LINEAR WATER WAVES

The book gives a self-contained and up-to-date account of mathematical results in the linear theory of water waves. The study of these waves has many applications, including naval archtecture, ocean engineering, and geophysical hydrodynamics. The book is divided into three sections that cover the linear boundary value problems serving as the approximate mathematical models for time-harmonic waves, ship waves on calm water, and unsteady waves, respectively. These problems are derived from physical assumptions set forth in the introductory chapter, in which the linearization procedure is also described for the nonlinear boundary conditions on the free surface. In the rest of the book, a plethora of mathematical techniques is applied for investigation of the problems. In particular, the reader will find integral equations based on Green's functions, various inequalities involving the kinetic and potential energy, and integral identities. These tools are applied for establishing conditions that provide the existence and uniqueness of solutions, and their asymptotic behavior at infinity and near singularities of the boundary of the water domain. Examples of nonuniqueness usually referred to as "trapped modes," are constructed with the help of the so-called inverse procedure. For time-dependent problems with rapidly stabilizing and high-frequency boundary data, the perturbation method is used for obtaining the asymptotic behavior as the perturbation parameter tends to a limiting value.

Linear Water Waves will serve as an ideal reference for those working in fluid mechanics and engineering, as well as a source of new applications for those interested in partial differential equations of mathematical physics.

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LINEAR WATER WAVES

A Mathematical Approach

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To our wives
Natasha, Tatyana, and Anastasia

Contents

Preface

> Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respect like sound and light; they have all the complications that waves can have.
>
> —*The Feynman Lectures on Physics*, Vol. 1, Section 51.4 (86)

The aim of the present book is to give a self-contained and up-to-date account of mathematical results in the linear theory of water waves. The study of different kinds of waves is of importance for various applications. For example, it is required for predicting the behavior of floating structures (immersed totally or partially) such as ships, submarines, and tension-leg platforms and for describing flows over bottom topography. Furthermore, the investigation of wave patterns of ships and other vehicles in forward motion is closely related to the calculation of the wave-making resistance and other hydrodynamic characteristics that are used in marine design. Another area of application is the mathematical modeling of unsteady waves resulting from such phenomena as underwater earthquakes, blasts, and the like.

The history of water wave theory is almost as old as that of partial differential equations. Their founding fathers are the same: Euler, Lagrange, Cauchy, Poisson. Further contributions were made by Stokes, Lord Kelvin, Kirchhoff, and Lamb, who constructed a number of explicit solutions. In the 20th century, Havelock, Kochin, Sretensky, Stoker, John, and others applied the Fredholm theory of boundary integral equations to the field of water waves.

There are several general expositions of the classical theory by Crapper [42], Lamb [179], Lighthill [201], Sretensky [310], Stoker [312], Wehausen and Laitone [354], and Whitham [359]. Various aspects of the linear theory of water waves were considered in works of Havelock and Ursell and can be found in their collected papers (see [111] and [342], respectively). Other works are focused on various applied aspects of the theory. In particular, Haskind [106], Mei [242], Newman [262, 263], and Wehausen [352] consider the wave–body interaction. Also, there is the very recent monograph by Linton and McIver [208] on the mathematical methods used in the theory of such interactions, but it mainly discusses mathematical techniques from the point of view of their applications in ocean engineering. Problems in the theory of ship waves and wave resistance are considered by Kostyukov [147], Bhattacharya [26], Timman, Hermans, and Hsiao [318], and Wehausen [353], but like [208] these works illuminate those problems in a way more appropriate for applied research. There are books by Debnath [46] and Ovsyannikov et al. [273] concerned with nonlinear waves. However, there is no monograph on the progress achieved in the more mathematical approach to the linear water-wave theory during the last few decades.

