THE LOGIC OF
CONCEPT EXPANSION

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Historical background

The history of mathematics and the sciences is replete with examples of the expansion of concepts. Nowadays we are witness to a growing interest in the history of mathematics which has given rise to a range of essays on the history of specific concepts and theories. In this chapter, I should like to concentrate on several turning points and dilemmas in the development of the idea of expanding concepts and domains. This will require tracing the emergence of expansions as a general process from specific examples, and distinguishing these developments from the history of other general and basic notions such as algebraic structures and deduction. At the end of this chapter I briefly survey the state of the art in the study of expansions in mathematical logic and philosophy.

EARLY DEBATES

Expansions of concepts began to occur in seventh-century India, with negative numbers, the irrational numbers, and the zero. In sixteenth-century Europe a great number of expansions occurred one after another, giving Western mathematics a unique status. The first signs of this phenomenon were apparently the introduction of the zero and the beginnings of algebra, which were brought to the West by the Arabs.

When Western mathematicians developed these ideas, they did not follow pure logic; in fact, they had to make some compromises on rigor. If they had not done so, their expansions would have been blocked by the ancient Greek conception of mathematics, just as this conception had first blocked the acceptance of the rational numbers and then of the irrational numbers. The Greek model prevented development in mathematics because it recognized only the natural numbers and required

that all mathematical developments be made according to rigid axioms such as those used in Euclidean geometry.[1]

At first there was great resistance to the negative numbers that were suggested as possible solutions for algebraic equations that apparently had none. Pascal, for example, thought that the very idea of negative numbers was nonsense, since he believed that subtracting any number from a smaller number must yield zero. Arnauld rejected the negative numbers because they violated basic laws that were true for positive numbers. If $a < b$, Arnauld argued, then $a:b$ can never be equal to $b:a$. It is therefore difficult to understand how, for example, $-1:1$ can be equal to $1:-1$.

Similar objections were offered against virtually all developments in modern mathematics.[2] The complex numbers especially were considered total nonsense, and were not accepted until the nineteenth century. Even though we now accept complex numbers as a matter of course, we can still understand these objections.[3] It seems to make no sense to assign a meaning to the square root of a number that cannot have one by definition. Doing so invites analogous questions, such as why we cannot define the immediate successor function on $\frac{1}{2}$ or study vector spaces with negative dimensions. The obvious answer to the first question – that the rational numbers are dense and so there is no meaning to a successor function for them – can no longer be given, since it seems analogous to the argument that we can prove that -1 has no square root. If we could add a whole new set of numbers such that their squares would be negative numbers, then why can we not add new numbers that would be the immediate successors of the fractions?[4]

The numbers that appeared as weird solutions to quadratic equations were variously called "sophistic," "inexplicable," or "impossible." These "nonsensical" numbers, however, proved extremely useful in solving not only problems in mathematics but also problems in physics (e.g., negative velocities and fractions of an hour, etc.). If it were not for the fact that the negative numbers had proved immediately useful, the objections to them could not have been set aside. Eventually it became clear that without this "nonsense" there would be no mathematics – or at least no

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[1] For example, Cavalieri, a student of Galileo's, consciously decided to abandon the rigid requirements of the ancient Greeks, leaving them to the philosophers.

[2] This point was noticed by Crowe (1992).

[3] An echo of these objections can be seen in students' difficulties in understanding number systems that are expansions of the natural numbers.

[4] A similar question can be found in Frege's argument against the formalists. See chapter 6 below.

modern mathematics. Moreover, without expansions it is hard to see how we could progress in physics. What would our formulas look like if we could not substitute rational numbers for the variables? How would we manage without the idea of negative velocity or vectors? The use of the products of mathematical expansions has increased considerably in modern physics. In the case of the complex numbers, for example, it is hard even to imagine how awkward and unwieldy our formulas would be without them.

The beginning of the attempt to understand complex numbers can be seen in the use of the term "imaginary." This term is a considerable improvement on such words as "nonsensical" or "absurd." Leibniz employed familiar ontological descriptions such as "existing only in the mind," which he also used for relations and anything else that is not an object. This description lies at the heart of his famous saying about imaginary numbers that they are "a fine and wonderful refuge of the divine spirit, almost an amphibian between being and nonbeing" (quoted from Klein 1939, p. 56).

