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MATHEMATICS AND MUSIC, A DUET

When I mention to someone my work on a book about mathematics and music, I tend to get one of two responses:

> "Yeah, they really are a lot alike, aren't they?"
>
> "What in the world could the one have to do with the other??"

Math and music *do* have much in common. At heart, abstract patterns form the stock-in-trade of both. To express these patterns, each field has developed its own symbolic language, used the world over regardless of nationality. And the two areas, although in different ways, combine the intellectual and the aesthetic in a wonderful blend. Unfortunately, nonmusicians often remain unaware of the rich intellectual content of music, and nonmathematicians likewise of the equally rich aesthetic side of math. That accounts for many of the skeptical reactions I mentioned above. An anecdote of Raymond Smullyan provides a different slant on these misperceptions. In his book, *5000 B.C.*, Smullyan told of a mathematician hearing people talk about connections between math and music. The mathematician, after looking puzzled, blurted out, "But I don't see the likeness; after all, mathematics is beautiful!" Such apocryphal stories notwithstanding, math and music do appeal to their practitioners in similar ways. To quote the nineteenth-century mathematician J. J. Sylvester, "May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? the soul of each the same!”

But beyond the generalities of shared traits, the two areas have many specific, direct links. Over the years, people have used math to describe, analyze, and create music. This book constitutes an informal smorgasbord of several such applications to music.

The Good Old Summary Time

One of the more obvious contacts between the two fields is through the middleman of physics. The acoustical side of music lends itself well to mathematical analysis. Chapter 2 lays out some of the basic mathematical relationships behind musical pitch, timbre, and overtones. In particular, I examine why some notes harmonize well with others. Attempts to incorporate such pleasing harmonies have led to various schemes for tuning musical scales. The third chapter compares some of these tuning systems.

These acoustical considerations deal with the raw material of music. In the visual arts, the parallel study would be the physics of light and color. But with visual art, mathematics also enters into the compositional aspects. Think of the theory of perspective drawing or the works of M. C. Escher. Similarly, math plays a role in musical composition. Sometimes composers have explicitly employed mathematical techniques in their works. At other times composers have built structures that, although designed by purely musical considerations, can profitably be described in mathematical terms. The math may not present itself as obviously as the geometry in a painting, but it can be just as vital.

The fourth and fifth chapters both draw on the mathematical area of group theory, though in different ways. The former chapter inspects the group theory behind some standard tricks of the composer’s trade, such as transposing a melody or playing it backwards. In Chapter 5, I concentrate on the old English tradition of ringing the changes on tower bells. Change-ringers had actually used many group-theoretic ideas in this specific context long before mathematicians developed the subject. For the next chapter the focus shifts to randomness and probability as used by various composers. From musical dice in Mozart's time to recent computer work, musicians have been intrigued by the artistic possibilities of chance. Chapter 7 investigates some structural qualities of music. Specifically, I discuss how the same patterns can appear at different levels within a piece. Earlier I compared music with visual art. Of course, the two sometimes interact, and math can help bind them together. Chapter 8 details various mathematical strategies for building music out of pictures. We will also take a look at dancing, where group theory returns to the scene.

Given all of the areas of overlap between math and music, it can be easy to get carried away and start seeing connections that aren't necessarily there. This danger looms especially large when a mathematical analysis is imposed on a musical composition after the fact. The final chapter contains some cautionary tales of dubious attempts to meld the two fields.

Mathematics and music go back a long way together. In the Western tradition, their partnership traces back at least 2500 years, when the Pythagoreans explored the math–music connection. The seven liberal arts of the Middle Ages consisted of the quadrivium, the path to knowledge, and the trivium, the path to eloquence. As I suggested at the beginning, many people these days associate the nature of music with the latter more than with the former. But to medieval scholars, music enjoyed good standing as a member of the quadrivium. Its mates there? Arithmetic, geometry, and astronomy. The interplay between math and music continues at present. Indeed, recent years have seen a proliferation of conference sessions and even entire conferences devoted to the links that connect mathematics and music. I hope that this book provides at least a taste of the fascinating results that occur when math and music combine forces.