1
Introduction

Turbulence is a ubiquitous phenomenon. Wherever fluids are set into motion turbulence tends to develop, as everyday experience shows us. When the fluid is electrically conducting, the turbulent motions are accompanied by magnetic-field fluctuations. However, conducting fluids are rare in our terrestrial world, where electrical conductors are usually solid. One of the rare examples of a fast-moving conducting fluid, which has been of some practical importance and concern and to which authors of theoretical studies sometimes referred, is, or better, was the flow of liquid sodium in the cooling ducts of a fast-breeder reactor. It is therefore not surprising that, in contrast to the broad scientific and technical literature on ordinary, i.e., hydrodynamic, turbulence, magnetic turbulence has not received much attention.

The most natural conducting fluid is an ionized gas, called a plasma. It is true that laboratory plasmas, which are confined by strong magnetic fields, notably in nuclear-fusion research, exhibit little dynamics, except in short disruptive pulses. Only the reversed-field pinch, a toroidal plasma discharge of relatively high plasma pressure, exhibits continuous magnetic activity, such that it is sometimes considered more as a convenient device for studying magnetic turbulence rather than as a particularly promising approach to controlled nuclear fusion.

Plasmas are, however, abundant in the extraterrestrial world. It is said that 99% of all material in the universe exists in the plasma state. This does not, however, mean that the plasma properties are always important. In fact, for understanding stellar evolution, during which conditions are mostly quasi-static, or galaxy formation, which is dominated by gravitational forces, the specific plasma properties and the presence of magnetic fields are not crucial and have usually been neglected, atomic and nuclear properties being more important. Only in certain processes have magnetic fields long been acknowledged to play a dominant part, such as in the dynamics of stellar atmospheres and in the generation of cosmic rays. In recent times, however, it appears that the omnipresence and active role of magnetic fields has also been recognized in those astrophysical objects which had formerly been treated as essentially nonmagnetic neutral fluids, for instance in accretion disks and in the interstellar medium. Here magnetic fluctuations provide essential clues regarding the intrinsic transport properties, namely transport of angular momentum in the former and gravitational collapse in the latter. These applications have led to a remarkable revival of interest in magnetic turbulence.

Though magnetic-field fluctuations occur on all scales, down to the smallest plasma modes, magnetic fields are most important at macroscales, i.e., mean wavelengths exceeding the internal plasma scale lengths, such as the ion gyroradius. In this regime magnetohydrodynamics (MHD) provides the appropriate framework, to which this book is restricted. This does not mean that the classical condition for a fluid approach, namely the smallness of the mean free path between interparticle collisions compared with gradient scales, must be satisfied. Indeed, in many dilute plasmas collisions are very rare, but there are other, collective, processes that play a similar dissipative role giving rise to effective (often called anomalous) transport coefficients. Also, in weakly ionized gases magnetic diffusion is not governed by classical resistivity, the friction between ions and electrons, but by ambipolar diffusion, the friction between ions and neutral species. Hence the huge values of the Reynolds numbers and other parameters characterizing a turbulent system, which are typical for astrophysical systems when they are calculated with the classical transport coefficients, should not be taken too seriously. In any case, the dissipation processes, independently of their nature, serve only as energy sinks, which cut off the spectrum of turbulent fluctuations at small scales but do not affect the main turbulence scales.

Since MHD turbulence is related to hydrodynamic turbulence, by following similar equations one may apply, and generalize, the formalism developed for the latter. Hence this book deals necessarily also with hydrodynamic turbulence. Turbulence theory has developed along two rather different lines, one oriented toward technical applications, the other focussing on the intrinsic turbulence properties. The practical importance of the first line is obvious – in popular view, turbulence is, indeed, considered more a technical problem than a physical phenomenon. The second line, which is naturally more interesting to a physicist, is characterized by certain approximations made for algebraic convenience as well as for conceptual clarity, which move the focus away from the practical aspects of turbulence. This is also reflected in the various treatises on turbulence theory published during the past few decades, for instance by Leslie (1973), by Lesieur (1997), and by Frisch (1995), which concentrate more on the formal developments in homogeneous incompressible turbulence theory. While the classical books, especially the two volumes by Monin and Yaglom (1975), contain also major parts dealing with observational results, in more recent works the emphasis has increasingly shifted toward discussion of results of computer simulations, which can be compared more directly with the theoretical models.