Leibniz suggested the notion of fiction, which helps us describe these peculiar numbers, but the problems they raise are not confined to algebra. The infinitesimals of the new calculus showed that the phenomenon is more general. Mathematicians now had far more power than ever before, but they did not know how to justify this power. While Leibniz's attitude to complex numbers was fairly clear, his attitude to the differentials was much more complicated, as these new entities have a natural interpretation in geometry as the slopes of tangents and in physics as expressions for instantaneous velocity. In this case too Leibniz could not decide if the new entities were fiction or reality.

But since Leibniz did not have a theory of fiction, it was not clear what status could be given to the mathematical entities that had forced themselves upon mathematicians. The differentials and the strange laws obeyed by these peculiar entities were soon subjected to harsh criticism. Berkeley attacked the theory of differentials, calling them "the ghosts of departed quantities." From our present viewpoint, it is hard to find anyone in the history of philosophy who contributed more to the development of rigorous standards in mathematics than Berkeley. More than any other philosopher of his day, he confronted the community of mathematicians and told them that they did not know what they were doing.

Not only were mathematicians unable to solve the problems raised by expansions, but, lacking a theory of mathematical fiction, they apparently could not even formulate clear questions to be answered. According to

Leibniz, for example, only monads exist – not the natural numbers, and not geometric figures. We could say that the square root of −1 is an abstract entity, but in what way is it more abstract than the number 7? For many years philosophers made a distinction between negative numbers, which can be given a fairly simple interpretation, and the square roots of such numbers, which seemed totally absurd, yet they did not have a clear theory according to which the former are only slightly problematic while the latter are seriously so.

Today it is easy for us to say that their problem was that they did not have an interpretation for the complex numbers. But this involves seeing their approach in the light of our modern views, which were accepted only after a paradigm shift. It took time to realize that the problem of the meaning of the complex numbers could be solved by giving an interpretation to all the symbols containing "$\sqrt{-1}$." It is not exactly obvious that we can eliminate the fictional aura of the complex numbers by declaring that the square root of −1 is a point on a plane or identifying it with the ordered pair (0, 1).[5] After all, the mathematicians who use points on a plane as an interpretation of the complex numbers do not actually mean that these numbers really *are* these points, nor does anyone believe that they are merely ordered pairs. These identifications involve conceptual difficulties associated with our basic understanding of what mathematics is.

From product to procedure

Putting aside the ontological status of the products of expansion for the moment, let us examine the actual procedure of expansion. Since fictions are produced by the human faculty of imagination, should we infer that the complex numbers are produced by this faculty? Or, if we do not consider the complex numbers to be fictions, should we say that we discover them? If we do consider them to be fictions, then are they created by the same human faculty that is responsible for creating fictional stories? And if we consider them to be discoveries, then are they discovered in the same way that we discover continents, as Frege said in his argument with the formalists (which we will discuss later)?

This last issue can be sharpened by an examination of early debates about the way to expand functions. These debates differed from those

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[5] F. J. Servois (1785–1864) criticized the geometric interpretation of complex numbers as being a geometrical mask applied to algebraic forms; the direct use of them seemed to him simple and more efficient.

about the differential calculus and the status of complex numbers in that they did not involve the acceptance of strange new objects that we have every reason to believe cannot possibly exist. Rather, the debates about the expansion of functions required changing the definitions of some of our concepts. This distinction led to another one. Although at that time no one had ever thought of the possibility of expanding the *number system* in different ways, it was well known that *functions* can be expanded in different ways. In each such case it was therefore necessary to determine the best way of expanding the function. No one asked whether a particular expansion of a function was a fiction or a discovery; mathematicians simply tried to find the best possible expansion. Indeed, the very existence of the stormy debates about which expansion is the best shows that the expansions of functions were not regarded as fictions.

