileged curves and surfaces can lead to solitonic equations. Thus, in Section 2.1, the classical sine-Gordon equation is arrived at by consideration of motions of an inextensible curve of constant curvature or torsion. In the latter case, the curve sweeps out a pseudospherical surface. In Section 2.2, the AKNS spectral problem for the sine-Gordon equation is derived via the $so(3)$–$su(2)$ isomorphism applied to its 3×3 Gauss-Weingarten representation. In Section 2.3, the discussion turns to privileged motions of pseudospherical surfaces which are associated with soliton equations said to be compatible with, or symmetries of, the sine-Gordon equation. Particular classes of motion of pseudospherical surfaces are considered. One is linked to a continuum version of an anharmonic lattice model which incorporates the important modified Korteweg-de Vries (mKdV) equation

$$\omega_t + \omega_{xxx} + 6\omega^2\omega_x = 0. \tag{0.7}$$

This mKdV equation, like the KdV equation (0.1) to which it is connected by the Miura transformation, is of considerable physical importance and arises, in particular, in plasma physics in the theory of the propagation of Alfvén waves.

Another important motion of pseudospherical surfaces, purely normal in character, is shown to produce a classical system due to Weingarten and Bianchi which may be found in Eisenhart's *A Treatise on the Differential Geometry of Curves and Surfaces* in connection with triply orthogonal systems of surfaces wherein one constituent family is pseudospherical. This system adopts the form

$$\begin{gathered}
\theta_{xyt} - \theta_x\theta_{yt}\cot\theta + \theta_y\theta_{xt}\tan\theta = 0,\\
\left(\frac{\theta_{xt}}{\cos\theta}\right)_x - \frac{1}{\rho}\left(\frac{1}{\rho}\sin\theta\right)_t - \frac{\theta_y\theta_{yt}}{\sin\theta} = 0,\\
\left(\frac{\theta_{yt}}{\sin\theta}\right)_y + \frac{1}{\rho}\left(\frac{1}{\rho}\cos\theta\right)_t + \frac{\theta_x\theta_{xt}}{\cos\theta} = 0,\\
\theta_{xx} - \theta_{yy} = \frac{1}{\rho^2}\sin\theta\cos\theta.
\end{gathered} \tag{0.8}$$

Bäcklund transformations for both the continuum lattice model and the above system are then shown to be induced by gauge transformations acting on an

AKNS representation. To conclude this chapter, in Section 2.4, the mKdV equation is generated via the motion of an inextensible curve of zero torsion. The motion of solitonic Dini surfaces is then investigated and triply orthogonal Weingarten systems of surfaces are thereby constructed.

In Chapter 3, the discussion turns to the classical surfaces of Tzitzeica which, like pseudospherical surfaces, emerge as having an underlying soliton connection. It was in the first decade of the twentieth century that the Romanian geometer Tzitzeica investigated the class of surfaces which is associated with the important nonlinear hyperbolic equation

$$(\ln h)_{\alpha\beta} = h - h^{-2}, \tag{0.9}$$

to be rediscovered some seventy years later in a solitonic context. In fact, the study by Tzitzeica of the surfaces associated with this equation may be said to have initiated the important subject of affine geometry. Therein, the Tzitzeica equation (0.9) describes the so-called affinsphären.

In Section 3.1, the class of surfaces Σ determined by the so-called Tzitzeica condition $\mathcal{K} = -c^2 d^4,\ c = \text{const}$ is introduced, wherein d is the distance from the origin to the tangent plane to Σ at a generic point. The linear representation of the Tzitzeica equation as originally set down by Tzitzeica is rederived and its dual is then used as a route to another important avatar of (0.9), namely the affinsphären equation

$$\left(\frac{R_u}{R^2 v^2}\right)_u = \left(\frac{R R_v}{v^2}\right)_v \tag{0.10}$$

as obtained by the German geometer Jonas in 1953. This integrable equation is then shown to arise naturally in a Lagrangian description of an anisentropic gasdynamics system for a certain three-parameter class of constitutive laws. In Section 3.2, a Bäcklund transformation for the construction of suites of Tzitzeica surfaces is derived in a geometric manner, and its connection with the classical Moutard transformation of 1878 is elucidated. The action of the Bäcklund transformation on the trivial seed solution $h = 1$ of the Tzitzeica equation (0.9) is then used to generate an affinsphäre with rotational symmetry. Tzitzeica surfaces corresponding to one- and two-soliton solutions of (0.9) are then constructed. In particular, a Tzitzeica surface corresponding to a breather solution is displayed.

It turns out that the Tzitzeica equation is embedded in another classical system which surprisingly has an even longer history. This solitonic system has become known as the two-dimensional Toda lattice model

$$(\ln h_n)_{uv} = -h_{n+1} + 2h_n - h_{n-1}, \quad n \in \mathbb{Z}. \tag{0.11}$$

This nonlinear differential-difference scheme, to be rediscovered almost a century later in modern soliton theory, is actually to be found in a treatise of Darboux published in 1887. There, it was derived in the iteration of what have become known as Laplace-Darboux transformations. The latter, like the contemporary Moutard transformation, arose in connection with the iterative reduction of linear hyperbolic equations to canonical form. They have interesting application to the theory of conjugate nets in the classical differential geometry of surfaces. This aspect of Laplace-Darboux transformations is described at length in Eisenhart's *Transformations of Surfaces*. Here, in Section 3.3, the notion of a Laplace-Darboux transformation is introduced along with key associated invariants. It is shown how application of a Laplace-Darboux transformation leads to the Toda lattice scheme (0.11). The Tzitzeica equation is then generated as a particular periodic Toda lattice. An invariance of the general two-dimensional Toda lattice model is presented which, in particular, preserves periodicity. It is then shown how Laplace-Darboux transformations may be applied iteratively to produce a suite of surfaces on which the parametric lines constitute conjugate nets.

