Part 1
Path Integrals

> "Yossarian? What kind of a name is Yossarian?"
> He had the facts at his finger tips. "It's Yossarian's name," he explained.
>
> J. HELLER, *Catch-22*

The path integral is a method of quantization which is equivalent to the operator formalism. It recovers the operator formalism in quantum mechanics and perturbation theory in quantum field theory (QFT).

The approach based on path integrals has several advantages over the operator formalism. It provides a useful tool for nonperturbative studies including:

(1) instantons,
(2) analogy with statistical mechanics,
(3) numerical methods.

A standard way of deriving the path integral is from the operator formalism:

$$\boxed{\text{operator formalism}} \quad \Longleftrightarrow \quad \boxed{\text{path integral}} \ .$$

We shall proceed in the opposite direction, following the original paper by Feynman [Fey51].

1
Operator calculus

The operator calculus developed by Feynman [Fey51] makes it possible to represent functions of (noncommuting) operators as path integrals, with the integrand being the path-ordered exponential of operators, the order of which is controlled by a parameter that varies along the trajectory. This procedure is termed *Feynman disentangling*. It is also applicable to functions of matrices (say, γ-matrices which are associated with a spinor particle). When applied to the evolution operator, this procedure results in the standard path-integral representation of quantum mechanics.

In this chapter we first demonstrate the general technique using the simplest example, a free propagator in Euclidean space, and then consider the path-integral representation of quantum mechanics, as well as propagators in an external electromagnetic field.

1.1 Free propagator

Let us first consider the simplest propagator of a free scalar field which is given in the operator formalism by the vacuum expectation value of the T-product*

$$G(x-y) \;=\; \langle 0|\, \boldsymbol{T}\boldsymbol{\varphi}(x)\, \boldsymbol{\varphi}(y)\, |0\rangle \tag{1.1}$$

with $\boldsymbol{\varphi}$ being the field-operator.

The T-product (1.1) obeys the equation

$$\left(-\partial^2 - m^2\right) G(x-y) \;=\; \mathrm{i}\,\delta^{(d)}(x-y)\,, \tag{1.2}$$

where $d = 4$ is the dimension of space-time, however the formulas are applicable at any value of d. In the operator formalism, Eq. (1.2) is a

* The ordered products of operators were introduced by Dyson [Dys49]. This paper and other classical papers on quantum electrodynamics are collected in the book edited by Schwinger [Sch58].

consequence of the free equations

$$\left.\begin{array}{rcl}\left(-\partial^2 - m^2\right)\boldsymbol{\varphi}(x)\big|\,0\right\rangle &=& 0\,,\\ \left\langle 0\,\big|\left(-\partial^2 - m^2\right)\boldsymbol{\varphi}(x) &=& 0\end{array}\right\} \tag{1.3}$$

and canonical equal-time commutators

$$\left.\begin{array}{rcl}\left[\boldsymbol{\varphi}(t,\vec{x})\,,\dot{\boldsymbol{\varphi}}(t,\vec{y})\right] &=& \mathrm{i}\,\delta^{(d-1)}(\vec{x}-\vec{y})\,,\\ \left[\boldsymbol{\varphi}(t,\vec{x})\,,\boldsymbol{\varphi}(t,\vec{y})\right] &=& 0\,.\end{array}\right\} \tag{1.4}$$

The delta-function $\delta^{(1)}(x_0 - y_0)$ emerges when $(\partial/\partial x_0)^2$ is applied to the operator of the T-product in (1.1).

Problem 1.1 Derive Eq. (1.2) in the operator formalism.

