Introduction

The concept of square root was expanded to include the negative numbers; the concept of power, originally defined only for the natural numbers, was expanded to include zero, fractions, and real and complex numbers; the logarithm function, which was originally defined only for positive numbers, was expanded to the negative numbers; in general, nearly every mathematical function has been expanded in a non-arbitrary way. But this is not only true of mathematics; in physics as well there are expansions of concepts that were originally defined only for a restricted range. The expansion of the concept of temperature to black holes, the notion of instantaneous velocity, the idea of imaginary time, and perhaps even the idea of determining the age of the universe are a few examples of this process. Metaphors and analogies can also be considered expansions of concepts beyond the sphere in which they were first used. Moreover, philosophy has always been suspected of expanding concepts beyond their legitimate range of applicability. It seems that every area that contains concepts also contains expansions of concepts.

Various incidental remarks about expansions of concepts that have taken place throughout the development of modern mathematics were made by Leibniz, Pascal, Bernoulli, and Gauss. The first attempts to deal with this phenomenon systematically, however, were George Peacock's (1791–1858) "principle of permanence of equivalent forms" and Peano's requirement that logical notation must leave room for functions to develop. With Frege a crucial turn took place. Frege took the phenomenon of expansions more seriously than any other logician, but his conclusion was unfavorable:

> It is all the more necessary to emphasize that logic cannot recognize as concepts quasi-conceptual constructions that are still fluid and have not yet been given definitive and sharp boundaries (Frege 1977a, vol. II, sec. 58).

Frege's opposition to the idea of expansions stemmed from his broad view of logic, based on the major principles that apply to linguistic expressions pertaining to science rather than fiction. And yet, in spite of Frege's objections, there is still a great deal of interest in the idea of expansions among logicians (e.g., Gödel, Hilbert, Robinson), model theorists, philosophers of mathematics (e.g., Lakatos), and general philosophers, most prominently Wittgenstein. Nevertheless, within the realm of logic, as far as it can be seen as an investigation of general rational principles and a discipline involving truth and language, one can say that Frege's view is still predominant. This book is an attempt to provide a systematic analysis of one type of non-arbitrary expansion of concepts, while taking Frege's objections into account.

Let us consider the expansion of the power function to include zero. This function was originally defined on the natural numbers as an abbreviation for the process of multiplying a number by itself *n* times, so how can we even consider what 2 to the "zeroth" power might be? The question is clearly odd, since it violates the very definition of the power function as an abbreviation of multiplication, and yet we have succeeded in giving the expression "2^0" a meaning. In order to do this, we have considered the laws that apply to the power function and expanded them to the present case. Thus we define $2^0 = 1$ because this is the only way to preserve these laws.

Another example is the expansion of the concept of number to infinite sets. This expansion, like the previous one, does not involve adding elements to the universe, as does the case of expanding the set of numbers to include the negative or the complex numbers.[1] In the present case it is the function "the number of elements in a set" that is expanded to include sets that are already known to exist on the basis of our axioms. The result of this expansion is the arithmetic of infinite numbers, which is a *sine qua non* for modern mathematics.

Now it may be possible to place metaphors, analogies, and vague concepts outside the realm of logic, as Frege does, but it is definitely undesirable to present a theory of logic in which this is the fate of the expansion of concepts of the sort illustrated here, since it is impossible to imagine modern mathematics and physics without such expansions. Here is the general structure of the argument I present in this book. While Frege claims that the idea of expansions detracts from the principles

[1] The reader might object at this point by claiming that the expansion of the notion of cardinal number is not of the same kind as the expansion of the power function. I discuss this issue in chapter 3.

of reference and sense, and that therefore there cannot be a logic that includes this process, I claim that there can be a logic that includes non-arbitrary expansions, and that there are convincing reasons to believe that a certain type of expansion expresses human rationality. Therefore, instead of allowing some principles to place this phenomenon outside logic, the principles must be changed so as to include this process. These changes will eventuate in a different conception of logic that is not confined to a general study of the space of reference and truth *after* they have already been consolidated, but also includes an analysis of how this space is established.