In fact, numerical computations have become an indispensable tool in turbulence research. Since statistical fluid theory is notoriously difficult to deal with analytically, a numerical treatment of the time-dependent primitive equations, now generally called numerical simulation, providing an exact solution (within controlled numerical-discreteness effects), is often the only method by which to check the validity of analytical modeling. The numerical scaling laws obtained by varying the values of parameters may be considered *the* solution of a turbulence problem, which should subsequently only be "understood" by invoking a simple intuitive model, or mechanism. The argument often given, namely that numerically attainable Reynolds numbers are simply too small to be useful for understanding real, high-Reynolds-number turbulence is gradually becoming academic, as progress in computer technology allows use of ever larger computational grids. To illustrate the progress in computer power, only 15 years ago two-dimensional (2D) simulations on a grid of 1024^2 points were considered the state of the art, whereas present-day supercomputers can handle the same linear resolution in three dimensions (3D), an increase by a factor of 10^3. Naturally the development of efficient numerical methods and codes and their exploitation has become a major activity for many turbulence theorists. For MHD turbulence numerical simulations play an even greater role than they do for hydrodynamic turbulence, since laboratory experiments are practically impossible and astrophysical systems, in particular solar-wind turbulence, the most important system of high-Reynolds-number MHD turbulence accessible to *in situ* measurements, are too complex to be directly comparable with theoretical results. We shall therefore frequently refer to such numerical studies.

Initially, interest in MHD turbulence focussed on the dynamo problem, notably in Batchelor's early paper (1950) and in the famous article by Steenbeck *et al.* (1966) reviewed later on in Moffatt's book (1978). A milestone in the fundamental scaling theory was the introduction of the Alfvén effect proposed independently by Iroshnikov (1964) and Kraichnan (1965b). This describes small-scale turbulent fluctuations as weakly interacting Alfvén waves propagating along the large-scale field. Because of the reduction of the corresponding spectral transfer the energy spectrum was predicted to be somewhat flatter, $k^{-3/2}$ instead of the Kolmogorov spectrum $k^{-5/3}$. This led to a long-standing debate about which process actually dominates the turbulence dynamics. From the theory side the importance of the Elsässer fields as basic dynamic variables, which incorporate the Alfvén-wave properties and naturally describe aligned states, seems to point to the fundamental role of the Alfvén effect. From the observation side, however, the energy spectrum of solar-wind turbulence was found to be clearly closer to a Kolmogorov law. The solution of this paradox lies in the intrinsic anisotropy of MHD turbulence emphasized by Goldreich and Sridhar (1995), who showed that the spectrum is more strongly developed perpendicularly to the local magnetic field, where the Alfvén effect is not operative. Hence a Kolmogorov-like transfer dynamics should dominate, which is also corroborated by results of recent numerical studies of 3D MHD turbulence.

There are, however, self-organization processes in MHD turbulence that have no hydrodynamic counterpart. These processes originate from certain selective decay properties arising from the presence of several ideal invariants with different decay rates, which have been studied intensively primarily by Montgomery and Matthaeus and their collaborators. Conservation of cross-helicity leads to highly correlated, or aligned, states, while conservation of magnetic helicity gives rise to the formation of force-free magnetic configurations as was first shown by Woltjer (1958) and generalized by Taylor (1974). A further facet of the latter process is the inverse cascade of the magnetic helicity, the excitation of increasingly larger magnetic scales, which is the basis of the nonlinear dynamo effect as noted by Pouquet *et al*. (1976).

The book consists of four parts. Chapters 2 and 3 discuss the properties of the MHD model and the transition from a smooth to a turbulent state; Chapters 4–7 deal with the various aspects of fully developed incompressible turbulence; Chapters 8 and 9 treat two extensions, namely 2D turbulence, which corresponds to the limit of a strong magnetic field, and, in a sense the opposite limit, supersonic turbulence. The last part, Chapters 10–12, considers three astrophysical applications, namely turbulence in the solar wind, in accretion disks, and in the interstellar medium.

