DYNAMICS OF THE ATMOSPHERE: A COURSE IN THEORETICAL METEOROLOGY

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M1

Algebra of vectors

M1.1 Basic concepts and definitions

A *scalar* is a quantity that is specified by its sign and by its magnitude. Examples are temperature, the specific volume, and the humidity of the air. Scalars will be written using Latin or Greek letters such as $a, b, \ldots, A, B, \ldots, \alpha, \beta, \ldots$. A *vector* requires for its complete characterization the specification of magnitude and direction. Examples are the velocity vector and the force vector. A vector will be represented by a boldfaced letter such as $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{A}, \mathbf{B}, \ldots$. A *unit vector* is a vector of prescribed direction and of magnitude 1. Employing the unit vector $\mathbf{e}_A$, the arbitrary vector $\mathbf{A}$ can be written as

$$\mathbf{A} = |\mathbf{A}|\,\mathbf{e}_A = A\mathbf{e}_A \implies \mathbf{e}_A = \frac{\mathbf{A}}{|\mathbf{A}|} \tag{M1.1}$$

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if they have the same magnitude and direction regardless of the position of their initial points,

that is $|\mathbf{A}| = |\mathbf{B}|$ and $\mathbf{e}_A = \mathbf{e}_B$. Two vectors are *collinear* if they are parallel or antiparallel. Three vectors that lie in the same plane are called *coplanar*. Two vectors always lie in the same plane since they define the plane. The following rules are valid:

$$\begin{aligned}
&\textit{the commutative law}: && \mathbf{A} \pm \mathbf{B} = \mathbf{B} \pm \mathbf{A}, \quad \mathbf{A}\alpha = \alpha\mathbf{A} \\
&\textit{the associative law}: && \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}, \quad \alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A} \\
&\textit{the distributive law}: && (\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}
\end{aligned} \tag{M1.2}$$

The concept of linear dependence of a set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ is closely connected with the dimensionality of space. The following definition applies: A set of N vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ of the same dimension is linearly dependent if there exists a set of numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$, not all of which are zero, such that

$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_N\mathbf{a}_N = 0 \tag{M1.3}$$

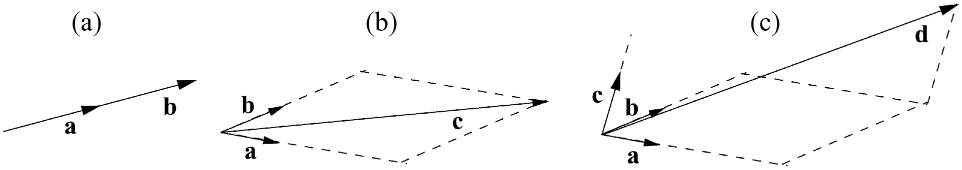

Fig. M1.1 Linear vector spaces: (a) one-dimensional, (b) two-dimensional, and (c) three-dimensional.

If no such numbers exist, the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ are said to be linearly independent. To get the geometric meaning of this definition, we consider the vectors **a** and **b** as shown in Figure M1.1(a). We can find a number $k \neq 0$ such that

$$\mathbf{b} = k\mathbf{a} \tag{M1.4a}$$

By setting $k = -\alpha/\beta$ we obtain the symmetrized form

$$\alpha\mathbf{a} + \beta\mathbf{b} = 0 \tag{M1.4b}$$

Assuming that neither α nor β is equal to zero then it follows from the above definition that two collinear vectors are linearly dependent. They define the one-dimensional *linear vector space*. Consider two noncollinear vectors **a** and **b** as shown in Figure M1.1(b). Every vector **c** in their plane can be represented by

$$\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b} \quad \text{or} \quad \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0 \tag{M1.5}$$

with a suitable choice of the constants k_1 and k_2. Equation (M1.5) defines a two-dimensional linear vector space. Since not all constants α, β, γ are zero, this formula insures that the three vectors in the two-dimensional space are linearly dependent. Taking three noncoplanar vectors **a**, **b**, and **c**, we can represent every vector **d** in the form

$$\mathbf{d} = k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c} \tag{M1.6}$$

in a three-dimensional linear vector space, see Figure M1.1(c). This can be generalized by stating that, in an N-dimensional linear vector space, every vector can be represented in the form

$$\mathbf{x} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + \cdots + k_N\mathbf{a}_N \tag{M1.7}$$

where the $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ are linearly independent vectors. Any set of vectors containing more than N vectors in this space is linearly dependent.

Table M1.1. *Extensive quantities of different degrees for the N-dimensional linear vector space*

Extensive quantity	Degree υ	Symbol	Number of vectors	Number of components
Scalar	0	B	0	$N^0 = 1$
Vector	1	$\mathbf{B}$	1	N^1
Dyadic	2	$\mathbb{B}$	2	N^2

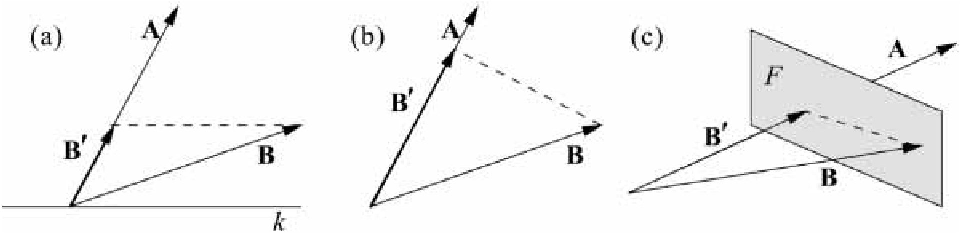

Fig. M1.2 Projection of a vector **B** onto a vector **A**.

We call the set of N linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ the *basis vectors* of the N-dimensional linear vector space. The numbers $k_1, k_2, \ldots, k_N$ appearing in (M1.7) are the *measure numbers* associated with the basis vectors. The term $k_i\mathbf{a}_i$ of the vector $\mathbf{x}$ in (M1.7) is the *component of this vector* in the direction $\mathbf{a}_i$.

A vector **B** may be projected onto the vector **A** parallel to the direction of a straight line k as shown in Figure M1.2(a). If the direction of the straight line k is not given, we perform an orthogonal projection as shown in part (b) of this figure. A projection in three-dimensional space requires a plane F parallel to which the projection of the vector **B** onto the vector **A** can be carried out; see Figure M1.2(c).

In vector analysis an *extensive quantity* of degree ν is defined as a homogeneous sum of general products of vectors (with no dot or cross between the vectors). The number of vectors in a product determines the degree of the extensive quantity. This definition may seem strange to begin with, but it will be familiar soon. Thus, a scalar is an extensive quantity of degree zero, and a vector is an extensive quantity of degree one. An extensive quantity of degree two is called a *dyadic*. Every dyadic $\mathbb{B}$ may be represented as the sum of three or more *dyads*. $\mathbb{B} = \mathbf{p}_1\mathbf{P}_1 + \mathbf{p}_2\mathbf{P}_2 + \mathbf{p}_3\mathbf{P}_3 + \cdots$. Either the *antecedents* $\mathbf{p}_i$ or the *consequents* $\mathbf{P}_i$ may be arbitrarily assigned as long as they are linearly independent. Our practical work will be restricted to extensive quantities of degree two or less. Extensive quantities of degree three and four also appear in the highly specialized literature. Table M1.1 gives a list of extensive quantities used in our work. Thus, in the three-dimensional linear vector space with $N = 3$, a vector consists of three and a dyadic of nine components.

