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Part 1

Mathematical tools

M1

Algebra of vectors

M1.1 Basic concepts and definitions

A *scalar* is a quantity that is specified by its sign and by its magnitude. Examples are temperature, the specific volume, and the humidity of the air. Scalars will be written using Latin or Greek letters such as $a, b, \ldots, A, B, \ldots, \alpha, \beta, \ldots$. A *vector* requires for its complete characterization the specification of magnitude and direction. Examples are the velocity vector and the force vector. A vector will be represented by a boldfaced letter such as $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{A}, \mathbf{B}, \ldots$. A *unit vector* is a vector of prescribed direction and of magnitude 1. Employing the unit vector $\mathbf{e}_A$, the arbitrary vector $\mathbf{A}$ can be written as

$$\mathbf{A} = |\mathbf{A}|\,\mathbf{e}_A = A\mathbf{e}_A \implies \mathbf{e}_A = \frac{\mathbf{A}}{|\mathbf{A}|} \tag{M1.1}$$

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if they have the same magnitude and direction regardless of the position of their initial points,

that is $|\mathbf{A}| = |\mathbf{B}|$ and $\mathbf{e}_A = \mathbf{e}_B$. Two vectors are *collinear* if they are parallel or antiparallel. Three vectors that lie in the same plane are called *coplanar*. Two vectors always lie in the same plane since they define the plane. The following rules are valid:

$$\begin{aligned}
&\textit{the commutative law:} && \mathbf{A} \pm \mathbf{B} = \mathbf{B} \pm \mathbf{A}, \quad \mathbf{A}\alpha = \alpha\mathbf{A} \\
&\textit{the associative law:} && \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}, \quad \alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A} \\
&\textit{the distributive law:} && (\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}
\end{aligned} \tag{M1.2}$$

The concept of linear dependence of a set of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ is closely connected with the dimensionality of space. The following definition applies: A set of N vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ of the same dimension is linearly dependent if there exists a set of numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$, not all of which are zero, such that

$$\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_N\mathbf{a}_N = 0 \tag{M1.3}$$

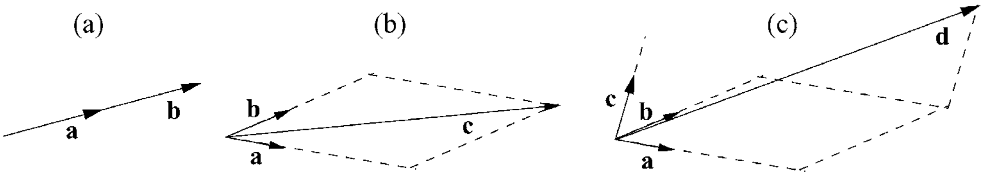

Fig. M1.1 Linear vector spaces: (a) one-dimensional, (b) two-dimensional, and (c) three-dimensional.

If no such numbers exist, the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ are said to be linearly independent. To get the geometric meaning of this definition, we consider the vectors **a** and **b** as shown in Figure M1.1(a). We can find a number $k \neq 0$ such that

$$\mathbf{b} = k\mathbf{a} \tag{M1.4a}$$

By setting $k = -\alpha/\beta$ we obtain the symmetrized form

$$\alpha\mathbf{a} + \beta\mathbf{b} = 0 \tag{M1.4b}$$

Assuming that neither α nor β is equal to zero then it follows from the above definition that two collinear vectors are linearly dependent. They define the one-dimensional *linear vector space*. Consider two noncollinear vectors **a** and **b** as shown in Figure M1.1(b). Every vector **c** in their plane can be represented by

$$\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b} \quad \text{or} \quad \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0 \tag{M1.5}$$

with a suitable choice of the constants k_1 and k_2. Equation (M1.5) defines a two-dimensional linear vector space. Since not all constants α, β, γ are zero, this formula insures that the three vectors in the two-dimensional space are linearly dependent. Taking three noncoplanar vectors **a**, **b**, and **c**, we can represent every vector **d** in the form

$$\mathbf{d} = k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c} \tag{M1.6}$$

in a three-dimensional linear vector space, see Figure M1.1(c). This can be generalized by stating that, in an N-dimensional linear vector space, every vector can be represented in the form

$$\mathbf{x} = k_1\mathbf{a}_1 + k_2\mathbf{a}_2 + \cdots + k_N\mathbf{a}_N \tag{M1.7}$$

where the $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ are linearly independent vectors. Any set of vectors containing more than N vectors in this space is linearly dependent.