Solution Let us apply the operator on the left-hand side (LHS) of Eq. (1.2) to the T-product which is defined by

$$\boldsymbol{T}\boldsymbol{\varphi}(x)\,\boldsymbol{\varphi}(y) \;=\; \theta(x_0 - y_0)\,\boldsymbol{\varphi}(x)\,\boldsymbol{\varphi}(y) + \theta(y_0 - x_0)\,\boldsymbol{\varphi}(y)\,\boldsymbol{\varphi}(x) \tag{1.5}$$

with

$$\theta(x_0 - y_0) \;=\; \left\{\begin{array}{l}1 \text{ for } x_0 \geq y_0\\ 0 \text{ for } x_0 < y_0\,.\end{array}\right. \tag{1.6}$$

Equation (1.3) implies a nonvanishing result to emerge only when $(\partial/\partial x_0)^2$ is applied to the operator of the T-product. One obtains

$$\begin{aligned}\left(-\partial^2 - m^2\right)\left\langle 0\right|\boldsymbol{T}\boldsymbol{\varphi}(x)\,\boldsymbol{\varphi}(y)\left|0\right\rangle &= -\frac{\partial}{\partial x_0}\left\langle 0\right|\boldsymbol{T}\dot{\boldsymbol{\varphi}}(x)\,\boldsymbol{\varphi}(y)\left|0\right\rangle\\ &= \delta^{(1)}(x_0 - y_0)\left\langle 0\right|\left[\boldsymbol{\varphi}(y)\,,\dot{\boldsymbol{\varphi}}(x)\right]\left|0\right\rangle\\ &= \mathrm{i}\,\delta^{(d)}(x-y)\,,\end{aligned} \tag{1.7}$$

where the canonical commutation relations (1.4) are used.

The explicit solution to Eq. (1.2) for the free propagator is well-known and is most simply given by the Fourier transform:

$$G(x-y) \;=\; \int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}p(x-y)}\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}. \tag{1.8}$$

An extra $\mathrm{i}\varepsilon$ (with $\varepsilon \to +0$) in the denominator is due to the T-product in the definition (1.1) and unambiguously determines the integral over p_0. The propagator (1.8) is known as the Feynman propagator that respects causality.

Problem 1.2 Perform the Fourier transformation of the free momentum-space propagator in the energy p_0:

$$G_\omega(t-t') \;=\; \int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}p_0}{2\pi}\,\mathrm{e}^{\mathrm{i}p_0(t-t')}\frac{\mathrm{i}}{p_0^2 - \omega^2 + \mathrm{i}\varepsilon}, \qquad \omega \;=\; \sqrt{\vec{p}^{\,2} + m^2}\,. \tag{1.9}$$

Solution The poles of the momentum-space propagator are at

$$p_0 \;=\; \pm\omega \mp \mathrm{i}\varepsilon\,. \tag{1.10}$$

For $t > t'$ ($t < t'$), the contour of integration can be closed in the upper (lower) half-plane which gives

$$\begin{aligned} G_\omega(t-t') \;&=\; \theta(t-t')\frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{2\omega} + \theta(t'-t)\frac{\mathrm{e}^{\mathrm{i}\omega(t-t')}}{2\omega} \\ &=\; \frac{\mathrm{e}^{-\mathrm{i}\omega|t-t'|}}{2\omega}\,. \end{aligned} \tag{1.11}$$

The Green function (1.11) obeys the equation

$$\left(-\frac{\partial^2}{\partial t^2} - \omega^2\right) G_\omega(t-t') \;=\; \mathrm{i}\,\delta^{(1)}(t-t') \tag{1.12}$$

and therefore coincides with the causal Green function for a harmonic oscillator with frequency ω.

Remark on operator notations

In mathematical language, the Green function $G(x-y)$ is termed the *resolvent* of the operator on the LHS of Eq. (1.2), and is often denoted as the matrix element of the inverse operator

$$G(x-y) \;=\; \left\langle y \left| \frac{\mathrm{i}}{-\partial^2 - m^2} \right| x \right\rangle. \tag{1.13}$$

The operators act in an infinite-dimensional Hilbert space, the elements of which in Dirac's notation [Dir58] are the *bra* and *ket* vectors $\langle g|$ and $|f\rangle$, respectively. The coordinate representation emerges when these vectors are chosen to be the eigenstates of the position operator $\boldsymbol{x}_\mu$:

$$\boldsymbol{x}_\mu |x\rangle \;=\; x_\mu |x\rangle\,. \tag{1.14}$$

These basis vectors obey the completeness condition

$$\int \mathrm{d}^d x\, |x\rangle\langle x| \;=\; 1\,, \tag{1.15}$$

while the wave functions, associated with $\langle g|$ and $|f\rangle$, are given by

$$\langle g \,|\, x\rangle \;=\; g(x)\,, \qquad \langle x \,|\, f\rangle \;=\; f(x)\,. \tag{1.16}$$