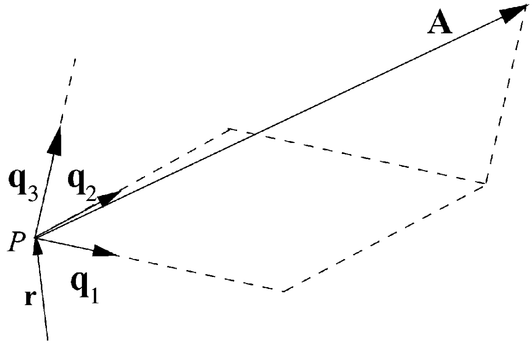

Fig. M1.3 The general vector basis $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ of the three-dimensional space.

M1.2 Reference frames

The representation of a vector in component form depends on the choice of a particular coordinate system. A *general vector basis* at a given point in three-dimensional space is defined by three arbitrary linearly independent basis vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ spanning the space. In general, the basis vectors are neither orthogonal nor unit vectors; they may also vary in space and in time.

Consider a position vector $\mathbf{r}$ extending from an arbitrary origin to a point P in space. An arbitrary vector $\mathbf{A}$ extending from P is defined by the three basis vectors $\mathbf{q}_i, i = 1, 2, 3$, existing at P at time t, as shown in Figure M1.3 for an oblique coordinate system. Hence, the vector $\mathbf{A}$ may be written as

$$\mathbf{A} = A^1\mathbf{q}_1 + A^2\mathbf{q}_2 + A^3\mathbf{q}_3 = \sum_{k=1}^{3} A^k\mathbf{q}_k \tag{M1.8}$$

where it should be observed that the so-called *affine measure numbers* A^1, A^2, A^3 carry superscripts, and the basis vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ carry subscripts. This type of notation is used in the *Ricci calculus,* which is the tensor calculus for nonorthonormal coordinate systems. Furthermore, it should be noted that there must be an equal number of upper and lower indices.

Formula (M1.8) can be written more briefly with the help of the familiar *Einstein summation convention* which omits the summation sign:

$$\mathbf{A} = A^1\mathbf{q}_1 + A^2\mathbf{q}_2 + A^3\mathbf{q}_3 = A^n\mathbf{q}_n \tag{M1.9}$$

We will agree on the following notation: Whenever an index (subscript or superscript) $m, n, p, q, r, s, t,$ is repeated in a term, we are to sum over that index from 1 to 3, or more generally to N. In contrast to the summation indices $m, n, p, q, r, s, t,$ the letters i, j, k, l are considered to be "free" indices that are used to enumerate equations. Note that summation is not implied even if the free indices occur twice in a term or even more often.

A special case of the general vector basis is the *Cartesian vector basis* represented by the three orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, or, more conveniently, $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$. Each of these three unit vectors has the same direction at all points of space. However, in rotating coordinate systems these unit vectors also depend on time. The arbitrary vector $\mathbf{A}$ may be represented by

$$\begin{aligned} &\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} = A^n\mathbf{i}_n = A_n\mathbf{i}_n \\ \text{with} \quad &A_x = A^1 = A_1, \quad A_y = A^2 = A_2, \quad A_z = A^3 = A_3 \end{aligned} \tag{M1.10}$$

In the Cartesian coordinate space there is no need to distinguish between upper and lower indices so that (M1.10) may be written in different ways. We will return to this point later.

Finally, we wish to define the *position vector* $\mathbf{r}$. In a Cartesian coordinate system we may simply write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x^n\mathbf{i}_n = x_n\mathbf{i}_n \tag{M1.11}$$

In an oblique coordinate system, provided that the same basis exists everywhere in space, we may write the general form

$$\mathbf{r} = q^1\mathbf{q}_1 + q^2\mathbf{q}_2 + q^3\mathbf{q}_3 = q^n\mathbf{q}_n \tag{M1.12}$$

where the q^i are the measure numbers corresponding to the basis vectors $\mathbf{q}_i$. The form (M1.12) is also valid along the radius in a spherical coordinate system since the basis vectors do not change along this direction.

A different situation arises in case of curvilinear coordinate lines since the orientations of the basis vectors change with position. This is evident, for example, on considering the coordinate lines (lines of equal latitude and longitude) on the surface of a nonrotating sphere. In case of curvilinear coordinate lines the position vector $\mathbf{r}$ has to be replaced by the differential expression $d\mathbf{r} = dq^n\,\mathbf{q}_n$. Later we will discuss this topic in the required detail.

M1.3 Vector multiplication

M1.3.1 The scalar product of two vectors

By definition, the coordinate-free form of the scalar product is given by

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos(\mathbf{A}, \mathbf{B}) \tag{M1.13}$$

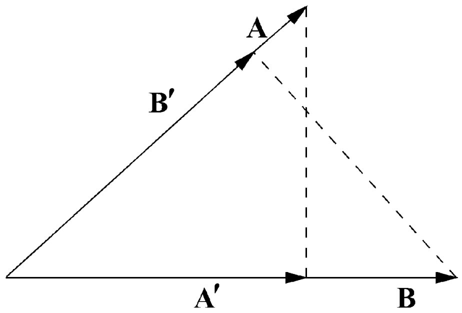

Fig. M1.4 Geometric interpretation of the scalar product.

If the vectors $\mathbf{A}$ and $\mathbf{B}$ are orthogonal the expression $\cos(\mathbf{A}, \mathbf{B}) = 0$ so that the scalar product vanishes. The following rules involving the scalar product are valid:

$$\begin{aligned} &\textit{the commutative law}: && \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \\ &\textit{the associative law}: && (k\mathbf{A})\cdot\mathbf{B} = k(\mathbf{A}\cdot\mathbf{B}) = k\mathbf{A}\cdot\mathbf{B} \\ &\textit{the distributive law}: && \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \end{aligned} \tag{M1.14}$$

Moreover, we recognize that the scalar product, also known as the dot product or inner product, may be represented by the orthogonal projections

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}'||\mathbf{B}|, \qquad \mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}'| \tag{M1.15}$$

whereby the vector $\mathbf{A}'$ is the projection of $\mathbf{A}$ on $\mathbf{B}$, and $\mathbf{B}'$ is the projection of $\mathbf{B}$ on $\mathbf{A}$; see Figure M1.4.