Table M1.1. *Extensive quantities of different degrees for the N-dimensional linear vector space*

Extensive quantity	Degree ν	Symbol	Number of vectors	Number of components
Scalar	0	B	0	$N^0 = 1$
Vector	1	$\mathbf{B}$	1	N^1
Dyadic	2	$\mathbb{B}$	2	N^2

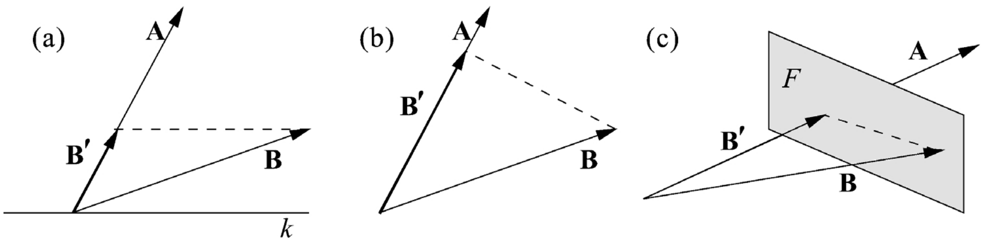

Fig. M1.2 Projection of a vector **B** onto a vector **A**.

We call the set of N linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ the *basis vectors* of the N-dimensional linear vector space. The numbers $k_1, k_2, \ldots, k_N$ appearing in (M1.7) are the *measure numbers* associated with the basis vectors. The term $k_i\mathbf{a}_i$ of the vector $\mathbf{x}$ in (M1.7) is the *component of this vector* in the direction $\mathbf{a}_i$.

A vector **B** may be projected onto the vector **A** parallel to the direction of a straight line k as shown in Figure M1.2(a). If the direction of the straight line k is not given, we perform an orthogonal projection as shown in part (b) of this figure. A projection in three-dimensional space requires a plane F parallel to which the projection of the vector **B** onto the vector **A** can be carried out; see Figure M1.2(c).

In vector analysis an *extensive quantity* of degree ν is defined as a homogeneous sum of general products of vectors (with no dot or cross between the vectors). The number of vectors in a product determines the degree of the extensive quantity. This definition may seem strange to begin with, but it will be familiar soon. Thus, a scalar is an extensive quantity of degree zero, and a vector is an extensive quantity of degree one. An extensive quantity of degree two is called a *dyadic*. Every dyadic $\mathbb{B}$ may be represented as the sum of three or more *dyads*. $\mathbb{B} = \mathbf{p}_1\mathbf{P}_1 + \mathbf{p}_2\mathbf{P}_2 + \mathbf{p}_3\mathbf{P}_3 + \cdots$. Either the *antecedents* $\mathbf{p}_i$ or the *consequents* $\mathbf{P}_i$ may be arbitrarily assigned as long as they are linearly independent. Our practical work will be restricted to extensive quantities of degree two or less. Extensive quantities of degree three and four also appear in the highly specialized literature. Table M1.1 gives a list of extensive quantities used in our work. Thus, in the three-dimensional linear vector space with $N = 3$, a vector consists of three and a dyadic of nine components.

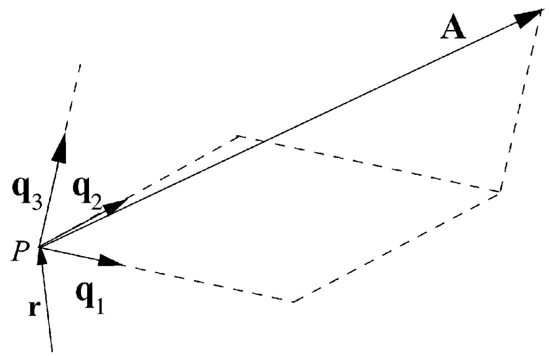

Fig. M1.3 The general vector basis $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ of the three-dimensional space.