These wave functions appear in the expansions

$$|f\rangle \;=\; \int \mathrm{d}^d x\, f(x)|x\rangle\,, \qquad \langle g| \;=\; \int \mathrm{d}^d y\, g(y)\langle y|\,. \tag{1.17}$$

The action of a linear operator $\boldsymbol{O}$ on the bra and ket vectors in Hilbert space is determined by its matrix element $\langle y\,|\boldsymbol{O}|\,x\rangle$, which is also known as the *kernel* of the operator $\boldsymbol{O}$ and is denoted by

$$\langle y\,|\boldsymbol{O}|\,x\rangle \;=\; O(y,x)\,. \tag{1.18}$$

Using the expansion (1.17), one obtains

$$\langle g\,|\boldsymbol{O}|\,f\rangle \;=\; \int \mathrm{d}^d x \int \mathrm{d}^d y\, g(y)\, O(y,x)\, f(x)\,. \tag{1.19}$$

Since the kernel of the unit operator is the delta-function,

$$\langle y\,|\mathbf{1}|\,x\rangle \;=\; \langle y|x\rangle \;=\; \delta^{(d)}(x-y)\,, \tag{1.20}$$

the formula

$$\langle y\,|\boldsymbol{O}|\,x\rangle \;=\; \boldsymbol{O}\,\delta^{(d)}(x-y) \tag{1.21}$$

can also be written down as a direct consequence of Eq. (1.20), where the operator $\boldsymbol{O}$ on the right-hand side (RHS) acts on the variable x.

Therefore, when the operator acts on a function $f(x)$, the result is expressed via the kernel by the standard formula

$$\boldsymbol{O}f(y) \;\equiv\; \langle y\,|\boldsymbol{O}|\,f\rangle \;=\; \int \mathrm{d}^d x\, O(y,x)\, f(x)\,. \tag{1.22}$$

Equation (1.21) is obviously reproduced when f is substituted by a delta-function, while Eq. (1.19) takes the form

$$\langle g\,|\boldsymbol{O}|\,f\rangle \;=\; \int \mathrm{d}^d x\, g(x)\, \boldsymbol{O}f(x)\,. \tag{1.23}$$

If space-time is approximated by a discrete set of points, then the operator $\boldsymbol{O}$ is approximated by a matrix with elements $\langle y\,|\boldsymbol{O}|\,x\rangle$.

1.2 Euclidean formulation

Equation (1.8) can be obtained alternatively by inverting the operator on the LHS of Eq. (1.2). Before doing that, it is convenient to make an analytic continuation in the time-variable t, and to pass to the Euclidean formulation of quantum field theory (QFT) where one substitutes

$$t \;=\; -\mathrm{i}\,x_4\,. \tag{1.24}$$

The four-momentum operator in Minkowski space reads as

$$\boldsymbol{p}^\mu_{\mathrm{M}} \;=\; \mathrm{i}\,\partial^\mu_{\mathrm{M}} \;\equiv\; \left(\mathrm{i}\frac{\partial}{\partial t}, -\mathrm{i}\frac{\partial}{\partial \vec{x}}\right) \qquad \boxed{\text{Minkowski space}}\,, \tag{1.25}$$

while its Euclidean counterpart is given by

$$\boldsymbol{p}^{\mu}_{\rm E} \;=\; -{\rm i}\,\partial^{\mu}_{\rm E} \;\equiv\; \left(-{\rm i}\frac{\partial}{\partial\vec{x}}, -{\rm i}\frac{\partial}{\partial x_4}\right) \qquad \boxed{\text{Euclidean space}}\,. \tag{1.26}$$

These two formulas together with Eq. (1.24) yield

$$E \;\equiv\; p_0 \;=\; -{\rm i}\,p_4 \tag{1.27}$$

for the relation between energy and the fourth component of the Euclidean four-momentum.