Chapter 2 introduces the MHD theory and discusses special approximations, such as incompressibility and the Boussinesq approximation. The MHD equations exhibit a number of ideal conservation laws reflecting the constraints on the turbulence dynamics which are relaxed only by dissipative effects. We then briefly outline the properties of static equilibrium configurations, either magnetic or gravitational, and the different types of linear waves, which arise due to magnetic tension, pressure, and stratification. Finally the MHD equations are formulated in terms of the Elsässer fields, which are particularly well suited to MHD turbulence.

Chapter 3 deals with the transition to turbulence, namely how random motions are generated from a smooth flow. We first discuss the character of the singular solutions developing in the ideal equations, in particular the generation of finite-time singularities. This problem has aroused considerable interest, though its connection to the real, dissipative turbulence is somewhat loose. To date no general mathematical answer has been found, but there are strong indications that the hydrodynamic Euler equations do, indeed, give rise to finite-time singularities, if the initial state is not too symmetric, whereas in the MHD case no finite-time singularities seem to exist. This difference can be pinned down to the structure of the solution at the location of the singularity, filamentary in hydrodynamics and sheet-like in MHD. The development of small-scale random fluid motions is described by the effect of some instability of the quasi-singular structures of the ideal solution. When the velocity shear, pressure gradient, and current density exceed certain thresholds, the fluid becomes Kelvin–Helmholtz unstable, Rayleigh–Taylor unstable or unstable against tearing. We discuss these instabilities and their nonlinear evolution in some detail.

In Chapters 4–7 we consider incompressible turbulence. Chapter 4 focusses on macroscopic properties. We first discuss the Reynolds equations. Here the effects of the small-scale fluctuations are contained in the turbulent Reynolds and Maxwell tensors in the momentum equation and the electromotive force and turbulent resistivity in the induction equation, for which phenomenological expressions based on the mixing-length concept are used. Self-organization processes in MHD turbulence are caused by selective decay. Since magnetic helicity, and, to a lesser extent, also cross-helicity, decay much more slowly than does the turbulence energy, the decay of the latter leads to relaxed states, depending on the initial state either a linear force-free magnetic configuration or an Alfvénic state, in which the velocity and magnetic field are aligned. The energy-decay law is a characteristic property of the turbulence. For finite magnetic helicity the decay is controlled by selective decay, in particular kinetic energy decays more rapidly than does magnetic energy, $E^K \sim t^{-1}$ and $E^M \simeq E \sim t^{-0.5}$. If the magnetic helicity is negligible, the turbulence remains macroscopically self-similar during the decay, $E^K \sim E^M$, and the energy decay is faster, $E \sim t^{-1}$, since the turbulence is less constrained.

High-Mach-number turbulence carries a wide range of spatial scales, which exhibit characteristic scaling properties, notably the internal-range energy spectrum examined in Chapter 5. The scaling behavior becomes particularly transparent if the turbulence is not affected by the inhomogeneity of the global system, but can be considered statistically homogeneous and, possibly, also isotropic, which is the framework of most theories dealing with the intrinsic turbulence properties. To understand the spectral-transfer processes it is helpful to look at the statistical equilibrium states of the nondissipative system truncated in Fourier space, which are called absolute equilibrium states, from which the preferential spectral-transfer, or cascade, direction of a spectral quantity can be read off directly, for instance the inverse cascade of the magnetic helicity. We then derive, in a phenomenological way, the energy spectrum in dissipative MHD turbulence by assuming that there is either a local, Kolmogorov-type, transfer process leading to a $k^{-5/3}$ spectrum or a spectral transfer controlled by the Alfvén effect, which gives rise to the IK spectrum $k^{-3/2}$. In accounting for the inherent spectral anisotropy the latter is essentially ruled out. Numerical simulations of 3D MHD turbulence corroborate the Goldreich–Sridhar concept of a Kolmogorov inertial-range spectrum, which is independent of the amount of magnetic helicity present. Only at small wavenumbers, for which the inverse cascade of the magnetic helicity enhances the magnetic-energy spectrum, is there a difference between helical and nonhelical turbulence.

Two-point closure theory, which we treat in Chapter 6, gives a description of turbulence derived from the basic fluid equations instead of purely phenomenological arguments. Though closure theory cannot treat higher-order correlations correctly because of the basic quasi-normal approximation, it provides, in principle, a self-consistent dynamical theory of the evolution of the two-point correlations, the various spectral quantities in Fourier space. In practice the equations are made tractable by additional phenomonological assumptions, which lead to the EDQNM model which is usually considered. We discuss the MHD closure theory, in particular the special cases of helical and of correlated turbulence.

