Balanced models in Geophysical Fluid Dynamics: Hamiltonian formulation, constraints and formal stability

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1 Introduction

Most fluid systems, such as the three-dimensional compressible Euler equations, are too complicated to yield general analytical solutions, and approximation methods are needed to make progress in understanding aspects of particular flows. This chapter reviews derivations of approximate or reduced geophysical fluid equations which result from combining perturbation methods with preservation of the variational or Hamiltonian structure. Preservation of this structure ensures that analogues of conservation laws in the original 'parent' equations of motion are preserved. Although formal accuracy in terms of a small parameter may be achieved with conservative asymptotic perturbation methods, asymptotic solutions are expected to diverge on longer time scales. Nevertheless, perturbation methods combined with preservation of the variational or Hamiltonian structure are hypothesized to be useful in a climatological sense because conservation laws associated with this structure remain to constrain the reduced fluid dynamics.

Variational and Hamiltonian formulations of fluid flows are of interest when effects of forcing and dissipation are of secondary importance, which is often the case on scales shorter than characteristic damping times or when nonlinearities remain dominant on longer time scales. Variational or Hamiltonian methods form a unifying framework to analyze various fluid phenomena. Applications of these methods include the systematic derivation and use of wave-activity conservation laws, classical linear and nonlinear stability theorems, saturation bounds on the growth of instabilities, statistical mechanics of geophysical fluid dynamics and conservative numerical integration (e.g. Fjørtoft 1950; Holloway 1986; Holm *et al.* 1985; McLachlan 1995; Morrison 1998; Salmon 1988a, 1998; Shepherd 1990, 1994; Vladimirov 1987, 1989).

In geophysical fluid dynamics the above-mentioned approximate or reduced models are generally called *balanced* models because certain types of waves have been eliminated relative to ones present in the original 'parent' dynamics; e.g. an incompressible fluid is balanced relative to a compressible one because sound waves have been eliminated through the constraint of incompressibility. Elimination of certain types of waves can often be formalized through scaling, yielding relevant small parameters, and perturbation analysis. A well known example is the elimination of acoustic waves in the reduction from compressible to incompressible dynamics in which the Mach number, the ratio between a characteristic velocity scale and the speed of sound, is the relevant small parameter. Balanced equations thus result from singular perturbation methods, or equivalent approaches, which simplify the equations with essential singular terms and reduce the order (for example in time) of the system of differential equations. Although a perturbative approach appears to be most rigorous, one always has to realize that small parameters are a result of a scaling of the equations. This scaling tends to be a non-rigorous process, because although there may be a dominant characteristic time or spatial scale in the flow other scales can be excited and remain present due to nonlinear interactions. As an alternative to a formal perturbative approach, certain types of waves in the flow may be eliminated by imposing constraints based on observed characteristics or special insights in the fluid dynamical behaviour, which in light of the non-rigorous aspects of scaling often results into reduced systems of similar accuracy as the ones obtained via formal scaling and perturbation methods. This alternative, apparently less accurate, approach for finding constraints goes along with the observation that the notion of 'balance' and the accuracy of solutions of balanced systems (analytical or numerical) hold often surprisingly well outside the realm of asymptotic perturbation theory. Examples of the numerical accuracy of solutions of geophysical balanced models are found in the context of coastal dynamics in Allen and Newberger (1993), in atmospheric dynamics in McIntyre and Norton (2000) and perhaps even in surf-zone dynamics where breaking waves on beaches lead to low Froude number balanced along-shore currents (e.g. Özkan-Haller 1997).