The passage to Euclidean space results in changing the Minkowski signature of the metric $g_{\mu\nu}$ to the Euclidean one:*

$$\begin{gathered}(+---) \;\longrightarrow\; (++++)\\ \boxed{\text{Minkowski signature}} \;\longrightarrow\; \boxed{\text{Euclidean signature}}\,.\end{gathered} \tag{1.28}$$

As such, one finds

$$p^2_{\rm M} = p_0^2 - \vec{p}^{\,2} \;\longrightarrow\; -p^2_{\rm E} = -\vec{p}^{\,2} - p_4^2\,. \tag{1.29}$$

The exponent in the Fourier transformation changes analogously:

$$-p_\mu x^\mu = -Et + \vec{p}\vec{x} \;\longrightarrow\; p^{\mu}_{\rm E}x^{\mu}_{\rm E} = \vec{p}\vec{x} + p_4 x_4\,. \tag{1.30}$$

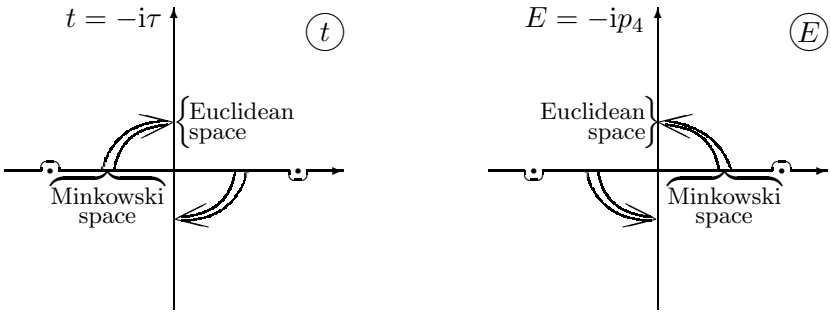
This reproduces the standard Fourier transformation in Euclidean space

$$\left.\begin{aligned} f(p) &= \int {\rm d}^d x\, {\rm e}^{-{\rm i}px} f(x)\,,\\ f(x) &= \int \frac{{\rm d}^d p}{(2\pi)^d}\, {\rm e}^{{\rm i}px} f(p)\,. \end{aligned}\right\} \tag{1.31}$$

We shall use the same notation v^μ for a four-vector in Minkowski and Euclidean spaces:

$$\left.\begin{aligned} v^{\mu}_{\rm M} &= (v_0, \vec{v}) \qquad \boxed{\text{Minkowski space}}\,,\\ v^{\mu}_{\rm E} &= (\vec{v}, v_4) \qquad \boxed{\text{Euclidean space}}\,, \end{aligned}\right\} \tag{1.32}$$

* An older generation will be familiar with the Euclidean notation which is used throughout the book by Akhiezer and Berestetskii [AB69]. In contrast, the two canonical books on quantum field theory by Bogoliubov and Shirkov [BS76] and by Bjorken and Drell [BD65] use the Minkowskian notation instigated by Feynman. The modern generation of textbooks on quantum field theory includes those by Brown [Bro92] and Weinberg [Wei98].



(a) (b)

Fig. 1.1. Direction of Wick's rotation from Minkowski to Euclidean space (indicated by the arrows) for (a) time and (b) energy. The dots represent singularities of a free propagator in (a) coordinate and (b) momentum spaces. The contours of integration in Minkowski space are associated with causal Green functions. They can obviously be deformed in the directions of the arrows.

with

$$v_0 \;=\; -iv_4\,. \tag{1.33}$$

The only difference resides in the metric. We do not distinguish between upper and lower indices in Euclidean space.

Using Eqs. (1.24) and (1.26), we see that in Euclidean space Eq. (1.2) takes the form

$$\left(-\partial^2 + m^2\right) G(x-y) \;=\; \delta^{(d)}(x-y) \tag{1.34}$$

with a positive sign in front of m^2.