M1.2 Reference frames

The representation of a vector in component form depends on the choice of a particular coordinate system. A *general vector basis* at a given point in three-dimensional space is defined by three arbitrary linearly independent basis vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ spanning the space. In general, the basis vectors are neither orthogonal nor unit vectors; they may also vary in space and in time.

Consider a position vector $\mathbf{r}$ extending from an arbitrary origin to a point P in space. An arbitrary vector $\mathbf{A}$ extending from P is defined by the three basis vectors $\mathbf{q}_i$, $i = 1, 2, 3$, existing at P at time t, as shown in Figure M1.3 for an oblique coordinate system. Hence, the vector $\mathbf{A}$ may be written as

$$\mathbf{A} = A^1\mathbf{q}_1 + A^2\mathbf{q}_2 + A^3\mathbf{q}_3 = \sum_{k=1}^{3} A^k\mathbf{q}_k \tag{M1.8}$$

where it should be observed that the so-called *affine measure numbers* A^1, A^2, A^3 carry superscripts, and the basis vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ carry subscripts. This type of notation is used in the *Ricci calculus*, which is the tensor calculus for nonorthonormal coordinate systems. Furthermore, it should be noted that there must be an equal number of upper and lower indices.

Formula (M1.8) can be written more briefly with the help of the familiar *Einstein summation convention* which omits the summation sign:

$$\mathbf{A} = A^1\mathbf{q}_1 + A^2\mathbf{q}_2 + A^3\mathbf{q}_3 = A^n\mathbf{q}_n \tag{M1.9}$$

We will agree on the following notation: Whenever an index (subscript or superscript) m, n, p, q, r, s, t, is repeated in a term, we are to sum over that index from 1 to 3, or more generally to N. In contrast to the summation indices m, n, p, q, r, s, t, the letters i, j, k, l are considered to be "free" indices that are used to enumerate equations. Note that summation is not implied even if the free indices occur twice in a term or even more often.

A special case of the general vector basis is the *Cartesian vector basis* represented by the three orthogonal unit vectors **i**, **j**, **k**, or, more conveniently, $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$. Each of these three unit vectors has the same direction at all points of space. However, in rotating coordinate systems these unit vectors also depend on time. The arbitrary vector **A** may be represented by

$$\begin{aligned} &\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} = A^n\mathbf{i}_n = A_n\mathbf{i}_n \\ \text{with} \quad &A_x = A^1 = A_1, \quad A_y = A^2 = A_2, \quad A_z = A^3 = A_3 \end{aligned} \tag{M1.10}$$

In the Cartesian coordinate space there is no need to distinguish between upper and lower indices so that (M1.10) may be written in different ways. We will return to this point later.

Finally, we wish to define the *position vector* **r**. In a Cartesian coordinate system we may simply write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x^n\mathbf{i}_n = x_n\mathbf{i}_n \tag{M1.11}$$

In an oblique coordinate system, provided that the same basis exists everywhere in space, we may write the general form

$$\mathbf{r} = q^1\mathbf{q}_1 + q^2\mathbf{q}_2 + q^3\mathbf{q}_3 = q^n\mathbf{q}_n \tag{M1.12}$$

where the q^i are the measure numbers corresponding to the basis vectors $\mathbf{q}_i$. The form (M1.12) is also valid along the radius in a spherical coordinate system since the basis vectors do not change along this direction.

A different situation arises in case of curvilinear coordinate lines since the orientations of the basis vectors change with position. This is evident, for example, on considering the coordinate lines (lines of equal latitude and longitude) on the surface of a nonrotating sphere. In case of curvilinear coordinate lines the position vector **r** has to be replaced by the differential expression $d\mathbf{r} = dq^n\,\mathbf{q}_n$. Later we will discuss this topic in the required detail.