The history of numerical weather prediction also nicely illustrates the use of balanced models (e.g. Daley 1991). The first numerical weather prediction model was the barotropic quasi-geostrophic equation (see section 3.5 for a f-plane version), which crudely describes the motion of vortical structures and Rossby waves (e.g. Gill 1982) in a one-layer fluid. In this model, gravity waves and acoustic waves have been eliminated or filtered, and the Rossby number (the ratio of the local Earth's rotation time scale to the advective time scale) and aspect ratio (between vertical and horizontal spatial scales and velocity fields, respectively) are the relevant small parameters used in the approximation. In the 1960's the hydrostatic primitive equations (see section 3.2 for a planar version) replaced the (barotropic and baroclinic) quasi-geostrophic numerical weather prediction models. In these hydrostatic equations only acoustic waves have been filtered (except for the boundary-trapped Lamb mode, e.g. Gill (1982)); it was nevertheless still necessary to initialize or balance the data such as to eliminate spurious high-amplitude gravity waves. The concept of balance remains crucial in the initialization and interpretation phase of numerical weather prediction.

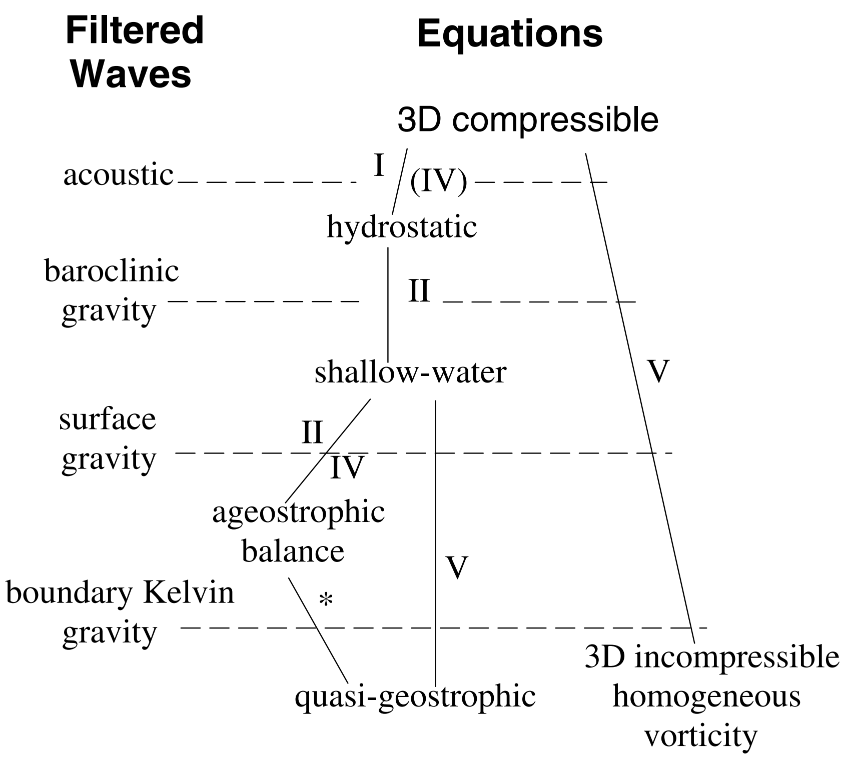

Figure 1: Sketch of the fluid systems considered in section 3. Connecting solid lines, going down, indicate the approximation route and Roman numerals the followed approximation approach. I–V denote various singular approximation methods defined in section 2 and 3 while '*' is a regular leading-order Rossby-number expansion. The left column under the heading 'Filtered Waves' indicates the wave types filtered in the approximation *between* two fluid systems (dashed horizontal lines).

Theoretical analysis and numerical process studies of balanced models have greatly advanced our understanding in meteorology and oceanography and (nearly) inviscid fluid models are often the first ones to be studied (e.g. consider the analysis of quasi-geostrophic systems in Pedlosky (1987); and the analysis of cyclogenesis in various balanced systems in Snyder *et al.* (1991) and Maraki *et al.* 1999). A systematic derivation of reduced models with conservation laws has been and is important to understand geophysical flows. This chapter gives an account of some of the recent progress in deriving these conservative, geophysical balanced models.