This point can be sharpened even further. The best-known debate of this sort was undoubtedly the one between Johann Bernoulli and Leibniz on the way to expand $\log(-1)$. As this debate is important for the entire issue of expansions, it is worth discussing it in detail. Bernoulli claimed that:

$$\log x = \int \frac{dx}{x} = \int \frac{d(-x)}{-x} = \log(-x).$$

From this he deduced the equation

$$\log x = \log(-x)$$

and therefore

$$\log(-1) = \log 1 = 0.$$

Another proof could be brought for this claim. If we denote $\log(-1)$ by x we get

$$0 = x + x,$$

since

$$0 = \log(1) = \log[(-1) \times (-1)] = \log(-1) + \log(-1).$$

Therefore

$$\log(-1) = 0.$$

Leibniz opposed this definition, claiming that the axiom involving differentials that Bernoulli was trying to use for expanding the logarithm function is not valid for the logarithms of negative numbers. One of the arguments he presented in his lengthy correspondence with Bernoulli

is the following. If log(−1) were 0 or any other real number, then the logarithm of the square root of −1 would also be 0, since the logarithm of the square root of a number is always half the logarithm of the number itself, and half of 0 is 0. This result seemed absurd to Leibniz, but he nevertheless took the trouble to point out additional problems with Bernoulli's expansion.[6]

This problem was decided by Euler who, in his own words, was "tortured" by the "paradoxes" that he had to face in his attempt to discover the right answer. Euler "guessed" it from taking into consideration the analytical properties of the power and the sine and cosine functions. He ended up with the following definition:

$$\log(-1) = \pi i + 2k\pi \quad \text{for } k = 1, 2, 3 \ldots$$

See Kline (1972, vol. II, pp. 406–11) for a short description of the debate.

This debate, unlike the one about fiction, was based on the use of reason. The question of what log(−1) should be was a real problem that bothered mathematicians for a long time. They did not treat the question as similar to "What is the logarithm of the moon?" Both Bernoulli and Euler believed that there was a true value of log(−1), even though they did not know what it was, and even though, from our viewpoint, they had no idea what sort of number it should be or even if such a number could be defined. Even if they may have seen themselves as dealing with fictions, they attacked the problem just as if it were completely realistic.

Now although the arguments used in this debate appealed to reason, they were not based on strict logic. Indeed, as is the case with all functions, the expansion of the logarithm cannot be deduced from the original definition of a logarithm. In general, the mathematicians had a *tentative* definition of the function, but when they looked for an expansion they were actually going counter to this definition. Just as the mathematical objects we add are in a no man's land between the real and the imaginary, so the arguments we use in expanding functions lie somewhere between deductions and analogies.

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[6] The reason why it seemed absurd to him was never specified – I can only speculate that it had to do with his requirement that every function must be one-to-one. Compare this with my remarks in the appendix to chapter 4.

Leibniz attempted to prove that log(−1) is nonsense by substituting $x = (-2)$ in

$$\log(1 + x) = x - \frac{x}{2} + \frac{x}{3} + \cdots$$

which yields:

$$\log(-1) = -2 - \frac{4}{2} - \frac{8}{3} - \cdots$$

We can better understand the procedure of expanding functions if we ask ourselves whether other rational beings with the same arithmetic that we have would also have formulated their higher mathematics in the same way. Let us say that we manage to send a spaceship to a planet of Alpha Centauri and we discover that its inhabitants use arithmetic. Does this give us reason to predict that if we return there a thousand years later we will find them using negative numbers, perhaps even complex numbers? And if we find that they use the power function, is it also probable that they will expand it to the zero? It seems that there is some basis for believing that we will find all these developments.

This idea is sharpened through the following radical example. Suppose we were to remove the 7 from the set of natural numbers. Although we can easily describe such an operation, we clearly feel a resistance to considering the remaining set legitimate. We want to say that if some tribe had an arithmetic that forbade adding 7 to any other number, someday a member of that tribe would rebel and begin adding 7s. Moreover, if some community had an operation that was close to our own addition but not exactly the same, our intuition is that they would eventually come to use the same addition operation that we do. We feel that this is a natural process and we sense its power when we consider the unnatural situation of this hypothetical tribe.

This special power of the way we formulate and expand mathematical operations makes it sensible to call such expansions "forced expansions of concepts." Using this expression enables us to avoid the dilemma of whether discovery or fiction is involved here, as well as the use of terms such as "deduction" or "analogy." But at this stage we do not understand the nature of this process; we have merely found a name for it.