Chapter 7 deals with the higher-order statistics of turbulence in order to obtain a more detailed picture of the spatial structure. Fluid turbulence is not strictly self-similar, as small-scale eddies are increasingly sparsely distributed, a property which is called intermittency. To familiarize the reader with the general concept of intermittency, we first present some examples demonstrating the difference between self-similar and intermittent behavior. Structure functions $S^{(n)}(l)$, in particular the set of scaling exponents ζ_n, $S^{(n)} \sim l^{\zeta_n}$, describe the statistical distribution of the turbulent structures. In many turbulent systems, however, for which Reynolds numbers are rather modest, the scaling range of the structure functions, especially for higher orders, is too short to yield clear values of the exponents. The scaling range is often significantly broadened when one considers $S^{(n)}(S^{(3)})$ instead of $S^{(n)}(l)$, a property called extended self-similarity, which yields surprisingly accurate values of the relative scaling exponents ζ_n/ζ_3. Third-order structure functions, in turn, satisfy some exact relations derived directly from the fluid equations, Kolmogorov's four-fifths law in hydrodynamic turbulence, Yaglom's four-thirds law for an advected turbulent scalar field, and a similar relation for a third-order structure function of the Elsässer fields in MHD turbulence. Since no further exact relations seem to exist and no approximate scheme is known to date from which to obtain further information about the scaling properties directly from the fluid equations, the only viable approach consists of phenomenological modeling using some physical picture of the turbulence dynamics. We discuss in some detail two different models, the log-normal model and the log-Poisson model, the latter of which gives good agreement with experimental measurements in hydrodynamic turbulence and, *mutatis mutandis*, reproduces equally well the results for MHD turbulence obtained from numerical simulations. By varying the strength of a mean magnetic field B_0 simulations can also be made to show the transition from globally isotropic 3D MHD turbulence for $B_0 = 0$ to 2D turbulence for $B_0 \to \infty$.

Two-dimensional turbulence is considered in Chapter 8. We first discuss the hydrodynamic case, which has attracted much interest. In the presence of two ideal invariants, the energy and the enstrophy, the turbulence dynamics is dominated by a self-organization process, the buildup of large-scale structures due to the inverse cascade of the energy. An analogous process occurs in 2D MHD turbulence, for which now the energy exhibits a direct cascade but the mean-square magnetic potential exhibits an inverse cascade, which leads to the formation of large-scale magnetic structures. The free decay of turbulence proceeds in a macroscopically self-similar way with the asymptotic energy-decay law $E \sim t^{-1}$. The spatial scaling properties are different from those observed in 3D MHD turbulence, scaling exponents being generally lower, in particular $\zeta_3 < 1$, and the energy spectrum is flatter than the Kolmogorov spectrum, roughly consistent with the IK spectrum $k^{-3/2}$. Scaling properties are, however, not uniform in the sense that all structure functions of the same order have the same scaling exponent, since the exponent of the energy flux, a particular type of third-order structure function, is again unity.

The first part in Chapter 9 is devoted to compressible, in particular supersonic, turbulence, which is important in view of many astrophysical applications. Turbulence consists of both eddy motions and shock waves, and, as a rule of thumb, the inertial-range scales are dominated by eddy motions – the energy spectrum follows a Kolmogorov law –, while dissipation occurs mainly through shock waves. Also density fluctuations exhibit a Kolmogorov spectrum, but their spatial distribution is highly intermittent. Except for the case of a strong mean field, the dynamics is dominated by the supersonic flows, while magnetic effects are less important, the field being mainly advected, forming filamentary structures similar to those of the density. The second part of the chapter deals with turbulent convection in stratified systems. We consider first turbulence in the Boussinesq approximation, for which simple spectral laws can be derived, in particular for passive-scalar turbulence, and then add compressibility and magnetic fields. Because of the complexity of these systems only fully dynamic simulations can provide reliable information.