In Chapter 4, we focus upon the NLS equation (0.3). The latter seems to have escaped the attention of the geometers of the nineteenth century even though it has a simple geometric origin in the evolution of an inextensible curve moving through space with speed $\boldsymbol{v} = \kappa \boldsymbol{b}$, where κ is its curvature and $\boldsymbol{b}$ its binormal. In Section 4.1, the NLS equation is derived in a geometric manner, and soliton surfaces corresponding to single soliton and breather solutions are presented along with general geometric properties and the connection to the Heisenberg spin equation

$$\boldsymbol{S}_t = \boldsymbol{S} \times \boldsymbol{S}_{ss}, \quad \boldsymbol{S}^2 = 1, \tag{0.12}$$

where t is time and s is arc length. In Section 4.2, a solitonic system linked to the NLS equation, namely the Pohlmeyer-Lund-Regge model,

$$\begin{gathered}\theta_{\xi\xi} - \theta_{\eta\eta} - \epsilon^2 \cos\theta \sin\theta + \left(\phi_\xi^2 - \phi_\eta^2\right) \cos\theta \ \operatorname{cosec}^3\theta = 0, \\ \left(\phi_\xi \cot^2\theta\right)_\xi = \left(\phi_\eta \cot^2\theta\right)_\eta\end{gathered} \tag{0.13}$$

is also derived in a geometric manner. This system arises in the study of relativistic vortices. It is shown to be related, in turn, to the sharpline self-induced transparency (SIT) system

$$\begin{aligned}\chi_{tx} &= \sin\chi + \nu_t \nu_x \tan\chi, \\ \nu_{tx} &= -\nu_x \chi_t \cot\chi - \nu_t \chi_x (\cos\chi \sin\chi)^{-1}\end{aligned} \tag{0.14}$$

which stems from the unpumped Maxwell-Bloch system

$$
\begin{gathered}
E_x = P, \quad P_t = EN, \\
N_t = -\frac{1}{2}(\bar{E}P + E\bar{P}), \qquad N^2 + P\bar{P} = 1.
\end{gathered}
\tag{0.15}
$$

In the above, E and $P = e^{i\nu}\sin\chi$ denote, in turn, the slowly varying amplitudes of the electric field and polarisation, while $N = \cos\chi$ is the atomic inversion. The unpumped Maxwell-Bloch system is likewise shown to be linked to the stimulated Raman scattering (SRS) system

$$
A_{1X} = -SA_2, \quad A_{2X} = \bar{S}A_1, \quad S_T = A_1\bar{A}_2, \tag{0.16}
$$

where A_1, A_2 are the electric field amplitudes of the pump and Stokes waves, respectively. The connection between the SIT and SRS systems and the NLS equation is then established via the compatibility of the latter with the unpumped Maxwell-Bloch system. Thus, an appropriate time evolution of the eigenfunction pair in the AKNS representation for the NLS equation produces the system (0.15). In geometric terms, this unpumped Maxwell-Bloch system arises out of certain motions of Hasimoto surfaces in the same way as the mKdV equation or Weingarten system come from appropriate compatible motions of pseudospherical surfaces. In Section 4.3, the NLS equation is derived in an alternative manner via a geometric formulation originally developed in a kinematic analysis of certain hydrodynamical motions by Marris and Passman in 1969. The auto-Bäcklund transformation for the NLS equation is derived in this representation at the level of the generation of Hasimoto surfaces. Spatially periodic solutions of 'smoke-ring' type are thereby generated.

Chapter 5 is concerned with yet another classical class of surfaces which have a soliton connection, namely isothermic surfaces. These surfaces seem to have their origin in work by Lamé in 1837 motivated by problems in heat conduction. An important subclass of isothermic surfaces were subsequently investigated in a paper by Bonnet in 1867. These Bonnet surfaces admit non-trivial families of isometries which leave invariant the principal curvatures κ_1 and κ_2 and, accordingly, both the Gaussian curvature $\mathcal{K} = \kappa_1\kappa_2$ and mean curvature $\mathcal{M} = (\kappa_1 + \kappa_2)/2$. In Section 5.1, the Gauss-Mainardi-Codazzi system associated with isothermic surfaces parametrised in curvature coordinates is set down, namely

$$
\begin{gathered}
\theta_{xx} + \theta_{yy} + \kappa_1\kappa_2 e^{2\theta} = 0, \\
\kappa_{1y} + (\kappa_1 - \kappa_2)\theta_y = 0, \qquad \kappa_{2x} + (\kappa_2 - \kappa_1)\theta_x = 0
\end{gathered}
\tag{0.17}
$$

and a reduction originally obtained by Calapso in 1903 is made to the single