The first chapter of the present book describes some important milestones in the discussion of non-arbitrary expansions of concepts that preceded Frege's view. Expansions, as noted by Felix Klein (1939), have forced themselves on mathematicians since the sixteenth century, compelling them to give up the rigid standards they had inherited from the Greek mathematicians. Philosophers such as Leibniz used terms like "fiction" to describe what was happening, but this was insufficient. During the nineteenth century, when rigor gradually resumed a place of importance in mathematics, there was a systematic attempt to conceptualize the idea of expansions, as presented in Peacock's "principle of permanence of equivalent forms." This attempt transferred the issue from the products of the expansion to the *process* of expansion itself. Peacock claimed that the symbolic algebra obtained from the expansion of arithmetic is logically independent of arithmetic, yet suggested by it. How an expansion of a realm can be "suggested" by the existing realm has not, however, been analyzed properly. Apparently this lack is due to the fact that discussions in logic are generally centered on deduction, which involves closed realms, thus marginalizing the issue of the expansion of concepts. But the most cursory survey shows that there is an abundance of logical, mathematical, and philosophical material that is continually raising the idea of expansions as a logical and philosophical issue which naturally invites a more comprehensive discussion.

Frege was aware of these attempts to conceptualize the process of expansions through the work of Hankel and Peano, but he nevertheless rejected the entire notion of forced expansions. Chapter 2 discusses Frege's objections, extracting three arguments from his criticism of the legitimacy of the expansion of concepts. The first argument is based on Frege's mathematical realism, the second on his principle that concepts must be defined everywhere, and the third on his extensionalism. The third argument claims that if sentences are to be analyzed into components that refer to things in such a way that the truth-value of any sentence is a function of what its components denote, then the idea of the development of concepts must be rejected. This last argument is the most important one, presenting a challenge for any alternative picture.

Chapter 3 describes what is involved in the procedure of forced expansion. First, I examine the view, which seems to be supported by Frege's writings, that there are no expansions of concepts, only the replacement of one concept by a different one. I claim that this formulation does not capture the whole complexity of non-arbitrary expansions of concepts. Then I present an explication of the phenomenon that makes use of the concept of *truth in a model*, based on adjusting Tarski's definition of truth to our needs. On this basis I distinguish between external expansions, in which elements are added to the realm under discussion, and internal expansions, in which a function is applied to a new realm. Many expansions, both within and outside of mathematics, are of the second type. I also propose a distinction between two types of internal expansion – forced expansions and strongly forced expansions – which I make use of later on. I then present two logics of expansion which were analyzed by Saharon Shelah, who determined that one of them is complete and the other is not.

Chapter 4 discusses the claim that the procedure of expanding functions in accordance with constraints is a fundamental rational process. I begin the chapter with three types of support for this claim. One sort of forced expansion is identical to deduction. Thus, even though expansions and deductions must be distinguished, the procedure of non-arbitrary expansion is a refinement of deduction, in a sense that is explained in the chapter. The procedure of expansion in accordance with constraints can be found in many different areas and on various levels (e.g., sentences, objects, concepts), as is required of a logical operation. I also call attention to the fact that when we ask people to complete partially filled matrices, as is often done in intelligence testing, we are actually asking them to perform a forced expansion. The fact that such tests are generally accepted as a way of revealing people's intellectual capacity shows that we see a connection between forced expansions and rational procedures. On the basis of these types of support I examine the normativity of forced expansions. The chapter ends with an appendix on what makes an expansion fruitful. Even though there is no a priori test for answering this question, I suggest that reflecting on fruitful expansions that have been made in the past, to see what gave them this desirable property, can provide us with general guidelines for evaluating new expansions that suggest themselves to us.

Chapter 5 proposes a picture of the relation between concepts and their expansions that is based on the discussion in the previous chapters. Frege's third argument against the idea of the development of concepts (from chapter 2) can be countered in such a way that his realism and extensionalism are preserved. According to the proposed picture, the range of a concept is not given all at once but is composed of stages that are connected in a treelike structure. The transition from one stage of a concept to another can be formulated as a sentence (whose logic was analyzed in chapter 3). This provides an amendment of Frege's formulations about what happens in the expansion of concepts. Thus it should not be claimed, for example, that Gauss only attempted to grasp the concept of number while Cantor actually did so, as implied by what Frege said on the subject, but rather that Gauss grasped one stage of the concept, thus initiating a link with the concept as a whole, while Cantor, who expanded the concept of number, grasped a more advanced stage. Nevertheless, the picture presented here is an extensionalist one. Instead of attributing a single truth-value to sentences, we attribute a whole tree of truth-values to them – at different stages of concepts the truth-value of sentences containing these concepts is subject to change.