Variational and Hamiltonian formulations, perturbative approaches based on slaving, and several constrained variational or Hamiltonian approximation approaches are introduced, and denoted by numerals I to V (Fig. 1), at first in section 2 for finite-dimensional systems because they facilitate a more succinct exposition of the essentials. (The more technical mathematical aspects of infinite-dimensional Hamiltonian systems are not considered here, see e.g. Marsden and Ratiu 1994.) Section 2 also contains several examples of finite-dimensional conservative fluid models. It additionally introduces the powerful energy-Casimir method which can be used to derive stability criteria for steady states of (non-canonical) Hamiltonian systems. In section 3 the Hamiltonian approximation approaches I–V are applied to various fluid models (Fig. 1) starting from the compressible Euler equations and finishing with the barotropic quasi-geostrophic and higher-order geostrophically balanced equations. The presentation of fluid examples runs in parallel with the general finite-dimensional treatment in section 2 which facilitates comparisons. In addition, I quote or derive stability criteria for all fluid examples. These criteria are summarized in Table 1 in the summary and discussion.

2 Finite-dimensional systems

Two variational principles, Hamilton's principle and its related action principle, are introduced in section 2.1. This action principle follows from Hamilton's principle via a Legendre transformation and yields Hamilton's equations of motion. Hamilton's equations open the route to the definition of the more general Poisson systems in section 2.2. Systematic approximations are introduced in section 2.3 using slaving principles and singular perturbations. These approximations yield constraints which will be imposed in various but related ways on variational and Hamiltonian formulations in section 2.4. A unified abstract treatment combining the derivation of constraints and balanced Hamiltonian dynamics is presented in section 2.5 together with a discussion of its limitations, which appear so severe that only the leading-order theory presented in section 2.6 seems to be applicable in practice. Finally, a review of the energy-Casimir method concerning stability criteria for steady states of Hamiltonian systems can be found in section 2.7.

2.1 Variational principles

2.1.1 Hamilton's principle

The equations of motion for a classical-mechanical system with generalized coordinates $q^i(t)$ and velocities $\dot{q}^i(t) \equiv dq^i(t)/dt$ as functions of time t follow from Hamilton's principle (e.g. Lanczos 1970, Arnol'd 1989, Marsden and Ratiu 1994)

$$\delta A[q^i] = \lim_{\epsilon \to 0} \frac{A[q^i + \epsilon\, \delta q^i] - A[q^i]}{\epsilon} = 0 \tag{2.1}$$

with the action $A[q^i]$ defined by

$$A[q^i] = \int_{t_0}^{t_1} \mathrm{d}t\, L(q^i, \dot{q}^i, t) \tag{2.2}$$

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and its endpoint conditions by $\delta q^i(t_0) = \delta q^i(t_1) = 0$, where L is the Lagrangian and $i = 1, \ldots, K$. The familiar Euler–Lagrange equations appear when variations in Hamilton's principle (2.1) are performed and when the endpoint conditions are used to eliminate terms arising after integration by parts in time. They have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}. \tag{2.3}$$

A variety of dynamical systems can be derived from Hamilton's principle. For example, mono-atomic fluids consisting of N classical point particles, each with unit mass $\mathsf{m} = 1$, constitute a dynamical system with (generalized) positions q^i and velocities $\dot{q}^i$ (for $i = 1, \ldots, K = d\,N$ and with dimension d). Its dynamics is given by (2.3) for a Lagrangian $L = T - V$ being the kinetic energy T minus the potential energy V of the atoms. Alternatively, the dynamics (2.3) may be considered as the discretization of a continuous description of a fluid in terms of fluid parcels with unit mass $\mathsf{m} = 1$, (generalized) positions q^i and velocities $\dot{q}^i$. (Salmon (1983) uses such a discrete description of fluid parcels, along with an approximate representation of the potential energy, to perform numerical integrations of a blob of shallow water. Brenier (1996) provides another intriguing geometrical model of fluid parcel motion). More concretely, let us consider the following two finite-dimensional examples.

Example 1: Dynamics of a particle of unit mass in three spatial dimensions with position $\mathbf{q} = (q^1, q^2, q^3)^T = (x, y, z)^T$ (now $K = 3$) and potential energy $V(x, y, z)$ follows from Hamilton's principle as

$$\ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = -\frac{\partial V}{\partial y}, \quad \ddot{z} = -\frac{\partial V}{\partial z}, \tag{2.4}$$

which we recognize as Newton's equations of motion with a conservative force.