M1.3 Vector multiplication

M1.3.1 The scalar product of two vectors

By definition, the coordinate-free form of the scalar product is given by

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos(\mathbf{A},\mathbf{B}) \tag{M1.13}$$

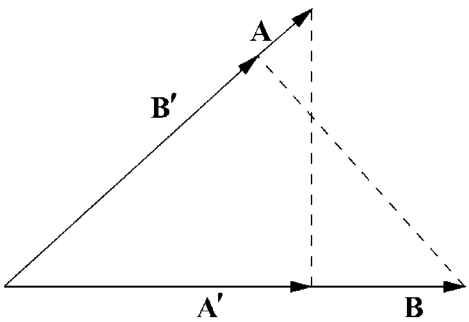

Fig. M1.4 Geometric interpretation of the scalar product.

If the vectors $\mathbf{A}$ and $\mathbf{B}$ are orthogonal the expression $\cos(\mathbf{A}, \mathbf{B}) = 0$ so that the scalar product vanishes. The following rules involving the scalar product are valid:

$$\begin{aligned} &\textit{the commutative law}: && \mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \\ &\textit{the associative law}: && (k\mathbf{A})\cdot\mathbf{B} = k(\mathbf{A}\cdot\mathbf{B}) = k\mathbf{A}\cdot\mathbf{B} \\ &\textit{the distributive law}: && \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \end{aligned} \tag{M1.14}$$

Moreover, we recognize that the scalar product, also known as the dot product or inner product, may be represented by the orthogonal projections

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}'||\mathbf{B}|, \qquad \mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}'| \tag{M1.15}$$

whereby the vector $\mathbf{A}'$ is the projection of $\mathbf{A}$ on $\mathbf{B}$, and $\mathbf{B}'$ is the projection of $\mathbf{B}$ on $\mathbf{A}$; see Figure M1.4.

The component notation of the scalar product yields

$$\begin{aligned} \mathbf{A}\cdot\mathbf{B} = {} & A^1B^1\mathbf{q}_1\cdot\mathbf{q}_1 + A^1B^2\mathbf{q}_1\cdot\mathbf{q}_2 + A^1B^3\mathbf{q}_1\cdot\mathbf{q}_3 \\ & + A^2B^1\mathbf{q}_2\cdot\mathbf{q}_1 + A^2B^2\mathbf{q}_2\cdot\mathbf{q}_2 + A^2B^3\mathbf{q}_2\cdot\mathbf{q}_3 \\ & + A^3B^1\mathbf{q}_3\cdot\mathbf{q}_1 + A^3B^2\mathbf{q}_3\cdot\mathbf{q}_2 + A^3B^3\mathbf{q}_3\cdot\mathbf{q}_3 \end{aligned} \tag{M1.16}$$

Thus, in general the scalar product results in nine terms. Utilizing the Einstein summation convention we obtain the compact notation

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A^m\mathbf{q}_m\cdot B^n\mathbf{q}_n = A^mB^n\mathbf{q}_m\cdot\mathbf{q}_n = A^mB^ng_{mn}} \tag{M1.17}$$

The quantity g_{ij} is known as the covariant *metric fundamental quantity* representing an element of a covariant tensor of rank two or order two. This tensor is called the *metric tensor* or the *fundamental tensor*. The expression "covariant" will be described later. Since $\mathbf{q}_i\cdot\mathbf{q}_j = \mathbf{q}_j\cdot\mathbf{q}_i$ we have the identity

$$\boxed{g_{ij} = g_{ji}} \tag{M1.18}$$

On substituting for **A**, **B** the unit vectors of the Cartesian coordinate system, we find the well-known orthogonality conditions for the Cartesian unit vectors

$$\mathbf{i}\cdot\mathbf{j}=0,\qquad \mathbf{i}\cdot\mathbf{k}=0,\qquad \mathbf{j}\cdot\mathbf{k}=0 \tag{M1.19}$$

or the normalization conditions

$$\mathbf{i}\cdot\mathbf{i}=1,\qquad \mathbf{j}\cdot\mathbf{j}=1,\qquad \mathbf{k}\cdot\mathbf{k}=1 \tag{M1.20}$$