In the last three chapters dealing with some applications we restrict our consideration to astrophysical topics, since in laboratory plasma devices MHD turbulence usually does not occur and the only exception, the reversed-field pinch, has already been reviewed in several treatises (e.g., Biskamp, 1993a). In Chapter 10 we consider the turbulence in the solar wind. This is the only system of high-Reynolds-number MHD turbulence which is accessible to *in situ* observations. Much data has been accumulated from several spacecraft sampling the interplanetary space at various radii and latitudes. Here many properties of homogeneous turbulence theory are recovered. In spite of the large Mach number of the solar wind, compression effects are not dominant in the turbulence. In general inhomogeneities due to radial expansion of the wind and mixing of different types of wind by solar rotation complicate conditions considerably, leaving important observations unexplained.

Chapter 11 treats accretion disks, a widespread phenomenon in astrophysics, in which magnetic turbulence should be present, since it is the only conceivable mechanism for transport of angular momentum, the very agent of accretion. We first give a brief introduction to the main properties of accretion disks before considering more closely the origin of the presumed turbulence. Since in the disk material rotates essentially in Keplerian orbits, the system is hydrodynamically shear-flow stable, even nonlinearly. Turbulence can, however, be excited through the Balbus–Hawley instability when we allow the presence of a weak but finite magnetic field. Numerical simulations indicate that this mechanism may account for the observed accretion rates.

The last chapter deals with the interstellar medium, where the presence of turbulence can be observed directly. As in the previous two chapters, we first give a general overview of the properties of the interstellar medium, in particular its densest parts, the molecular clouds. Observations indicate the occurrence of highly supersonic irregular flows. These act as an effective pressure preventing rapid gravitational collapse and thus explaining the observed long lifetimes and low star-formation rates. The role of the magnetic field, which is random, at least on the larger scales, is still being debated, but it appears that it provides an additional stabilizing effect against contraction. Another interesting feature is the coupling between the magnetic field and the gas, which is only very weakly ionized; while classical resistivity would give rise to almost perfect coupling, in reality the coupling is rather loose because of ambipolar diffusion.

In a book spanning various rather different fields, the individual topics cannot all be treated in depth. We have therefore included numerous references to more specialized reviews and also, of course, the appropriate citations of the original work. However, the number of these citations had to be restricted in order not to diminish the readability of the text. Concerning notation, I have tried as far as possible to avoid denoting different things by the same symbol, giving an explicit warning in the few exceptional cases dictated by traditional notation. I use cgs units in the original equations, because they are still rather common in astrophysics and because of personal preference, but, whenever convenient, introduce suitable normalizations in order to write the equations in nondimensional forms.

2

Magnetohydrodynamics

Magnetohydrodynamics, or MHD in short, describes the macroscopic behavior of an electrically conducting fluid – usually an ionized gas called a plasma –, which forms the basis of this book. By macroscopic we mean spatial scales larger than the intrinsic scale lengths of the plasma, such as the Debye length λ_D and the Larmor radii ρ_j of the charged particles.[1] In this chapter we first derive, in a heuristic way, the dynamic equations of MHD and discuss the local thermodynamics (Section 2.1). Since most astrophysical systems rotate more or less rapidly, it is useful to write the momentum equation also in a rotating reference frame, where inertial forces appear (Section 2.2). Then some convenient approximations are introduced, in particular incompressiblity and, for a stratified system, the Boussinesq approximation (Section 2.3). In MHD theory the ideal invariants, i.e., integral quantities that are conserved in an ideal (i.e., nondissipative) system, play a crucial role in turbulence theory; these are the energy, the magnetic helicity, and the cross-helicity (Section 2.4). Though this book deals with turbulence, it is useful to obtain an quick overview of magnetostatic equilibrium configurations, which are more important in plasmas than stationary flows are in hydrodynamics (Section 2.5). Also the zoology of linear modes, the small-amplitude oscillations about an equilibrium, is richer than that in hydrodynamics (Section 2.6). Finally, in Section 2.7 we introduce the Elsässer fields, which constitute the basic dynamic quantities in MHD turbulence. In this chapter we write the equations in dimensional form, using Gaussian units, to emphasize the physical meaning of the various terms. At the end nondimensionalization in terms of the Alfvén time is introduced, which will be used throughout most of the following.

[1] The fluid approximation also requires that the mean free path is smaller than the gradient scales. For motions perpendicular to the magnetic field, however, the Larmor radius assumes the role of the mean free path.