The chapter concludes with a brief discussion of two related issues. When Wittgenstein presented his notion of family resemblance he made use of the phenomenon of the expansion of concepts to obviate the need for a definition (of the word "game"). The idea presented here, however, does not necessitate the abandonment of the notion of definition, even if this is understood as the search for a common essence. Sometimes a definition can be found only as the result of an expansion, since the search for the laws that determine the explication makes it possible to distinguish between the features of the concept and the features that determine the expansion (this issue is discussed further in chapter 7).

The second discussion involves a short description of two attempts to offer a picture of expansions without adhering to Frege's principles. The first suggestion is to abandon the idea of a concept as a tree of stages and to recognize only the stages composing the tree. The second is to soften the distinction between one stage of a concept and the next by claiming that the process occurring within each stage already includes an expansion of the concept. This last suggestion raises the interesting idea that the logical analysis of concepts does not lead to their sets of extensions (as claimed by such thinkers as Quine and Tarski), but rather to seeing them as laws for expanding their extensions in a non-arbitrary way.

Chapter 6 discusses the debate between Frege and the formalists about the ontological status of the products of expansions. In this debate Frege the realist denies the formalists' postulates that "Familiar rules of calculation shall still hold, where possible, for the newly-introduced numbers," and "If no contradiction is anywhere encountered, the introduction of the new numbers is held to be justified" (Frege 1980, sec. 97). The discussion in the previous chapters makes it possible to suggest a compromise here, one that preserves the major intuitions of the debaters. I propose that the rational and negative numbers and the like are produced by expanding the identity relation. In other words, we can set up objects in two stages: first we establish a set of mere formal equations that follow from laws such as the commutative and the associative laws, and then we expand our quantifiers to the new "objects." This raises the following question: if the mathematical objects are obtained from a system given by expansions, where do we start? The answer I propose is that we do not start with the objects at all (not even in the system of natural numbers, as Kronecker postulated when he claimed that they were created by God, while the others were the work of man). We begin with ordinals that do not denote objects, but are themselves obtained as an expansion of the laws of deduction for quantifiers. It seems to me that this provides a method of developing a view of ordinals similar to that of Benacerraf without using the notion of structure, a notion recently proposed in the philosophy of mathematics.

Chapter 7 investigates one of Gödel's arguments – one that attempts to use the process of forced expansion to deduce the independent existence of concepts as well as a capacity to perceive them:

> If there is nothing sharp to begin with, it is hard to understand how, in many cases, a vague concept can uniquely determine a sharp one without even the *slightest* freedom of choice (quoted from Wang 1996, p. 233).

I examine the relation between this argument and Gödel's more familiar argument for realism, which relies on the fact that the axioms of set theory are forced on us. After a short comparison with Charles Parsons' (1995) analysis of Gödel's view on perception, I propose an amendment to Gödel's argument for the objective reality of concepts.

Chapter 8 studies the implications that the idea of stretching concepts has for the notion of thought. I shall argue against the idea that a sentence expresses either a complete thought or none at all. Instead, we have to admit a category of inchoate thoughts which correspond to what we grasp before assigning the thought a truth-value by a non-arbitrary expansion. Unlike complete thoughts, inchoate ones are not independent of what would count as a justification of them, and we cannot demand, as Frege does, a sharp dichotomy between judging and grasping them. I end this chapter with a short comparison of my view with that of Wittgenstein on the relation between mathematical theorems and their proofs, which criticizes Frege from a similar angle.

In chapter 9 I apply the results of chapter 8 to the analysis of the philosophical problem posed by the paradoxes of set theory, according to which the paradoxes stem from a careless expansion of concepts and laws. This takes me to an examination of the possible assumptions that lie behind the anxiety reflected in the expression "I was misled by language (reason/intuition)." These, I believe, are the assumptions that there is a clear-cut division between legitimate expressions and meaningless or problematic ones, and that logic cannot start without proposing such a division. Instead, I propose that we divide the expressions of a language into three groups, adding a third category of inchoate expressions. In this view, a paradox is a failed attempt to constitute a space of objects. The illusion here did not result from our incorrectly thinking that a certain expression was meaningful when it was actually meaningless; rather, it resulted from our failure in dealing with inchoate expressions.