PEACOCK'S FORGOTTEN PLANE

The first steps in treating the issue of expansions that are not merely comments about some particular expansion can be found only after Euler. One example is that of the late eighteenth-century philosopher Solomon Maimon. Maimon distinguishes between types of cognition. He adds what he calls symbolic cognition to Kant's a priori and empirical cognition:

In order to overcome this difficulty we need symbolic cognition, that is, first we substitute symbols for the things to be symbolized and then we replace each symbol by another symbol of equal force, and so on. In this way each new

formula creates a new truth. It is thus possible to discover truths, however hidden they may be, without much effort, even mechanically. But this creates a new difficulty, namely, sometimes we obtain symbolic combinations or formulas that have no real existence, that is, that do not denote any real object, such as imaginary numbers, tangents or the cosine of a straight angle . . . Although mathematics gains much from the new analysis . . . mathematicians who are not sufficiently careful encounter difficulties that were unknown to their predecessors (Maimon [1790] 1965, vol. II, p. 412).

In this passage Maimon describes a general problem, claiming that the differentials, the complex numbers, and expansions of functions are all part of the same topic. We begin with a set of symbols that have a denotation, but when the formalism is left to itself, it can be said to create symbols by following analogical constructions that yield nonsense. The Leibnizian trust in symbolization is therefore in need of a critique.

Maimon was perhaps the first to link the problem of meaningless symbols with other philosophical problems. For him, most philosophical problems are associated with the nature of language. Moreover, he rejects Kant's argument in the first *Critique*, in the second part of the first antinomy, where Kant argues against the finitude of the world on the assumption that "a beginning is an existence which is preceded by a time in which the thing does not exist" (Kant 1933, B-455). Maimon replies that if the world has a beginning then the expression "before the universe began" is like "the square root of -1" (Maimon [1794] 1965, vol. V, p. 241), and this, he thought, entails that Kant's argument is not valid.[7] I will also follow Maimon's direction, but only in chapter 9, after I develop the notion of inchoate thought in chapter 8.

Maimon had a well-developed theory of fiction, which was admired by the neo-Kantian Veihinger. He sees mathematical fictions involving the symbolic kind of knowledge as primary, and analyses metaphysical fiction in accordance with his analysis of the mathematical sort. Moreover, he is even willing to claim that metaphysical issues can be discussed through useful fictions the way mathematicians talk about "cos 0."

One important attempt to deal with the problems raised by expansions was that of the nineteenth-century mathematician George Peacock, which was developed in response to Berkeley's criticism. Peacock distinguished between arithmetical and symbolical algebra. The former deals only with positive quantities, and therefore does not permit

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7 Ironically, Stephen Hawking and James Hartle have recently proposed a theory which allows time to be imaginary. At the end of my argument I too come to a similar conclusion, but for this we will have to wait for chapter 9.

the subtraction of a number from a smaller number, or any operations or expressions that might produce complex numbers. Peacock considered arithmetical algebra to be a logically complete system, but he claimed it was not sufficient to capture the developments that had occurred in algebra since Vieta. Symbolical algebra, in contrast, ignores the requirement that symbols should stand for positive quantities, thus permitting any number to be represented. Symbols are abstract because of the necessity of representing something, although it might be possible to give them an interpretation later on. Peacock was the first to begin investigating the idea of a formal calculus in a fairly clear way:

> So that it may thus become essentially a science of symbols and their combinations, constructed upon its own rules, which may be applied to arithmetic and to all other sciences by interpretation: by this means, interpretation will *follow*, and not *precede*, the operations of algebra and their results (Peacock 1834, pp. 194–5).