Although the decades after World War II have brought a renewed interest in both mathematical and applied aspects of the theory, some fundamental questions still remained open. A number of (at the time) unsolved problems were listed by Ursell in 1992 [341]. Since then, substantial progress has been achieved. The new results and methods developed for obtaining them together with those dating from the 1970s and 1980s form the core of this book. We give an account of the state of the art in the field providing the reader with modern tools for further research. It is worth mentioning that these tools are not only applicable to problems of water waves but also have a much wider range of usage. Integral identities and energy inequalities for proving uniqueness theorems, the inverse procedure for constructing non-uniqueness examples, various versions of the integral equations method for solving boundary value problems, and asymptotic expansions for both transient and steady-state problems represent several of the techniques used in the book, and the list can be continued.

The book is arranged in three parts, each treating one of the main themes, which are, respectively, as follows: time-harmonic waves, waves caused by the uniform forward motion of a body on calm water, and unsteady waves. Also, there is an introductory chapter preceding Part 1 that is concerned with governing equations obtained on the basis of general dynamics of an inviscid incompressible fluid (water is the standard example of such a fluid). Linearized problems are derived there as well.

Part 1 is devoted to waves arising, in particular, in two closely related phenomena, which are radiation of waves by oscillating immersed bodies and scattering of incoming progressive waves by an obstacle (a floating body or variable bottom topography). Mathematically these phenomena give rise to a boundary value problem that is usually referred to as the water-wave problem. The difficulty of this problem stems from several facts. First, it is essential that the water domain is infinite. Second, there is a spectral parameter (it is related to the radian frequency of waves) in a boundary condition on a semi-infinite part of the boundary (referred to as the free surface of water). Above all, the free surface may consist of more than one component as occurs for a surface-piercing toroidal body. Thus the questions of solvability and uniqueness are far from being solved because usual tools applicable to other problems of mathematical physics fail in this case. The problem of uniqueness is particularly difficult, and it was placed first in Ursell's list of unsolved problems mentioned above. Different cases are possible, and we demonstrate in Part 1 that, for some geometries of the water domain, the so-called trapped modes (that is, nontrivial solutions of the homogeneous problem leading to non-uniqueness in the inhomogeneous problem) do exist for certain values of the spectral parameter whereas other geometries provide uniqueness for all frequencies.

Part 1 is divided into five chapters. In Chapter 1, we give an account of Green's functions in three and two dimensions. This material is frequently used in the sequel because, first of all, Green's function gives a key for proving the solvability theorem by reducing the water-wave problem to an integral equation on the wetted surface (contour) of an immersed body, or of a bottom obstruction (see Chapters 2 and 3). Second, Green's function is the tool that is applied in Chapter 4 for the construction of trapped waves, in other words, for examples of non-uniqueness in the water-wave problem.

Chapter 2 is concerned with those cases in which the free surface coincides with the whole horizontal plane. The application of the integral equation technique to the problem of a submerged body is developed in Section 2.1. It provides the solvability of the water-wave problem for all frequencies except possibly for a finite number of values. In Sections 2.2 and 2.3, sufficient conditions on the body shape and bottom profile are established that guarantee the unique solvability for all frequencies. Moreover, a certain auxiliary integral identity is derived for proving one of the uniqueness theorems. This identity finds further applications in Chapters 3 and 5.

In Chapter 3, semisubmerged bodies are allowed in the way that leaves no bounded components of the free surface. As in Chapter 2, we first apply the method of integral equations. However, the integral equation based on the source distribution over the wetted rigid surface gives rise to so-called irregular frequencies, that is, the frequencies at which the integral equation is not solvable for an arbitrary right-hand-side term. These values are not related to the water-wave problem and arise from the fact that a certain boundary value problem in the domain between the body surface and the free-surface plane has these values as eigenvalues. There are different ways that lead to other integral equations without irregular frequencies. We consider one of them in detail and give a survey of the others in Section 3.1. In Section 3.2, we present uniqueness theorems related to geometries under consideration. We begin with John's theorem and then consider extensions of John's method.