The component notation of the scalar product yields

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} = {} & A^1B^1\mathbf{q}_1\cdot\mathbf{q}_1 + A^1B^2\mathbf{q}_1\cdot\mathbf{q}_2 + A^1B^3\mathbf{q}_1\cdot\mathbf{q}_3 \\ & + A^2B^1\mathbf{q}_2\cdot\mathbf{q}_1 + A^2B^2\mathbf{q}_2\cdot\mathbf{q}_2 + A^2B^3\mathbf{q}_2\cdot\mathbf{q}_3 \\ & + A^3B^1\mathbf{q}_3\cdot\mathbf{q}_1 + A^3B^2\mathbf{q}_3\cdot\mathbf{q}_2 + A^3B^3\mathbf{q}_3\cdot\mathbf{q}_3 \end{aligned} \tag{M1.16}$$

Thus, in general the scalar product results in nine terms. Utilizing the Einstein summation convention we obtain the compact notation

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A^m\mathbf{q}_m\cdot B^n\mathbf{q}_n = A^mB^n\mathbf{q}_m\cdot\mathbf{q}_n = A^mB^ng_{mn}} \tag{M1.17}$$

The quantity g_{ij} is known as the covariant *metric fundamental quantity* representing an element of a covariant tensor of rank two or order two. This tensor is called the *metric tensor* or the *fundamental tensor*. The expression “covariant” will be described later. Since $\mathbf{q}_i\cdot\mathbf{q}_j = \mathbf{q}_j\cdot\mathbf{q}_i$ we have the identity

$$\boxed{g_{ij} = g_{ji}} \tag{M1.18}$$

On substituting for $\mathbf{A}$, $\mathbf{B}$ the unit vectors of the Cartesian coordinate system, we find the well-known orthogonality conditions for the Cartesian unit vectors

$$\mathbf{i}\cdot\mathbf{j}=0, \qquad \mathbf{i}\cdot\mathbf{k}=0, \qquad \mathbf{j}\cdot\mathbf{k}=0 \tag{M1.19}$$

or the normalization conditions

$$\mathbf{i}\cdot\mathbf{i}=1, \qquad \mathbf{j}\cdot\mathbf{j}=1, \qquad \mathbf{k}\cdot\mathbf{k}=1 \tag{M1.20}$$

For the special case of Cartesian coordinates, from (M1.16) we, therefore, obtain for the scalar product

$$\mathbf{A}\cdot\mathbf{B}=A_xB_x+A_yB_y+A_zB_z \tag{M1.21}$$

When the basis vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are oriented along the (x, y, z)-axes, the coordinates of their terminal points are given by

$$\mathbf{i}\colon \quad (1,0,0), \qquad \mathbf{j}\colon \quad (0,1,0), \qquad \mathbf{k}\colon \quad (0,0,1) \tag{M1.22}$$

This expression is the Euclidian three-dimensional space or the space of ordinary human life. On generalizing to the N-dimensional space we obtain

$$\mathbf{e}_1\colon \quad (1,0,\ldots,0), \qquad \mathbf{e}_2\colon \quad (0,1,\ldots,0), \qquad \ldots \qquad \mathbf{e}_N\colon \quad (0,0,\ldots,1) \tag{M1.23}$$

This equation is known as the Cartesian reference frame of the N-dimensional Euclidian space. In this space the generalized form of the position vector $\mathbf{r}$ is given by

$$\mathbf{r}=x^1\mathbf{e}_1+x^2\mathbf{e}_2+\cdots+x^N\mathbf{e}_N \tag{M1.24}$$

The length or the magnitude of the vector $\mathbf{r}$ is also known as the *Euclidian norm*

$$|\mathbf{r}|=\sqrt{\mathbf{r}\cdot\mathbf{r}}=\sqrt{(x^1)^2+(x^2)^2+\cdots+(x^N)^2} \tag{M1.25}$$

M1.3.2 The vector product of two vectors

In coordinate-free or invariant notation the vector product of two vectors is defined by

$$\mathbf{A}\times\mathbf{B}=\mathbf{C}=|\mathbf{A}|\,|\mathbf{B}|\sin(\mathbf{A},\mathbf{B})\,\mathbf{e}_C \tag{M1.26}$$

The unit vector $\mathbf{e}_C$ is perpendicular to the plane defined by the vectors $\mathbf{A}$ and $\mathbf{B}$. The direction of the vector $\mathbf{C}$ is defined in such a way that the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$

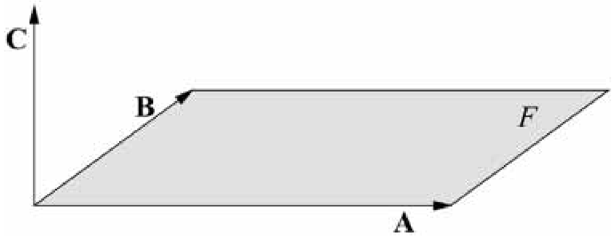

Fig. M1.5 Geometric interpretation of the vector or cross product.

form a right-handed system. The magnitude of **C** is equal to the area F of a parallelogram defined by the vectors **A** and **B** as shown in Figure M1.5. Interchanging the vectors **A** and **B** gives $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. This follows immediately from (M1.26) since the unit vector $\mathbf{e}_C$ now points in the opposite direction.

The following vector statements are valid:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ (k\mathbf{A}) \times \mathbf{B} &= \mathbf{A} \times (k\mathbf{B}) = k\mathbf{A} \times \mathbf{B} \\ \mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} \end{aligned} \tag{M1.27}$$

The component representation of the vector product yields

$$\boxed{\mathbf{A} \times \mathbf{B} = A^m \mathbf{q}_m \times B^n \mathbf{q}_n = \begin{vmatrix} \mathbf{q}_2 \times \mathbf{q}_3 & \mathbf{q}_3 \times \mathbf{q}_1 & \mathbf{q}_1 \times \mathbf{q}_2 \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{vmatrix}} \tag{M1.28}$$

By utilizing Cartesian coordinates we obtain the well-known relation

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{M1.29}$$

M1.3.3 The dyadic representation, the general product of two vectors

The general or *dyadic product* of two vectors **A** and **B** is given by

$$\Phi = \mathbf{AB} = (A^1\mathbf{q}_1 + A^2\mathbf{q}_2 + A^3\mathbf{q}_3)(B^1\mathbf{q}_1 + B^2\mathbf{q}_2 + B^3\mathbf{q}_3) \tag{M1.30}$$

It is seen that the vectors are not separated by a dot or a cross. At first glance this type of vector product seems strange. However, the advantage of this notation will

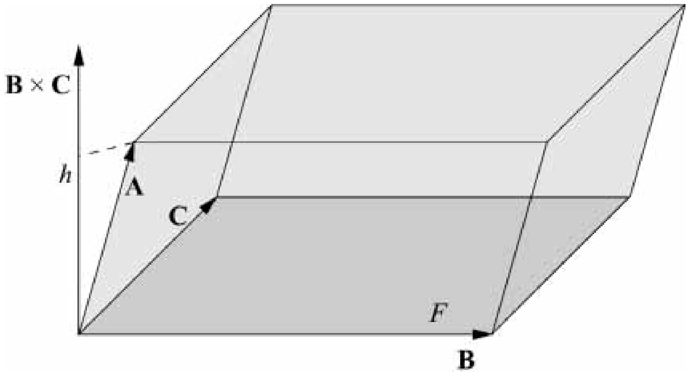

Fig. M1.6 Geometric representation of the scalar triple product.