For the special case of Cartesian coordinates, from (M1.16) we, therefore, obtain for the scalar product

$$\mathbf{A}\cdot\mathbf{B}=A_xB_x+A_yB_y+A_zB_z \tag{M1.21}$$

When the basis vectors **i**, **j**, **k** are oriented along the (x, y, z)-axes, the coordinates of their terminal points are given by

$$\mathbf{i}:\quad (1,0,0),\qquad \mathbf{j}:\quad (0,1,0),\qquad \mathbf{k}:\quad (0,0,1) \tag{M1.22}$$

This expression is the Euclidian three-dimensional space or the space of ordinary human life. On generalizing to the N-dimensional space we obtain

$$\mathbf{e}_1:\quad (1,0,\ldots,0),\qquad \mathbf{e}_2:\quad (0,1,\ldots,0),\qquad \ldots\qquad \mathbf{e}_N:\quad (0,0,\ldots,1) \tag{M1.23}$$

This equation is known as the Cartesian reference frame of the N-dimensional Euclidian space. In this space the generalized form of the position vector **r** is given by

$$\mathbf{r}=x^1\mathbf{e}_1+x^2\mathbf{e}_2+\cdots+x^N\mathbf{e}_N \tag{M1.24}$$

The length or the magnitude of the vector **r** is also known as the *Euclidian norm*

$$|\mathbf{r}|=\sqrt{\mathbf{r}\cdot\mathbf{r}}=\sqrt{(x^1)^2+(x^2)^2+\cdots+(x^N)^2} \tag{M1.25}$$

M1.3.2 The vector product of two vectors

In coordinate-free or invariant notation the vector product of two vectors is defined by

$$\mathbf{A}\times\mathbf{B}=\mathbf{C}=|\mathbf{A}|\,|\mathbf{B}|\sin(\mathbf{A},\mathbf{B})\,\mathbf{e}_C \tag{M1.26}$$

The unit vector $\mathbf{e}_C$ is perpendicular to the plane defined by the vectors **A** and **B**. The direction of the vector **C** is defined in such a way that the vectors **A**, **B**, and **C**

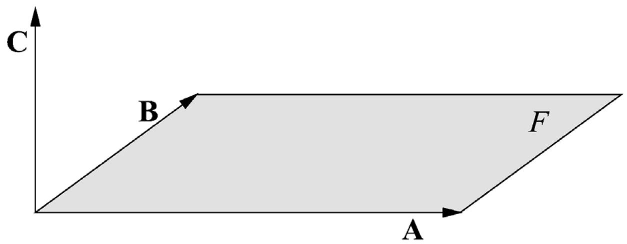

Fig. M1.5 Geometric interpretation of the vector or cross product.

form a right-handed system. The magnitude of $\mathbf{C}$ is equal to the area F of a parallelogram defined by the vectors $\mathbf{A}$ and $\mathbf{B}$ as shown in Figure M1.5. Interchanging the vectors $\mathbf{A}$ and $\mathbf{B}$ gives $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. This follows immediately from (M1.26) since the unit vector $\mathbf{e}_C$ now points in the opposite direction.

The following vector statements are valid:

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \\ (k\mathbf{A}) \times \mathbf{B} &= \mathbf{A} \times (k\mathbf{B}) = k\mathbf{A} \times \mathbf{B} \\ \mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} \end{aligned} \tag{M1.27}$$

The component representation of the vector product yields

$$\boxed{\mathbf{A} \times \mathbf{B} = A^m \mathbf{q}_m \times B^n \mathbf{q}_n = \begin{vmatrix} \mathbf{q}_2 \times \mathbf{q}_3 & \mathbf{q}_3 \times \mathbf{q}_1 & \mathbf{q}_1 \times \mathbf{q}_2 \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{vmatrix}} \tag{M1.28}$$