We must avoid trying to understand Peacock's idea in present-day terms, as if it were a formal syntactic investigation into the world of symbols. The modern concepts of logic and algebra tempt us to understand Peacock that way, and this may be the reason that historians see Peacock's view as the beginning of the concept of algebraic structure, which would later be generalized in Tarski's concept of a model. Peacock explains that symbolical algebra is not obtained from arithmetical algebra by deduction; what is most important for our present purposes, however, is Peacock's statement that the laws that are valid for algebraic research are *suggested* by arithmetical algebras. He presents the following formulation of the "principle of the permanence of equivalent forms":

> Whatever form is algebraically equivalent to another when expressed in general symbols, *must continue* to be equivalent, whatever those symbols denote.
>
> Whatever equivalent form is discoverable in arithmetical algebra considered as *the science of suggestion*, when the symbols are general in their form, though specific in their value, will continue to be an equivalent form when the symbols are general in their nature as well as in their form (Peacock 1834, pp. 198–9; my emphasis).

The question that interests us here is how to understand Peacock's "must continue." We have already mentioned his claim that it is not a matter of deduction, since symbolic algebra is independent of arithmetic. It is also known that Peacock did not accept Euler's view that these laws are

always valid for new cases. Still, we cannot infer that Peacock thought that arithmetic is only the motivation for the laws of algebra in the sense that these laws were abstracted from arithmetic and we are investigating them separately.

The laws that are obtained through Peacock's principle are not seen as empty formal laws but as *suggested* by arithmetical algebra, which is limited to the study of the positive numbers. This type of suggestion does not abrogate the autonomy of symbolical algebra, but it does require us to maintain the validity of all the laws we discovered in arithmetical algebra and extend them to the new cases as well. Peacock does not accept the claim that the motivation for the commutative law is the fact that the natural numbers obey it and we want to abstract and study this concept, the way we study the axioms of groups. Many textbooks justify mathematicians' interest in groups by the fact that there are many models that can be interpreted as obeying the axioms of groups. In Peacock's view, however, the commutative law has to be true of all numbers. This view can explain Hamilton's great difficulty years later in accepting the possibility of numbers that do not obey the commutative law. Hamilton tried to force this law on the new system that he developed for fifteen years before he could accept the necessity for abandoning it. If it had merely been a formal law, it would be very hard to explain Hamilton's insistence on trying to keep it in his new number system of quaternions.

Peacock's principle is an attempt to deal systematically with the phenomenon of expansions, but it is clearly insufficient. As can be seen from the example of Hamilton, it is not sensitive to the fact that some laws must be abandoned at times, since laws that are valid for arithmetic are not necessarily always valid outside of arithmetic. This does not mean, however, that we have to ignore all of Peacock's suggestions, some of which may be useful for our purposes.

Another problem with Peacock's principle is that it involves only a particular transition from arithmetic to algebra, but does not attempt to generalize to all the transitions from one system to another system that are suggested by it. Finally, and no less important, Peacock is not even aware of the possibility of analyzing the transition from one system to the other by formal devices.

The next development in the field of expansions was due to Duncan Gregory, in the first half of the nineteenth century, who came out against the idea that the laws extracted from the case of arithmetic have normative value. Gregory saw the laws abstracted from a mathematical structure as axioms that could be interpreted as applying to any class of

objects, just as the axioms of groups can be interpreted as applying to numbers, rotations of geometric objects, and the like. Boole, who was familiar with Gregory's work, applied his ideas to logic as well. He discovered a new interpretation of the laws of addition and multiplication as being applicable to what we now call disjunction and conjunction. Boole showed how the logical relations between propositions could be written in the form of algebraic equations. He thus treated the laws of logic as if they were the laws of some arithmetical structure.

A later version of Peacock's principle was formulated by Hankel (1867), but the two versions were separated by the vast developments that had taken place in all branches of mathematics, including the very meaning of mathematics itself, during the course of the nineteenth century. Some of these developments were Hamilton's provision of a basis for the complex numbers by identifying them with ordered pairs of real numbers, the discovery of various geometries, Cauchy's first attempts at providing a basis for the differential calculus, and the vast extension of our knowledge of algebra. The beautiful days of Euler were over and mathematicians had learned that there was no contradiction between the standards set by the Greeks and fruitful, useful mathematics. It was this attempt to adhere to standards that led to an even stronger demand for rigorous proofs than that of the Greeks, as can be seen in Hilbert's work on the foundations of geometry towards the end of nineteenth century, Dedekind's provision of a basis for the real numbers and the natural numbers, and Peano's and Frege's attempts to understand the concept of a proof.