Example 2: Euler–Lagrange equations for Lorenz's (1986) two-degree-of-freedom weather model with two coordinates $q \equiv q^1$ and $Q \equiv q^2$ (i.e. $K = 2$)

$$\ddot{q} - b\,\ddot{Q} + C\,\sin 2\,q = 0, \tag{2.5}$$

$$(1 + b^2)\,\ddot{Q} - b\,\ddot{q} + \frac{Q}{\epsilon^2} = 0, \tag{2.6}$$

readily follow from (2.1) and its endpoint conditions with Lagrangian

$$\begin{aligned} L(q, Q, \dot{q}, \dot{Q}) &= \frac{1}{2}\,\dot{q}^2 - b\,\dot{q}\,\dot{Q} + \frac{1}{2}\,(1 + b^2)\,\dot{Q}^2 \\ &\quad - \left[-\frac{1}{2}\,C\,\cos 2\,q + \frac{1}{2}\,\frac{Q^2}{\epsilon^2}\right], \end{aligned} \tag{2.7}$$

which is the kinetic minus potential (terms in square brackets) energy. The coupling parameter between the pendulum (2.5) and the harmonic oscillator

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(2.6) is b, ϵ a small parameter, and C is proportional to the square of the (linearized) frequency of a pendulum.

In section 3.1.1 three-dimensional equations of motion for a compressible fluid are shown to arise from a Hamilton's principle wherein the Lagrangian is a functional, i.e. an integral over space.

2.1.2 Action principle

The Lagrangian $L(q^i, \dot{q}^i, t)$ in (2.2) is non-singular if the determinant of the Jacobian of the transformation between the two coordinate pairs $\{q^i, \dot{q}^i\}$ and $\{q^i, p_i\}$ is nonzero $(i = 1, \ldots, K)$, in which conjugate momentum p_i is defined as

$$p_i \equiv \frac{\partial L(q^i, \dot{q}^i, t)}{\partial \dot{q}^i}. \tag{2.8}$$

In other words L is convex in $\dot{q}$. Consequently a Legendre transform

$$H(q^i, p_i, t) = p_i\, \dot{q}^i - L(q^i, \dot{q}^i, t) \tag{2.9}$$

is well defined (see Lanczos 1970, Arnol'd 1989, and Marsden and Ratiu 1994; also for a geometrical interpretation), and the Hamiltonian H is a function of the q^i, p_i, and t only. $\dot{q}^i(p_i, q^i, t)$ is now defined by the extremal conditions $\partial H/\partial \dot{q}^i = 0$. Under this transformation Hamilton's principle changes into the action principle

$$\delta \int_{t_0}^{t_1} \mathrm{d}t\, L(q^i, \dot{q}^i, t) = \delta \int_{t_0}^{t_1} \mathrm{d}t\, \Big\{ p_i\, \dot{q}^i - H(q^i, p_i, t) \Big\} = 0 \tag{2.10}$$

for variations δq^i and δp_i and endpoint conditions $\delta q^i(t_0) = \delta q^i(t_1) = 0$. Its variations yield $N \equiv 2K$ first-order equations, that is, Hamilton's equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{2.11}$$

Example 3: The action principle corresponding to Example 1 is

$$\delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ u\, \dot{x} + v\, \dot{y} + w\, \dot{z} - \left(\frac{1}{2}\,(u^2 + v^2 + w^2) + V(x, y, z) \right) \right\} = 0 \tag{2.12}$$

with $u = \partial L/\partial \dot{x}, v = \partial L/\partial \dot{y}$ and $w = \partial L/\partial \dot{z}$.

Example 4: We may verify that Hamilton's equations corresponding to the Euler–Lagrange equations for Lorenz's (1986) model of Example 2 follow from the action principle (2.10) with $N = 2$ and Hamiltonian

$$H = -\frac{1}{2}\, C\, \cos\, 2\, q + \frac{1}{2} \left(p^2 + \frac{Q^2}{\epsilon^2} + (P + b\, p)^2 \right) \tag{2.13}$$

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in which we have derived momenta $p \equiv p_1 = \dot{q} - b\,\dot{Q}$ and $P \equiv p_2 = (1 + b^2)\,\dot{Q} - b\,\dot{q}$ following (2.8). Conversely, we may derive $\dot{q}, \dot{Q}$ from the extremal conditions $\partial L/\partial p_i = 0$ with $L = \dot{q}^i\, p_i - H(q^i, p_i)$.