Chapter 4 deals with the case in which isolated portions of the free surface are present. This case is distinguished from the situations presented in Chapters 2 and 3, because examples of trapped waves involving such geometries have been constructed. In Section 4.1, we give two-dimensional examples as well as axisymmetric ones. They show that the exceptional values of frequency when the water-wave problem is not uniquely solvable do exist, at least for special geometries obtained by means of the so-called inverse procedure. We begin Section 4.2 with a number of geometric conditions providing uniqueness in the two-dimensional problem when either two bodies are symmetric about a vertical axis or the water domain has no mirror symmetry. Section 4.2 also deals with the uniqueness in the water-wave problem for a toroidal body. It occurs that for an axisymmetric toroid (similarly to the case of two symmetric cylinders), intervals of uniqueness alternate with intervals of possible non-uniqueness on the frequency half-axis. However, if more restrictions are imposed on the geometry, then it is possible to prove that some intervals of possible non-uniqueness are free of it.

A survey of results obtained in the extensive field of trapped waves periodic in a horizontal direction is given in Chapter 5. A short Section 5.1 contains a classification of such trapped waves. Edge waves are treated in Section 5.2. We present results on trapped modes above submerged cylinders and bottom protrusions in Section 5.3. Modes trapped by surface-piercing structures are considered in Section 5.4. The last section, Section 5.5, is concerned with trapped modes near vertical cylinders in channels.

Part 2 is concerned with waves caused by the uniform forward motion of a body on calm water, and these waves are usually referred to as ship waves. They are familiar to everybody because of their typical V pattern. The first mathematical explanation of this pattern appeared in 1887, when Lord Kelvin applied for this purpose his method of the stationary phase. Thus a clear evidence was given that *the linear theory* explains ship waves at least qualitatively. The boundary value problem describing ship waves is known as the Neumann–Kelvin problem, and as in the case of the water-wave problem the corresponding water domain is infinite, and there is a spectral parameter (related to the forward velocity) in the boundary condition on the free surface. The two problems are distinguished in both the free surface boundary conditions and conditions at infinity. The latter are unsymmetric and axially symmetric in the Neumann–Kelvin and water-wave problems, respectively.

Part 2 consists of three chapters, and as in Part 1 we begin with the three- and two-dimensional Green's functions for the Neumann–Kelvin problem (Chapter 6). It is worth mentioning that in Section 6.2 we give an asymptotic formula that describes the behavior of waves generated by Kelvin's source uniformly in all horizontal directions and with respect to depth.

The next two chapters, Chapters 7 and 8, are mainly concerned with the simpler two-dimensional Neumann–Kelvin problems for totally submerged and surface-piercing bodies, respectively. For the former case, necessary and sufficient conditions of the unique solvability are given for both infinite (Section 7.1) and finite (Section 7.2) depth of water. It is shown that these conditions hold for a circular cylinder in deep water, which is the only geometry when the problem is known to be uniquely solvable for all values of the forward velocity.

In the case of a surface-piercing cylinder, two supplementary conditions must be imposed and several sets of such conditions are possible. For one set of supplementary conditions considered in Section 8.1, the analogues of necessary and sufficient conditions from Chapter 7 are obtained, and they guarantee the unique solvability of the problem for surface-piercing cylinders. Other supplementary conditions are treated in Section 8.3, and some of them lead to the existence of trapped modes having finite energy. Examples of trapped modes are constructed in Section 8.4. In Section 8.5, we show that supplementary conditions of the first type guarantee that the unique solvability theorem holds for supercritical values of the forward velocity (that is, values exceeding a certain critical number depending on the water depth). Formulae for the total resistance of surface-piercing cylinders to the forward motion are derived in Section 8.2 for deep and shallow water, and these formulae generalize those obtained in Section 7.3 and expressing the wave-making resistance of totally submerged cylinders.