I then use this idea to counter Frege's pessimistic reaction to the paradoxes in set theory. I believe that the reason Frege despaired when he received Russell's postcard was due to his requirement that logic cannot operate without ensuring a reference for every proper name. Frege understood that this is an impossible demand, and so he became skeptical of the very possibility of logic. If, however, we adopt the conception of logic developed in this book, according to which logic is involved in the act of constituting the domain of objects, we can propose a different view, one much more hopeful than Frege's reaction to Russell's paradox.

In the current work I am only able to offer the preliminary steps towards a comprehensive study of expansions of concepts. I have confined myself to a logical analysis of this procedure and to its implications on the philosophy of logic. This is, however, a subject that can be delved into much more deeply. In the Epilogue I discuss a number of questions that I am leaving open in this book. I list several kinds of questions that could advance the study of conceptual changes. The most salient ones take the form of whether we should consider a particular domain to be one that was constituted by non-arbitrary expansions. These questions, many of which arose during the course of the book, involve such domains as the space of proofs (or some part of it), the arithmetic of large numbers, etc. Since such questions require special attention, I conclude the book with some methodological remarks about the way they should be dealt with.

CHAPTER I

Historical background

The history of mathematics and the sciences is replete with examples of the expansion of concepts. Nowadays we are witness to a growing interest in the history of mathematics which has given rise to a range of essays on the history of specific concepts and theories. In this chapter, I should like to concentrate on several turning points and dilemmas in the development of the idea of expanding concepts and domains. This will require tracing the emergence of expansions as a general process from specific examples, and distinguishing these developments from the history of other general and basic notions such as algebraic structures and deduction. At the end of this chapter I briefly survey the state of the art in the study of expansions in mathematical logic and philosophy.

EARLY DEBATES

Expansions of concepts began to occur in seventh-century India, with negative numbers, the irrational numbers, and the zero. In sixteenth-century Europe a great number of expansions occurred one after another, giving Western mathematics a unique status. The first signs of this phenomenon were apparently the introduction of the zero and the beginnings of algebra, which were brought to the West by the Arabs.

When Western mathematicians developed these ideas, they did not follow pure logic; in fact, they had to make some compromises on rigor. If they had not done so, their expansions would have been blocked by the ancient Greek conception of mathematics, just as this conception had first blocked the acceptance of the rational numbers and then of the irrational numbers. The Greek model prevented development in mathematics because it recognized only the natural numbers and required that all mathematical developments be made according to rigid axioms such as those used in Euclidean geometry.[1]

At first there was great resistance to the negative numbers that were suggested as possible solutions for algebraic equations that apparently had none. Pascal, for example, thought that the very idea of negative numbers was nonsense, since he believed that subtracting any number from a smaller number must yield zero. Arnauld rejected the negative numbers because they violated basic laws that were true for positive numbers. If $a < b$, Arnauld argued, then $a{:}b$ can never be equal to $b{:}a$. It is therefore difficult to understand how, for example, $-1{:}1$ can be equal to $1{:}-1$.

Similar objections were offered against virtually all developments in modern mathematics.[2] The complex numbers especially were considered total nonsense, and were not accepted until the nineteenth century. Even though we now accept complex numbers as a matter of course, we can still understand these objections.[3] It seems to make no sense to assign a meaning to the square root of a number that cannot have one by definition. Doing so invites analogous questions, such as why we cannot define the immediate successor function on $\frac{1}{2}$ or study vector spaces with negative dimensions. The obvious answer to the first question – that the rational numbers are dense and so there is no meaning to a successor function for them – can no longer be given, since it seems analogous to the argument that we can prove that -1 has no square root. If we could add a whole new set of numbers such that their squares would be negative numbers, then why can we not add new numbers that would be the immediate successors of the fractions?[4]

The numbers that appeared as weird solutions to quadratic equations were variously called "sophistic," "inexplicable," or "impossible." These "nonsensical" numbers, however, proved extremely useful in solving not only problems in mathematics but also problems in physics (e.g., negative velocities and fractions of an hour, etc.). If it were not for the fact that the negative numbers had proved immediately useful, the objections to them could not have been set aside. Eventually it became clear that without this "nonsense" there would be no mathematics – or at least no

[1] For example, Cavalieri, a student of Galileo's, consciously decided to abandon the rigid requirements of the ancient Greeks, leaving them to the philosophers.
[2] This point was noticed by Crowe (1992).
[3] An echo of these objections can be seen in students' difficulties in understanding number systems that are expansions of the natural numbers.
[4] A similar question can be found in Frege's argument against the formalists. See chapter 6 below.