become apparent later. On performing the dyadic multiplication we obtain

$$\boxed{\begin{aligned}\mathbf{\Phi} = \mathbf{AB} = A^1B^1\mathbf{q}_1\mathbf{q}_1 + A^1B^2\mathbf{q}_1\mathbf{q}_2 + A^1B^3\mathbf{q}_1\mathbf{q}_3 \\ + A^2B^1\mathbf{q}_2\mathbf{q}_1 + A^2B^2\mathbf{q}_2\mathbf{q}_2 + A^2B^3\mathbf{q}_2\mathbf{q}_3 \\ + A^3B^1\mathbf{q}_3\mathbf{q}_1 + A^3B^2\mathbf{q}_3\mathbf{q}_2 + A^3B^3\mathbf{q}_3\mathbf{q}_3\end{aligned}} \tag{M1.31}$$

In carrying out the general multiplication, we must be careful not to change the position of the basis vectors. The following statements are valid:

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}, \qquad \mathbf{AB} \neq \mathbf{BA} \tag{M1.32}$$

M1.3.4 The scalar triple product

The scalar triple product, sometimes also called the box product, is defined by

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = [\mathbf{A}, \mathbf{B}, \mathbf{C}] \tag{M1.33}$$

The absolute value of the scalar triple product measures the volume of the parallelepiped having the three vectors **A**, **B**, **C** as adjacent edges, see Figure M1.6. The height h of the parallelepiped is found by projecting the vector **A** onto the cross product $\mathbf{B} \times \mathbf{C}$. If the volume vanishes then the three vectors are coplanar. This situation will occur whenever a vector appears twice in the scalar triple product. It is apparent that, in the scalar triple product, any cyclic permutation of the factors leaves the value of the scalar triple product unchanged. A permutation that reverses the original cyclic order changes the sign of the product:

$$\begin{aligned}[\mathbf{A}, \mathbf{B}, \mathbf{C}] &= [\mathbf{B}, \mathbf{C}, \mathbf{A}] = [\mathbf{C}, \mathbf{A}, \mathbf{B}] \\ [\mathbf{A}, \mathbf{B}, \mathbf{C}] &= -[\mathbf{B}, \mathbf{A}, \mathbf{C}] = -[\mathbf{A}, \mathbf{C}, \mathbf{B}]\end{aligned} \tag{M1.34}$$

From these observations we may conclude that, in any scalar triple product, the dot and the cross can be interchanged without changing the magnitude and the sign of the scalar triple product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \tag{M1.35}$$

For the general vector basis the coordinate representation of the scalar triple product yields

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A^1 \mathbf{q}_1 + A^2 \mathbf{q}_2 + A^3 \mathbf{q}_3) \cdot \begin{vmatrix} \mathbf{q}_2 \times \mathbf{q}_3 & \mathbf{q}_3 \times \mathbf{q}_1 & \mathbf{q}_1 \times \mathbf{q}_2 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix} \tag{M1.36}$$

It is customary to assign the symbol $\sqrt{g}$ to the scalar triple product of the basis vectors:

$$\boxed{\sqrt{g} = \mathbf{q}_1 \cdot \mathbf{q}_2 \times \mathbf{q}_3} \tag{M1.37}$$

It is regrettable that the symbol g is also assigned to the acceleration due to gravity, but confusion is unlikely to occur. By combining equations (M1.36) and (M1.37) we obtain the following important form of the scalar triple product:

$$\boxed{[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \sqrt{g} \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix}} \tag{M1.38}$$

For the basis vectors of the Cartesian system we obtain from (M1.37)

$$\sqrt{g} = \mathbf{i} \cdot (\mathbf{j} \times \mathbf{k}) = 1 \tag{M1.39}$$

so that in the Cartesian coordinate system (M1.38) reduces to

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{M1.40}$$

In this expression, according to equation (M1.10), the components A^1, A^2, A^3, etc. have been written as A_x, A_y, A_z.

Without proof we accept the formula

$$[\mathbf{A}, \mathbf{B}, \mathbf{C}]^2 = \begin{vmatrix} \mathbf{A}\cdot\mathbf{A} & \mathbf{A}\cdot\mathbf{B} & \mathbf{A}\cdot\mathbf{C} \\ \mathbf{B}\cdot\mathbf{A} & \mathbf{B}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{C} \\ \mathbf{C}\cdot\mathbf{A} & \mathbf{C}\cdot\mathbf{B} & \mathbf{C}\cdot\mathbf{C} \end{vmatrix} \tag{M1.41}$$

which is known as the *Gram determinant*. The proof, however, will be given later. Application of this important formula gives

$$[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]^2 = \left(\sqrt{g}\right)^2 = \begin{vmatrix} \mathbf{q}_1\cdot\mathbf{q}_1 & \mathbf{q}_1\cdot\mathbf{q}_2 & \mathbf{q}_1\cdot\mathbf{q}_3 \\ \mathbf{q}_2\cdot\mathbf{q}_1 & \mathbf{q}_2\cdot\mathbf{q}_2 & \mathbf{q}_2\cdot\mathbf{q}_3 \\ \mathbf{q}_3\cdot\mathbf{q}_1 & \mathbf{q}_3\cdot\mathbf{q}_2 & \mathbf{q}_3\cdot\mathbf{q}_3 \end{vmatrix} = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = |g_{ij}| \tag{M1.42}$$

which involves all elements g_{ij} of the metric tensor. Comparison of (M1.37) and (M1.42) yields the important statement

$$\boxed{\mathbf{q}_1 \cdot (\mathbf{q}_2 \times \mathbf{q}_3) = \sqrt{g} = \sqrt{|g_{ij}|}} \tag{M1.43}$$

so that the scalar triple product involving the general basis vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ can easily be evaluated. This will be done in some detail when we consider various coordinate systems. Owing to (M1.43), $\sqrt{g}$ is called the *functional determinant* of the system.

M1.3.5 The vectorial triple product

At this point it will be sufficient to state the extremely important formula

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}} \tag{M1.44}$$

which is also known as the *Grassmann rule*. It should be noted that, without the parentheses, the meaning of (M1.44) is not unique. The proof of this equation will be given later with the help of the so-called reciprocal coordinate system.

M1.3.6 The scalar product of a vector with a dyadic

On performing the scalar product of a vector with a dyadic we see that the commutative law is not valid:

$$\mathbf{D} = \mathbf{A} \cdot (\mathbf{B}\mathbf{C}) = (\mathbf{A} \cdot \mathbf{B})\mathbf{C}, \qquad \mathbf{E} = (\mathbf{B}\mathbf{C}) \cdot \mathbf{A} = \mathbf{B}(\mathbf{C} \cdot \mathbf{A}) \tag{M1.45}$$

Whereas in the first expression the vectors $\mathbf{D}$ and $\mathbf{C}$ are collinear, in the second expression the direction of $\mathbf{E}$ is along the vector $\mathbf{B}$ so that $\mathbf{D} \neq \mathbf{E}$.