The passage to the Euclidean formulation is justified in perturbation theory where it is associated with the Wick rotation. The direction in which the rotation is performed is unambiguously prescribed by the $+i\varepsilon$ term in Eq. (1.8), and is depicted in Fig. 1.1. The variable $t = x_0$ rotates through $-\pi/2$, while $E = p_0$ rotates through $\pi/2$.

Figure 1.1a explains the sign in Eq. (1.24). Figure 1.1b and Eq. (1.27) implies that the integration over p_4 goes in the opposite direction, so that

$$\int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}p_0}{2\pi} \cdots \;=\; \mathrm{i} \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}p_4}{2\pi} \cdots\,. \tag{1.35}$$

Thus when passing into Euclidean variables, Eq. (1.8) becomes

$$G(x-y) \;=\; \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}p(y-x)} \frac{1}{p^2+m^2}. \tag{1.36}$$

Note that the RHS of Eq. (1.36) is nothing but the Fourier transform of the free momentum-space Euclidean propagator, and there is no need to retain an $i\varepsilon$ in the denominator since the integration prescription is now unambiguous.

It is now clear why we keep the same notation for the coordinate-space Green functions: the Feynman propagator in Minkowski space and the Euclidean propagator. They are the same analytic function of the time-variable.

Problem 1.3 Repeat the calculation of Problem 1.2 in Euclidean space.

Solution According to Eq. (1.36) we need to calculate

$$G_\omega(\tau-\tau') \;=\; \int\limits_{-\infty}^{+\infty} \frac{\mathrm{d}p_4}{2\pi}\, \mathrm{e}^{\mathrm{i}p_4(\tau'-\tau)} \frac{1}{p_4^2+\omega^2}\,. \tag{1.37}$$

The integral on the RHS can be calculated for $\tau > \tau'$ ($\tau < \tau'$) by closing the contour in the lower (upper) half-plane, and taking the residues at $p_4 = -\mathrm{i}\omega$ ($p_4 = \mathrm{i}\omega$), respectively. This yields

$$\begin{aligned} G_\omega(\tau-\tau') &= \theta(\tau-\tau')\frac{\mathrm{e}^{\omega(\tau'-\tau)}}{2\omega} + \theta(\tau'-\tau)\frac{\mathrm{e}^{\omega(\tau-\tau')}}{2\omega} \\ &= \frac{\mathrm{e}^{-\omega|\tau-\tau'|}}{2\omega}\,. \end{aligned} \tag{1.38}$$

The Euclidean Green function (1.38) can obviously be obtained from the Minkowskian one, Eq. (1.11), by the substitution

$$\tau \;=\; \mathrm{i}t\,, \qquad \tau' \;=\; \mathrm{i}t' \tag{1.39}$$

and vice versa. $G_\omega(\tau-\tau')$ obeys the equation

$$\left(-\frac{\partial^2}{\partial\tau^2}+\omega^2\right) G_\omega(\tau-\tau') \;=\; \delta^{(1)}(\tau-\tau') \tag{1.40}$$

and, therefore, is the Green function for a Euclidean harmonic oscillator with frequency ω.

As we shall see in a moment, the Euclidean formulation makes path integrals well-defined, and allows nonperturbative investigations analogous to statistical mechanics to be carried out. There are no reasons, however, why Minkowski and Euclidean formulations should always be equivalent nonperturbatively.

Remark on Euclidean γ-matrices

The γ-matrices in Minkowski space satisfy

$$\left\{\gamma^\mu_{\mathrm{M}}, \gamma^\nu_{\mathrm{M}}\right\} \;=\; 2\,g^{\mu\nu}\,\mathbb{I}\,, \tag{1.41}$$

where $\mathbb{I}$ denotes the unit matrix. Therefore, γ_0 is Hermitian while the Minkowskian spatial γ-matrices are anti-Hermitian.