Section 7.4 deals with the three-dimensional Neumann–Kelvin problem for a totally submerged body, and it is established that the problem is solvable for all values of the spectral parameter with a possible exception for a finite number of values. We note that less is known about the three-dimensional Neumann–Kelvin problem than is known about the two-dimensional one. For instance, there is no example of a totally submerged body for which the problem is uniquely solvable for all values of the forward velocity. One of the difficulties in this direction arises from the fact that the uniqueness of a solution having finite energy does not imply the uniqueness of an arbitrary solution, as the case is in the water-wave problem. Another important unsolved question in three dimensions is how to impose a supplementary condition on the contour, where a surface-piercing body intersects the free surface.

In Part 3, which consists of two chapters, we investigate unsteady wave motions that develop in time under various disturbances applied either to the free surface or beneath it. In addition, certain initial conditions must be given at the time moment $t = 0$. Such problems arise in oceanography (for example, when describing generation of tsunamis), as well as in ship research (in particular, in the theory of wave-making resistance). All unsteady problems may be divided into two large classes. One of them consists of problems describing waves on the surface of an unsteady flow, whereas problems in the second class deal with waves arising from disturbances that are motionless relative water, and that depend on time only.

We begin Part 3 with results on the uniqueness, existence, and smoothness of solution. They are presented in Chapter 9 and hold for both classes of problems mentioned above. It should be noted that these results are obtained under the essential restriction that the free surface coincides with the whole horizontal plane, and the rigid boundaries of the water domain are placed at a finite distance from the free surface. The case of rigid boundaries intersecting the free surface is still an open question. In the next chapter, Chapter 10, we are concerned with problems describing waves caused by rapidly stabilizing and high-frequency disturbances that are motionless relative water. For both cases we give an asymptotic analysis based on a two-scale expansion for the velocity potential, and this allows us to describe principal terms in asymptotics of hydrodynamic characteristics such as the free surface elevation, the force acting on submerged bodies, the energy of waves, and so on.

In the Bibliography, we tried to list as many works that were published after 1960 and that treat the mathematical aspects of water waves as we could. An extensive lists of papers published up to 1960 are given by Stoker [312] and Wehausen and Laitone [354], and an additional bibliography can be found in the survey papers published by Newman [263] and Wehausen [353] during the 1970s. The papers listed in our Bibliography are mostly described briefly in Bibliographical Notes (almost every chapter has such a title for its last section), but a few are not. Of course, despite our efforts, there are omissions in the Bibliography (this is inevitable when one is dealing with several hundreds of works published over several decades).

To complete the description of the book, we mention that parts are divided into chapters, which consist of sections that are divided into subsections (and some subsections are divided into subsubsections). The titles of chapters and sections are given on the top of even and odd pages respectively. The titles of sections and subsections are given as bold headlines and numbered by two and three numbers, respectively; for example, 4.2 is Section 2 in Chapter 4, and 2.4.2 is Subsection 2 in Section 2.4. The titles of subsubsections are numbered by four numbers and are not bold. Every chapter has independent numbering of formulae and figures; for example, (2.36) denotes the 36th formula in Chapter 2, and Fig. 2.3(a) refers to part (a) of the third figure in Chapter 2. Most of the references are collected in Bibliographical Notes, but this does not apply to review chapters and sections.

A substantial part of the book is based on authors' contributions to the theory. The presentation of material is mathematically rigorous, despite the fact that we usually avoid the lemma–theorem style. Instead, we adopt a more or less informal style, formulating, nevertheless, all proved assertions in italics.

The prerequisite for reading the book is a course in Mathematical Analysis, and a familiarity with Bessel functions and the Fourier transform. We assume also that the reader is aware of the elements of functional analysis (for example, the Fredholm alternative is widely used in the book).

The book is supposed to be a research monograph in applied mathematics. Some of its topics might be of interest to mathematicians who specialize in partial differential equations and spectral operator theory. We also hope it could be used as a reference book by experts in ocean engineering as well as an advanced text for applied and engineering mathematics graduate students.

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