M1.3.7 Products involving four vectors

Let us consider the expression $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D})$. Defining the vector $\mathbf{F} = \mathbf{C} \times \mathbf{D}$ we obtain the scalar triple product

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{F} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{F}) = \mathbf{A} \cdot [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] \tag{M1.46}$$

This equation results from interchanging the dot and the cross and by replacing the vector $\mathbf{F}$ by its definition. Application of the Grassmann rule (M1.44) yields

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot [(\mathbf{B} \cdot \mathbf{D})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{D}] = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{M1.47}$$

so that equation (M1.46) can be written as

$$\boxed{(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = \begin{vmatrix} \mathbf{A} \cdot \mathbf{C} & \mathbf{A} \cdot \mathbf{D} \\ \mathbf{B} \cdot \mathbf{C} & \mathbf{B} \cdot \mathbf{D} \end{vmatrix}} \tag{M1.48}$$

The vector product of four vectors may be evaluated with the help of the Grassmann rule:

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = (\mathbf{F} \cdot \mathbf{D})\mathbf{C} - (\mathbf{F} \cdot \mathbf{C})\mathbf{D} \quad \text{with} \quad \mathbf{F} = \mathbf{A} \times \mathbf{B} \tag{M1.49}$$

On replacing $\mathbf{F}$ by its definition and using the rules of the scalar triple product, we find the following useful expression:

$$\boxed{(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = [\mathbf{A}, \mathbf{B}, \mathbf{D}]\mathbf{C} - [\mathbf{A}, \mathbf{B}, \mathbf{C}]\mathbf{D}} \tag{M1.50}$$

M1.4 Reciprocal coordinate systems

As will be seen shortly, operations with the so-called reciprocal basis systems result in particularly convenient mathematical expressions. Let us consider two basis systems. One of these is defined by the three linearly independent basis vectors $\mathbf{q}_i$, $i = 1, 2, 3$, and the other one by the linearly independent basis vectors $\mathbf{q}^i$, $i = 1, 2, 3$. To have reciprocality for the basis vectors the following relation must be valid:

$$\boxed{\mathbf{q}_i \cdot \mathbf{q}^k = \mathbf{q}^k \cdot \mathbf{q}_i = \delta_i^k \quad \text{with} \quad \delta_i^k = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}} \tag{M1.51}$$

where δ_i^k is the Kronecker-delta symbol. Reciprocal systems are also called *contragredient systems*. As is customary, the system represented by basis vectors with the lower index is called *covariant* while the system employing basis vectors with an upper index is called *contravariant*. Therefore, $\mathbf{q}_i$ and $\mathbf{q}^i$ are called *covariant* and *contravariant basis vectors*, respectively.

Consider for example in (M1.51) the case $i = k = 1$. While the scalar product $\mathbf{q}_1 \cdot \mathbf{q}^1 = 1$ may be viewed as a normalization condition for the two systems, the scalar products $\mathbf{q}_1 \cdot \mathbf{q}^2 = 0$ and $\mathbf{q}_1 \cdot \mathbf{q}^3 = 0$ are conditions of orthogonality. Thus, $\mathbf{q}_1$ is perpendicular to $\mathbf{q}^2$ and to $\mathbf{q}^3$ so that we may write

$$\mathbf{q}_1 = C(\mathbf{q}^2 \times \mathbf{q}^3) \tag{M1.52a}$$

where C is a factor of proportionality. On substituting this expression into the normalization condition we obtain for C

$$\mathbf{q}^1 \cdot \mathbf{q}_1 = C\mathbf{q}^1 \cdot (\mathbf{q}^2 \times \mathbf{q}^3) = 1 \implies C = \frac{1}{\mathbf{q}^1 \cdot (\mathbf{q}^2 \times \mathbf{q}^3)} \tag{M1.52b}$$

so that (M1.52a) yields

$$\mathbf{q}_1 = \frac{\mathbf{q}^2 \times \mathbf{q}^3}{[\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3]} \tag{M1.52c}$$

We may repeat this exercise with $\mathbf{q}_2$ and $\mathbf{q}_3$ and find the general expression

$$\boxed{\mathbf{q}_i = \frac{\mathbf{q}^j \times \mathbf{q}^k}{[\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3]}} \tag{M1.53}$$

with i, j, k in cyclic order. Similarly we may write for $\mathbf{q}^1$, with D as the proportionality constant,

$$\mathbf{q}^1 = D(\mathbf{q}_2 \times \mathbf{q}_3), \quad \mathbf{q}_1 \cdot \mathbf{q}^1 = D\mathbf{q}_1 \cdot (\mathbf{q}_2 \times \mathbf{q}_3) = 1 \implies \mathbf{q}^1 = \frac{\mathbf{q}_2 \times \mathbf{q}_3}{[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]} \tag{M1.54}$$

Thus, the general expression is

$$\boxed{\mathbf{q}^i = \frac{\mathbf{q}_j \times \mathbf{q}_k}{[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]}} \tag{M1.55}$$

with i, j, k in cyclic order. Equations (M1.53) and (M1.55) give the explicit expressions relating the basis vectors of the two reciprocal systems.

Let us consider the special case of the Cartesian coordinate system with basis vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. Application of (M1.55) shows that $\mathbf{i}^j = \mathbf{i}_j$ since $[\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3] = 1$, so that in the Cartesian coordinate system there is no difference between covariant and contravariant basis vectors. This is the reason why we have written $A^i = A_i,\ i = 1, 2, 3$ in (M1.10).

Now we return to equation (M1.43). By replacing the covariant basis vector $\mathbf{q}_1$ with the help of (M1.52c) and utilizing (M1.48) we find

$$\begin{aligned}
\mathbf{q}_1 \cdot (\mathbf{q}_2 \times \mathbf{q}_3) &= \frac{(\mathbf{q}^2 \times \mathbf{q}^3) \cdot (\mathbf{q}_2 \times \mathbf{q}_3)}{[\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3]} \\
&= \frac{1}{[\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3]} \begin{vmatrix} \mathbf{q}^2 \cdot \mathbf{q}_2 & \mathbf{q}^2 \cdot \mathbf{q}_3 \\ \mathbf{q}^3 \cdot \mathbf{q}_2 & \mathbf{q}^3 \cdot \mathbf{q}_3 \end{vmatrix} = \frac{1}{[\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3]}
\end{aligned} \tag{M1.56}$$

From (M1.51) it follows that the value of the determinant in (M1.56) is equal to 1. Since $\mathbf{q}_1 \cdot (\mathbf{q}_2 \times \mathbf{q}_3) = \sqrt{g}$ we immediately find

$$\boxed{[\mathbf{q}^1, \mathbf{q}^2, \mathbf{q}^3] = \frac{1}{\sqrt{g}}} \tag{M1.57}$$

Thus, the introduction of the contravariant basis vectors shows that (M1.43) and (M1.57) are inverse relations.