By utilizing Cartesian coordinates we obtain the well-known relation

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{M1.29}$$

M1.3.3 The dyadic representation, the general product of two vectors

The general or *dyadic product* of two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\Phi = \mathbf{AB} = (A^1\mathbf{q}_1 + A^2\mathbf{q}_2 + A^3\mathbf{q}_3)(B^1\mathbf{q}_1 + B^2\mathbf{q}_2 + B^3\mathbf{q}_3) \tag{M1.30}$$

It is seen that the vectors are not separated by a dot or a cross. At first glance this type of vector product seems strange. However, the advantage of this notation will

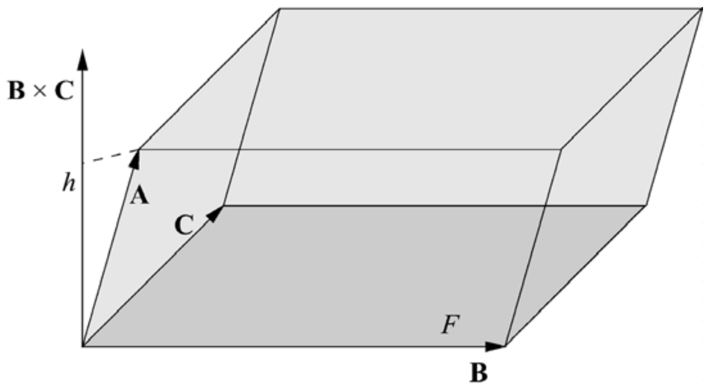

Fig. M1.6 Geometric representation of the scalar triple product.

become apparent later. On performing the dyadic multiplication we obtain

$$\boxed{\begin{aligned}\mathbf{\Phi} = \mathbf{AB} = A^1B^1\mathbf{q}_1\mathbf{q}_1 + A^1B^2\mathbf{q}_1\mathbf{q}_2 + A^1B^3\mathbf{q}_1\mathbf{q}_3 \\ + A^2B^1\mathbf{q}_2\mathbf{q}_1 + A^2B^2\mathbf{q}_2\mathbf{q}_2 + A^2B^3\mathbf{q}_2\mathbf{q}_3 \\ + A^3B^1\mathbf{q}_3\mathbf{q}_1 + A^3B^2\mathbf{q}_3\mathbf{q}_2 + A^3B^3\mathbf{q}_3\mathbf{q}_3\end{aligned}} \qquad \text{(M1.31)}$$

In carrying out the general multiplication, we must be careful not to change the position of the basis vectors. The following statements are valid:

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}, \qquad \mathbf{AB} \neq \mathbf{BA} \qquad \text{(M1.32)}$$

M1.3.4 The scalar triple product

The scalar triple product, sometimes also called the box product, is defined by

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = [\mathbf{A}, \mathbf{B}, \mathbf{C}] \qquad \text{(M1.33)}$$

The absolute value of the scalar triple product measures the volume of the parallelepiped having the three vectors **A**, **B**, **C** as adjacent edges, see Figure M1.6. The height h of the parallelepiped is found by projecting the vector **A** onto the cross product $\mathbf{B} \times \mathbf{C}$. If the volume vanishes then the three vectors are coplanar. This situation will occur whenever a vector appears twice in the scalar triple product. It is apparent that, in the scalar triple product, any cyclic permutation of the factors leaves the value of the scalar triple product unchanged. A permutation that reverses the original cyclic order changes the sign of the product:

$$\begin{aligned}[\mathbf{A}, \mathbf{B}, \mathbf{C}] &= [\mathbf{B}, \mathbf{C}, \mathbf{A}] = [\mathbf{C}, \mathbf{A}, \mathbf{B}] \\ [\mathbf{A}, \mathbf{B}, \mathbf{C}] &= -[\mathbf{B}, \mathbf{A}, \mathbf{C}] = -[\mathbf{A}, \mathbf{C}, \mathbf{B}]\end{aligned} \qquad \text{(M1.34)}$$