As far as I know, Peano was the first mathematician after Peacock to discuss the phenomenon of expansions in mathematics. He represented an improvement over Peacock in that he understood the importance of the process of expansion for the formalization of mathematics in general. Peano worked, in parallel with Frege, on creating a logical notation for presenting mathematical proofs; in one of his letters to Frege he expressed the idea that a notation must be structured in such a way as to include the possibility of expanding mathematical functions (quoted in Frege 1977a, vol. II, footnote to sec. 58; I shall return to this issue in chapter 5). The symbols denoting functions would always be open to the possibility of further expansions, while the symbols denoting objects would be closed. As we shall see, in this view even the identity sign is not unambiguous, but must be seen as capable of development. This attempt of Peano's did not actually result in a clear formalism that describes the dynamics of such development; even Peano's system for writing proofs

was not accepted by the community of logicians, who preferred Frege's system. Frege was aware of Peano's project when he expressed his opposition to the whole idea of expansions, and it may be this very awareness that made his opposition so strong.

THE CURRENT STATE

At the present time, all the mathematical objects that had previously been called "nonsense," or "fictions," as well as Leibniz's amphibious creatures, can all be placed within a space of unproblematic objects. Moreover, thanks to Gödel's completeness theorem, we know that if there is an object for a first-order theory, and this theory is consistent, then there must be a model for this theory that is made up of sets. At the present time there are no number-words or other mathematical expressions that supposedly have no reference, and mathematicians today appear certain that they will be able to find some reference for any symbol that crops up in the future.

Moreover, not only the products of expansion but also the procedure itself has been discussed from time to time. Robinson's notion of model completeness, Cohen's notion of forcing and van Frassen's (1966) concept of supervaluation can all be viewed as echoes of Peacock's principle of the permanence of forms. In fact, even Hilbert's program of viewing the relation between talk about infinity and talk about finite domains as similar to the relation between the complex numbers and the real ones is an echo of Peacock's principle. Mathematical logic thus supplies a set of different notions of expansion. However, such expansions stem from mathematical rather than philosophical interests.[8]

In modern philosophy we constantly find new types of expansions of concepts and principles. Brouwer's argument against classical logic is that its acceptance of the law of the excluded middle is the result of an intuitive expansion of a logic which is valid in the finite case. He insists, however, that a more careful study of mathematical objects will show that the laws of classical logic are not valid for infinite totalities. (This shows that Brouwer is assuming that he himself is not expanding the logic of finite aggregates in a different way from that of the classical logicians, but rather uncovering the correct laws of such aggregates, an assumption which obviously needs to be thoroughly examined.)

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[8] Robinson's interest in model completeness derives from his desire to contribute to the "metamathematics of algebra" (Robinson 1955, preface). Cohen's notion of forcing concerns the theory of sets and questions of the independence of axioms, and is less about philosophical or logical issues.

Expansions are connected with intuitionism in Hilbert's view as well. Hilbert admitted that we expand those axioms of classical logic that were found to be correct in the case of finitistic mathematics, but he considered this expansion a legitimate one. Just as the commutative law, which was valid for the natural numbers, could be expanded to the complex numbers as well, so the law of the excluded middle could be expanded from finitary statements of intuitive number theory to ideal statements that apparently refer to infinite totalities. Hilbert considered this analogy to be a way of establishing the certitude of classical mathematics. (He did not, however, even consider the possibility of expanding the laws of classical logic differently, so as to arrive at a logic incompatible with the classical one.)

Another thesis that makes use of the idea of expansions can be found in the analysis of antinomies in philosophy and the attempt to understand the paradoxes of set theory. This thesis, which, as we shall see in chapter 9, was adopted by the most important twentieth-century logicians, asserts that paradoxes are the result of the incautious expansion of our concepts.[9] Kant's analysis, which I presented in the Preface, claims that every concept has a clear range of applicability, and stretching it beyond this range leads to antinomies. Thus antinomies are seen as a sign that a concept has been stretched too far. Russell's theory of types is a Kantian move with the aim of avoiding paradoxes.