Often it is desirable to work with unit vectors having the same directions as the selected three linearly independent basis vectors. The desired relationships are

$$\mathbf{e}_i = \frac{\mathbf{q}_i}{|\mathbf{q}_i|} = \frac{\mathbf{q}_i}{\sqrt{\mathbf{q}_i \cdot \mathbf{q}_i}} = \frac{\mathbf{q}_i}{\sqrt{g_{ii}}}, \qquad \mathbf{e}^i = \frac{\mathbf{q}^i}{|\mathbf{q}^i|} = \frac{\mathbf{q}^i}{\sqrt{\mathbf{q}^i \cdot \mathbf{q}^i}} = \frac{\mathbf{q}^i}{\sqrt{g^{ii}}} \tag{M1.58}$$

While the scalar product of the covariant basis vectors $\mathbf{q}_i \cdot \mathbf{q}_j = g_{ij}$ defines the elements of the *covariant metric tensor*, the *contravariant metric tensor* is defined by the elements $\mathbf{q}^i \cdot \mathbf{q}^j = g^{ij}$, and we have

$$\boxed{\mathbf{q}^i \cdot \mathbf{q}^j = \mathbf{q}^j \cdot \mathbf{q}^i = g^{ij} = g^{ji}} \tag{M1.59}$$

Owing to the symmetry relations $g_{ij} = g_{ji}$ and $g^{ij} = g^{ji}$ each metric tensor is completely specified by six elements.

Some special cases follow directly from the definition (M1.13) of the scalar product. In case of an orthonormal system, such as the Cartesian coordinate system, we have

$$g_{ij} = g_{ji} = g^{ij} = g^{ji} = \delta_i^j \tag{M1.60}$$

As will be shown later, for any orthogonal system the following equation applies:

$$g_{ii} g^{ii} = 1 \tag{M1.61}$$

While in the Cartesian coordinate system the metric fundamental quantities are either 0 or 1, we cannot give any information about the g_{ij} or g^{ij} unless the coordinate system is specified. This will be done later when we consider various physical situations.

In the following we will give examples of the efficient use of reciprocal systems. Work is defined by the scalar product $dA = \mathbf{K} \cdot d\mathbf{r}$, where $\mathbf{K}$ is the force and $d\mathbf{r}$ is the path increment. In the Cartesian system we obtain a particularly simple result:

$$\mathbf{K} \cdot d\mathbf{r} = (K_x \mathbf{i} + K_y \mathbf{j} + K_z \mathbf{k}) \cdot (dx\, \mathbf{i} + dy\, \mathbf{j} + dz\, \mathbf{k}) = K_x\, dx + K_y\, dy + K_z\, dz \tag{M1.62}$$

consisting of three work contributions in the directions of the three coordinate axes. For specific applications it may be necessary, however, to employ more general coordinate systems. Let us consider, for example, an oblique coordinate system with contravariant components and covariant basis vectors of $\mathbf{K}$ and $d\mathbf{r}$. In this case work will be expressed by

$$\begin{aligned} \mathbf{K} \cdot d\mathbf{r} &= (K^1 \mathbf{q}_1 + K^2 \mathbf{q}_2 + K^3 \mathbf{q}_3) \cdot (dq^1 \mathbf{q}_1 + dq^2 \mathbf{q}_2 + dq^3 \mathbf{q}_3) \\ &= K^m\, dq^n\, \mathbf{q}_m \cdot \mathbf{q}_n = K^m\, dq^n\, g_{mn} \end{aligned} \tag{M1.63}$$

Expansion of this expression results in nine components in contrast to only three components of the Cartesian coordinate system. A great deal of simplification is achieved by employing reciprocal systems for the force and the path increment. As in the case of the Cartesian system, work can then be expressed by using only three terms:

$$\begin{aligned} \mathbf{K} \cdot d\mathbf{r} &= (K_1 \mathbf{q}^1 + K_2 \mathbf{q}^2 + K_3 \mathbf{q}^3) \cdot (dq^1\, \mathbf{q}_1 + dq^2\, \mathbf{q}_2 + dq^3\, \mathbf{q}_3) \\ &= K_m\, dq^n\, \mathbf{q}^m \cdot \mathbf{q}_n = K_m\, dq^n\, \delta_n^m = K_1\, dq^1 + K_2\, dq^2 + K_3\, dq^3 \end{aligned} \tag{M1.64a}$$

or

$$\begin{aligned}\mathbf{K}\cdot d\mathbf{r} &= (K^1\mathbf{q}_1 + K^2\mathbf{q}_2 + K^3\mathbf{q}_3)\cdot(dq_1\,\mathbf{q}^1 + dq_2\,\mathbf{q}^2 + dq_3\,\mathbf{q}^3)\\ &= K^m\,dq_n\,\mathbf{q}_m\cdot\mathbf{q}^n = K^m\,dq_n\,\delta_m^n = K^1\,dq_1 + K^2\,dq_2 + K^3\,dq_3\end{aligned} \tag{M1.64b}$$

Finally, utilizing reciprocal coordinate systems, it is easy to give the proof of the Grassmann rule (M1.44). Let us consider the expression $\mathbf{D} = \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. According to the definition (M1.26) of the vector product, $\mathbf{D}$ is perpendicular to $\mathbf{A}$ and to $(\mathbf{B}\times\mathbf{C})$. Therefore, $\mathbf{D}$ must lie in the plane defined by the vectors $\mathbf{B}$ and $\mathbf{C}$ so that we may write

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \lambda\mathbf{B} + \mu\mathbf{C} \tag{M1.65}$$

where λ and μ are unknown scalars to be determined. To make use of the properties of the reciprocal system, we first set $\mathbf{B} = \mathbf{q}_1$ and $\mathbf{C} = \mathbf{q}_2$. These two vectors define a plane oblique coordinate system. To complete the system we assume that the vector $\mathbf{q}_3$ is a unit vector orthogonal to the plane spanned by $\mathbf{q}_1$ and $\mathbf{q}_2$. Thus, we have

$$\mathbf{B} = \mathbf{q}_1, \qquad \mathbf{C} = \mathbf{q}_2, \qquad \mathbf{e}_3 = \frac{\mathbf{q}_1\times\mathbf{q}_2}{|\mathbf{q}_1\times\mathbf{q}_2|} \tag{M1.66}$$

and

$$\mathbf{q}_1\cdot(\mathbf{q}_2\times\mathbf{e}_3) = \mathbf{e}_3\cdot(\mathbf{q}_1\times\mathbf{q}_2) = \mathbf{e}_3\cdot\mathbf{e}_3\,|\mathbf{q}_1\times\mathbf{q}_2| = |\mathbf{q}_1\times\mathbf{q}_2| \tag{M1.67}$$

According to (M1.55), the coordinate system which is reciprocal to the $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ system is given by

$$\mathbf{q}^1 = \frac{\mathbf{q}_2\times\mathbf{e}_3}{|\mathbf{q}_1\times\mathbf{q}_2|}, \qquad \mathbf{q}^2 = \frac{\mathbf{e}_3\times\mathbf{q}_1}{|\mathbf{q}_1\times\mathbf{q}_2|}, \qquad \mathbf{e}^3 = \frac{\mathbf{q}_1\times\mathbf{q}_2}{|\mathbf{q}_1\times\mathbf{q}_2|} = \mathbf{e}_3 \tag{M1.68}$$