Analogously, the Euclidean γ-matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} \;=\; 2\,\delta_{\mu\nu}\,\mathbb{I}\,, \tag{1.42}$$

so that all of them are Hermitian. We compose them from 2×2 matrices as

$$\gamma_4 \;=\; \gamma_0 \;=\; \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \tag{1.43}$$

and

$$\vec{\gamma} \;=\; \begin{pmatrix} 0 & -\mathrm{i}\vec{\sigma} \\ \mathrm{i}\vec{\sigma} & 0 \end{pmatrix}, \tag{1.44}$$

where $\vec{\sigma}$ are the usual Pauli matrices. Note that the Euclidean spatial γ-matrices differ from the Minkowskian ones by a factor of i.

The free Dirac equation in Euclidean space reads as

$$\left(\hat{\partial} + m\right)\psi \;=\; 0\,, \qquad \hat{\partial} \;=\; \gamma_\mu\partial_\mu \tag{1.45}$$

or

$$(\mathrm{i}\hat{\boldsymbol{p}} + m)\,\psi \;=\; 0 \tag{1.46}$$

with $\boldsymbol{p}$ given by Eq. (1.26).

1.3 Path-ordering of operators

There are no problems in defining a function of an operator A, say via the Taylor series. For instance,

$$\mathrm{e}^A \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} A^n. \tag{1.47}$$

However, it is more complicated to define a function of several noncommuting operators (or matrices), e.g. A and B having

$$[A, B] \;\neq\; 0\,, \tag{1.48}$$

since the order of operators is now essential. In particular, one has

$$\mathrm{e}^{A+B} \;\neq\; \mathrm{e}^A\,\mathrm{e}^B, \tag{1.49}$$

so that the law of addition of exponents fails. Certainly, the exponential on the LHS is a well-defined function of $A + B$, but since A and B

are intermixed in the Taylor expansion, this expansion is of little use in practice. We would like to have an expression where all Bs are written, say, to the right of all As. Generically, this is a problem of representing a symmetric ordering of operators via a normal ordering.

This can be achieved by the following formal trick [Fey51].

Let us write

$$\begin{aligned} \mathrm{e}^{A+B} &= \lim_{M\to\infty}\left[1+\frac{1}{M}(A+B)\right]^M \\ &= \lim_{M\to\infty}\underbrace{\left[1+\frac{1}{M}(A+B)\right]\cdots\left[1+\frac{1}{M}(A+B)\right]}_{M\text{ times}}. \end{aligned} \tag{1.50}$$

The structure of the product on the RHS prompts us to introduce an index i running from 1 to M and replace $(A+B)$ in each multiplier by (A_i+B_i). Therefore, one writes

$$\begin{aligned} \mathrm{e}^{A+B} &= \lim_{M\to\infty}\prod_{i=1}^{M}\left[1+\frac{1}{M}(A_i+B_i)\right] \\ &= \lim_{M\to\infty}\left[1+\frac{1}{M}(A_M+B_M)\right]\cdots\left[1+\frac{1}{M}(A_1+B_1)\right], \end{aligned} \tag{1.51}$$

where the index i controls the order of the operators which are all treated *differently*. The ordering is such that the larger i is, the later the operator with the index i acts. This order of operators is prescribed by quantum mechanics, where initial and final states are represented by ket and bra vectors, respectively.

Equation (1.51) can be rewritten as

$$\mathrm{e}^{A+B} = \boldsymbol{P}\lim_{M\to\infty}\exp\left[\frac{1}{M}\sum_{i=1}^{M}(A_i+B_i)\right], \tag{1.52}$$

where the symbol $\boldsymbol{P}$ denotes the ordering operation. There is no ambiguity on the RHS of Eq. (1.52) concerning ordering A_i and B_i with the same index i, since such terms are $\mathcal{O}\big(M^{-2}\big)$ and are negligible as $M\to\infty$.

To describe the continuum limit as $M\to\infty$, one introduces the continuum variable $\sigma = i/M$ which belongs to the interval $[0,1]$. The continuum limit of Eq. (1.52) reads as

$$\mathrm{e}^{A+B} = \boldsymbol{P}\exp\left\{\int_0^1 \mathrm{d}\sigma\,[A(\sigma)+B(\sigma)]\right\}, \tag{1.53}$$