Another philosophical discussion which raises the issue of developments of concepts is in the attempt to understand the nature of scientific revolutions and the conceptual changes they involve. Such conceptual changes are more than changes in the extension of concepts. Nevertheless, there is an analogy between these conceptual changes and the ones we are discussing here. This analogy involves a variety of issues – e.g., the question whether the theory of relativity has changed our concept of space to the point that it is incommensurable with Newton's concepts is analogous to the question of whether Cantor's concept of number is a conceptual shift from Gauss's notion. A more subtle connection has to do with Putnam's discussion of the possibility of changing the laws of logic due to empirical findings. Putnam (1975a, 1994) claims that Einstein's scientific revolution, which has been empirically confirmed, did not change our geometrical concepts, although it contains results

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[9] Sometimes this thesis is presented as a way of resolving the paradoxes. This is how I read Kripke's criticism of Tarski. Kripke claims that the concept of truth should be seen not as belonging to a meta-language, but as a predicate that is not defined on all sentences, so that there are some sentences, such as the liar sentence, to which the truth predicate cannot be expanded.

that we could not understand before Einstein. In the same way, it is possible that a physical theory will force us to abandon certain logical truths. In such cases, an example of which Putnam believes to have occurred in quantum mechanics, the change in logic should not be interpreted as a change in the meaning of the logical operators, but rather as the refutation of a logical principle. The controversy about this issue, which has been going on since the 1970s, involves problems in the philosophy of language which I touch upon briefly later in this book (in chapters 5 and 8).

Mark Steiner (1998) recently used the idea of forced expansions to suggest a new formulation of the issue of the applicability of mathematics to physics. He asks how concepts obtained through non-arbitrary expansions, determined by pure mathematical considerations, can find natural applications in physics. Steiner presents examples of cases where people attempted unsuccessfully to apply a concept to a physical system, and then found that a forced expansion would enable the concept to fit the system better. This provides Steiner with an argument against naturalism in science.

Another application of expansions can be found in Manders (1989). Manders criticizes the discussion of ontology in mathematics for being too closely tied to physics. He proposes that mathematical objects emerge out of internal mathematical considerations, in order to simplify our study of the original systems. Manders offers a formal criterion for distinguishing a fruitful expansion from a useless one, and recommends examining its implications for epistemology.

Expansions and other changes of concepts are also associated with important issues in Wittgenstein's work. Perhaps the most important are the notion of family resemblance and the idea that a proof in mathematics changes the concepts involved in it and determines the meaning of the conclusion, both of which are discussed in detail below. Wittgenstein also presents a methodological recommendation to compare changes in language with changes in mathematics (1958, no. 23). He says this explicitly as well:

What does a man do when he constructs (invents) a new language; on what principle does he operate? For this principle is the concept of "language." Does every newly constructed language broaden (alter) the concept of language? – Consider its relationship to the earlier concept: that depends on how the earlier concept was established. – Think of the relation of complex numbers to the earlier concept of number; and again of the relation of a *new* multiplication

to the general concept of the multiplication of cardinal numbers, when two particular (perhaps very large) cardinal numbers are written down for the first time and multiplied together (Wittgenstein 1974, p. 115).

This remark of Wittgenstein's suggests seeing computations involving large numbers as the result of an expansion. This suggestion, which needs to be examined carefully, raises some general questions. When can a given area be seen as one that is constituted by expansions of concepts? Can such a position be held for empirical claims as well? Can logical deduction be considered a type of non-arbitrary expansion? In general, what would count as an answer to a question of this sort?

Whatever the answer to such questions, Wittgenstein is clearly saying that the expansions that occur in modern mathematics are not solely an issue for mathematics, but should be discussed in a wider setting. I take this as a strong recommendation to tie certain developments in mathematical logic to basic philosophical questions.

I could cite even more examples of topics where the issue of expansions arises, but it is not my intention to present a list of all the occurrences of this issue in modern philosophy. I have chosen the most prominent examples – the ones that show that this is not a procedure that is confined to mathematics – with the hope that the reader will keep them in mind while reading the rest of this book.

In summary, the products of expansions, such as the complex numbers, lie between fiction and reality, while the procedure of expansion falls somewhere between deductions and analogies. Peacock went a step further when he spoke about suggestions. But the story is actually rather more complicated, as we shall see in the following chapters.