The determination of λ and μ follows from scalar multiplication of $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\times(\mathbf{q}_1\times\mathbf{q}_2) = \lambda\mathbf{q}_1 + \mu\mathbf{q}_2$ by the reciprocal basis vectors $\mathbf{q}^1$ and $\mathbf{q}^2$:

$$\begin{aligned}\lambda &= \left[\mathbf{A}\times(\mathbf{q}_1\times\mathbf{q}_2)\right]\cdot\mathbf{q}^1 = \mathbf{A}\times(\mathbf{q}_1\times\mathbf{q}_2)\cdot\frac{(\mathbf{q}_2\times\mathbf{e}_3)}{|\mathbf{q}_1\times\mathbf{q}_2|}\\ &= (\mathbf{A}\times\mathbf{e}_3)\cdot(\mathbf{q}_2\times\mathbf{e}_3) = \mathbf{A}\cdot\mathbf{q}_2 = \mathbf{A}\cdot\mathbf{C}\end{aligned} \tag{M1.69a}$$

Analogously we obtain

$$\mu = \left[\mathbf{A}\times(\mathbf{q}_1\times\mathbf{q}_2)\right]\cdot\mathbf{q}^2 = (\mathbf{A}\times\mathbf{e}_3)\cdot(\mathbf{e}_3\times\mathbf{q}_1) = -\mathbf{A}\cdot\mathbf{q}_1 = -\mathbf{A}\cdot\mathbf{B} \tag{M1.69b}$$

Substitution of λ and μ into (M1.65) gives the final result

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \tag{M1.70}$$

M1.5 Vector representations

The vector $\mathbf{A}$ may be represented with the help of the covariant basis vectors $\mathbf{q}_i$ or $\mathbf{e}_i$ and the contravariant basis vectors $\mathbf{q}^i$ or $\mathbf{e}^i$ as

$$\mathbf{A} = A^m \mathbf{q}_m = A_m \mathbf{q}^m = \overset{*}{A}{}^m \mathbf{e}_m = \overset{*}{A}_m \mathbf{e}^m \tag{M1.71}$$

The invariant character of $\mathbf{A}$ is recognized by virtue of the fact that we have the same number of upper and lower indices. In addition to the *contravariant* and *covariant* measure numbers A^i and A_i of the basis vectors $\mathbf{q}_i$ and $\mathbf{q}^i$ we have also introduced the *physical measure numbers* $\overset{*}{A}{}^i$ and $\overset{*}{A}_i$ of the unit vectors $\mathbf{e}_i$ and $\mathbf{e}^i$. In general the contravariant and covariant measure numbers do not have uniform dimensions. This becomes obvious on considering, for example, the spherical coordinate system which is defined by two angles, which are measured in degrees, and the radius of the sphere, which is measured in units of length. Physical measure numbers, however, are uniformly dimensioned. They represent the lengths of the components of a vector in the directions of the basis vectors. The formal definitions of the physical measure numbers are

$$\overset{*}{A}{}^i = A^i\, |\mathbf{q}_i| = A^i \sqrt{g_{ii}}, \qquad \overset{*}{A}_i = A_i |\mathbf{q}^i| = A_i \sqrt{g^{ii}} \tag{M1.72}$$

Now we will show what consequences arise by interpreting the measure numbers vectorially. Scalar multiplication of $\mathbf{A} = A^n \mathbf{q}_n$ by the reciprocal basis vector $\mathbf{q}^i$ yields for A^i

$$\boxed{\mathbf{A} \cdot \mathbf{q}^i = A^m \mathbf{q}_m \cdot \mathbf{q}^i = A^m \delta^i_m = A^i} \tag{M1.73}$$

so that

$$\mathbf{A} = A^m \mathbf{q}_m = \mathbf{A} \cdot \mathbf{q}^m \mathbf{q}_m \tag{M1.74}$$

This expression leads to the introduction to the *unit dyadic* $\mathbb{E}$,

$$\mathbb{E} = \mathbf{q}^m \mathbf{q}_m \tag{M1.75a}$$

This very special dyadic or unit tensor of rank two has the same degree of importance in tensor analysis as the unit vector in vector analysis. The unit dyadic $\mathbb{E}$ is indispensable and will accompany our work from now on. In the Cartesian coordinate system the unit dyadic is given by

$$\mathbb{E} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2 + \mathbf{i}_3\mathbf{i}_3 \tag{M1.75b}$$

We repeat the above procedure by representing the vector $\mathbf{A}$ as $\mathbf{A} = A_m \mathbf{q}^m$. Scalar multiplication by $\mathbf{q}_i$ results in

$$\boxed{\mathbf{A} \cdot \mathbf{q}_i = A_m \mathbf{q}^m \cdot \mathbf{q}_i = A_i} \tag{M1.76}$$

and the equivalent definition of the unit dyadic $\mathbb{E}$

$$\mathbf{A} = A_m \mathbf{q}^m = \mathbf{A} \cdot \mathbf{q}_m \mathbf{q}^m \implies \mathbb{E} = \mathbf{q}_m \mathbf{q}^m \tag{M1.77}$$

Of particular interest is the scalar product of two unit dyadics:

$$\begin{aligned} \mathbb{E} \cdot \mathbb{E} &= \mathbf{q}^m \mathbf{q}_m \cdot \mathbf{q}^n \mathbf{q}_n = \mathbf{q}^m \delta^n_m \mathbf{q}_n = \mathbf{q}^m \mathbf{q}_m = \mathbb{E} \\ \mathbb{E} \cdot \mathbb{E} &= \mathbf{q}_m \mathbf{q}^m \cdot \mathbf{q}_n \mathbf{q}^n = \mathbf{q}_m \delta^m_n \mathbf{q}^n = \mathbf{q}_m \mathbf{q}^m = \mathbb{E} \end{aligned} \tag{M1.78}$$

From these expressions we obtain additional representations of the unit dyadic that involve the metric fundamental quantities g_{ij} and g^{ij}:

$$\mathbb{E} \cdot \mathbb{E} = \mathbf{q}^m \mathbf{q}_m \cdot \mathbf{q}_n \mathbf{q}^n = g_{mn} \mathbf{q}^m \mathbf{q}^n = \mathbf{q}_m \mathbf{q}^m \cdot \mathbf{q}^n \mathbf{q}_n = g^{mn} \mathbf{q}_m \mathbf{q}_n \tag{M1.79}$$

Again it should be carefully observed that each expression contains an equal number of subscripts and superscripts to stress the invariant character of the unit dyadic. We collect the important results involving the unit dyadic as

$$\boxed{\mathbb{E} = \mathbf{q}^m \mathbf{q}_m = \delta^n_m \mathbf{q}^m \mathbf{q}_n = \mathbf{q}_m \mathbf{q}^m = \delta^m_n \mathbf{q}_m \mathbf{q}^n = g_{mn} \mathbf{q}^m \mathbf{q}^n = g^{mn} \mathbf{q}_m \mathbf{q}_n} \tag{M1.80}$$

Scalar multiplication of $\mathbb{E}$ in two of the forms of (M1.80) with $\mathbf{q}_i$ results in

$$\begin{aligned} \mathbb{E} \cdot \mathbf{q}_i &= (\mathbf{q}_m \mathbf{q}^m) \cdot \mathbf{q}_i = \mathbf{q}_m \delta^m_i = \mathbf{q}_i \\ &= (g_{mn} \mathbf{q}^m \mathbf{q}^n) \cdot \mathbf{q}_i = g_{mn} \mathbf{q}^m \delta^n_i = g_{im} \mathbf{q}^m \end{aligned} \tag{M1.81}$$

Hence, we see immediately that

$$\boxed{\mathbf{q}_i = g_{im} \mathbf{q}^m} \tag{M1.82}$$

This very useful expression is known as the *raising rule* for the index of the basis vector $\mathbf{q}_i$. Analogously we multiply the unit dyadic by $\mathbf{q}^i$ to obtain

$$\mathbb{E} \cdot \mathbf{q}^i = (\mathbf{q}^m \mathbf{q}_m) \cdot \mathbf{q}^i = \mathbf{q}^i = (g^{mn} \mathbf{q}_m \mathbf{q}_n) \cdot \mathbf{q}^i = g^{im} \mathbf{q}_m \tag{M1.83}$$

and thus

$$\boxed{\mathbf{q}^i = g^{im} \mathbf{q}_m} \tag{M1.84}$$

which is known as the *lowering rule* for the index of the contravariant basis vector $\mathbf{q}^i$.

With the help of the unit dyadic we are in a position to find additional important rules of tensor analysis. In order to avoid confusion, it is often necessary to replace a letter representing a summation index by another letter so that the letter representing a summation does not occur more often than twice. If the replacement is done properly, the meaning of any mathematical expression will not change. Let us consider the expression

$$\mathbb{E} = \mathbf{q}_r \mathbf{q}^r = g_{rm} g^{rn} \mathbf{q}^m \mathbf{q}_n = \delta_m^n \mathbf{q}^m \mathbf{q}_n \tag{M1.85}$$

Application of (M1.82) and (M1.84) gives the expression to the right of the second equality sign. For comparison purposes we have also added one of the forms of (M1.80) as the final expression in (M1.85). It should be carefully observed that the summation indices m, n, r occur twice only.

To take full advantage of the reciprocal systems we perform a scalar multiplication first by the contravariant basis vector $\mathbf{q}^i$ and then by the covariant basis vector $\mathbf{q}_j$, yielding

$$(\mathbb{E} \cdot \mathbf{q}^i) \cdot \mathbf{q}_j = g_{rm} g^{rn} \delta_n^i \delta_j^m = \delta_m^n \delta_n^i \delta_j^m \tag{M1.86}$$

from which it follows immediately that

$$g_{rj} g^{ri} = \delta_j^i \tag{M1.87a}$$

By interchanging i and j, observing the symmetry of the fundamental quantities, we find

$$g_{ir} g^{rj} = \delta_i^j \quad \text{or} \quad (g_{ij})(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow (g_{ij}) = (g^{ij})^{-1} \tag{M1.87b}$$

Hence, the matrices (g_{ij}) and (g^{ij}) are inverse to each other. Owing to the symmetry properties of the metric fundamental quantities, i.e. $g_{ij} = g_{ji}$ and $g^{ij} = g^{ji}$, we need six elements only to specify either metric tensor. In case of an orthogonal system $g_{ij} = 0$, $g^{ij} = 0$ for $i \neq j$ so that (M1.87a) reduces to

$$g_{ii} g^{ii} = 1 \tag{M1.88}$$

thus verifying equation (M1.61). At this point we must recall the rule that we do not sum over repeated free indices i, j, k, l.

Next we wish to show that, in an orthonormal system, there is no difference between contravariant and covariant basis vectors. The proof is very simple:

$$\mathbf{e}_i = \frac{\mathbf{q}_i}{\sqrt{g_{ii}}} = \frac{g_{in}}{\sqrt{g_{ii}}} \mathbf{q}^n = \sqrt{g_{ii}} \mathbf{q}^i = \frac{\mathbf{q}^i}{\sqrt{g^{ii}}} = \mathbf{e}^i \tag{M1.89}$$

Here use of the raising rule has been made. With the help of (M1.89) it is easy to show that there is no difference between contravariant and covariant physical measure numbers. Utilizing (M1.71) we find

$$\mathbf{A} \cdot \mathbf{e}^i = \mathbf{A} \cdot \mathbf{e}_i \implies \overset{*}{A}{}^n \mathbf{e}_n \cdot \mathbf{e}^i = \overset{*}{A}_n \mathbf{e}^n \cdot \mathbf{e}_i \implies \overset{*}{A}{}^i = \overset{*}{A}_i \tag{M1.90}$$

M1.6 Products of vectors in general coordinate systems

There are various ways to express the dyadic product of vector **A** with vector **B** by employing covariant and contravariant basis vectors:

$$\mathbf{AB} = A^m B^n \mathbf{q}_m \mathbf{q}_n = A_m B_n \mathbf{q}^m \mathbf{q}^n = A_m B^n \mathbf{q}^m \mathbf{q}_n = A^m B_n \mathbf{q}_m \mathbf{q}^n \tag{M1.91}$$

This yields four possibilities for formulating the scalar product $\mathbf{A} \cdot \mathbf{B}$:

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= A^m B^n \mathbf{q}_m \cdot \mathbf{q}_n = A^m B^n g_{mn} = A_m B_n \mathbf{q}^m \cdot \mathbf{q}^n = A_m B_n g^{mn} \\ &= A_m B^n \mathbf{q}^m \cdot \mathbf{q}_n = A_m B^m = A^m B_n \mathbf{q}_m \cdot \mathbf{q}^n = A^m B_m\end{aligned} \tag{M1.92}$$

[1] The last two forms with mixed basis vectors (covariant and contravariant) are more convenient since the sums involve the evaluation of only three terms. In contrast, nine terms are required for the first two forms since they involve the metric fundamental quantities.

There are two useful forms in which to express the vector product $\mathbf{A} \times \mathbf{B}$. From the basic definition (M1.28) and the properties of the reciprocal systems (M1.55) we obtain

$$\mathbf{A} \times \mathbf{B} = A^m \mathbf{q}_m \times B^n \mathbf{q}_n = \sqrt{g} \begin{vmatrix} \mathbf{q}^1 & \mathbf{q}^2 & \mathbf{q}^3 \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{vmatrix} \tag{M1.93}$$

where all measure numbers are of the contravariant type. If it is desirable to express the vector product in terms of covariant measure numbers we use (M1.53) and (M1.57). Thus, we find

$$\mathbf{A} \times \mathbf{B} = A_m \mathbf{q}^m \times B_n \mathbf{q}^n = \frac{1}{\sqrt{g}} \begin{vmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{M1.94}$$

The two forms involving mixed basis vectors are not used, in general.

On performing the scalar triple product operation (M1.33) we find

$$\begin{aligned}[\mathbf{A}, \mathbf{B}, \mathbf{C}] &= A^m \mathbf{q}_m \cdot (B^n \mathbf{q}_n \times C^r \mathbf{q}_r) \\ &= A^m \sqrt{g} \mathbf{q}_m \cdot \begin{vmatrix} \mathbf{q}^1 & \mathbf{q}^2 & \mathbf{q}^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix} = \sqrt{g} \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix}\end{aligned} \tag{M1.95}$$

[1] For the scalar product $\mathbf{A} \cdot \mathbf{A}$ we usually write $\mathbf{A}^2$.