Sometimes dynamical systems do not arise from Hamilton's principle or from a related action principle in terms of generalized coordinates and momenta, but rather from an action principle in terms of some variables z. Consider the action principle

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ a_m(z)\, \frac{dz^m}{dt} - H(z^m) \right\} \tag{2.14}$$

with endpoint variations $\delta z^m(t_0) = \delta z^m(t_1) = 0$, Hamiltonian H, functions $a_m(z)$ of z, $m = 1, \ldots, N$ and $N = 2K$. Variation (2.14) with respect to δz^n yields the equations

$$\tilde{K}_{nm} \frac{dz^m}{dt} = \frac{\partial H}{\partial z^n}, \tag{2.15}$$

where it is assumed that

$$\tilde{K}_{nm} \equiv \frac{\partial a_m}{\partial z^n} - \frac{\partial a_n}{\partial z^m} \tag{2.16}$$

is a non-singular tensor. If $z = \{q^i, p_i\}$ and

$$\tilde{\mathbf{K}} = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \tag{2.17}$$

then (2.14) equals (2.10); here $\mathbf{I}$ is the $K \times K$ unit matrix. Since $\tilde{\mathbf{K}}$ is invertible we may define a tensor $\mathbf{J} \equiv (\tilde{\mathbf{K}})^{-1}$ and rewrite (2.15) as generalized Hamiltonian equations

$$\frac{dz^i}{dt} = J^{ij} \frac{\partial H}{\partial z^j}, \tag{2.18}$$

which include the canonical Hamilton equations (2.11). Since $\mathbf{J}$ is non-singular, transformations $\{z^m\} \to \{q^i, p_i\}$ may be defined, at least locally, by virtue of Darboux's theorem (see e.g. Arnol'd 1989) such that $\mathbf{J}$ takes the canonical form

$$\mathbf{J}^c = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}. \tag{2.19}$$

If global canonical, so-called Darboux, coordinates exist, then (2.14) may be rewritten in the form (2.10). Action principle (2.14) often provides a more convenient description than Hamilton's principle or the canonical action principle (2.10) when (non-canonical) variables z are more meaningful or when global canonical coordinates are difficult to define.

Example 5: An action principle (2.14) for Lorenz's (1986) model is

$$\delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ x_3 \frac{d\phi}{dt} - \epsilon (x_5 + b\, x_3) \frac{dx_4}{dt} - H \right\} = 0 \tag{2.20}$$

with respect to variations $\delta z = \delta\phi, \delta x_3, \delta x_4$, and δx_5, respectively, subject to endpoint conditions $\delta\phi(t_{0,1}) = \delta x_4(t_{0,1}) = 0$, and with Hamiltonian

$$H = -\frac{1}{2} C \cos 2\phi + \frac{1}{2} (x_3^2 + x_4^2 + x_5^2) \tag{2.21}$$

(Bokhove and Shepherd 1996). The action principle (2.20) yields Lorenz's (1986) model in a reduced format

$$\begin{aligned} \frac{d\phi}{dt} &= x_3 - b\, x_5, \quad \frac{dx_3}{dt} = -C \sin 2\phi, \\ \frac{dx_4}{dt} &= -\frac{x_5}{\epsilon}, \quad \frac{dx_5}{dt} = \frac{x_4}{\epsilon} + b\, C \sin 2\phi. \end{aligned} \tag{2.22}$$

Variational principle (2.20) is identical to the variational principle (2.10) in Example 4 when we make the identification $q = \phi$, $p = x_3$, $Q = \epsilon\, x_4$ and $P = -(x_5 + b\, x_3)$.

A Lagrangian action principle for three-dimensional compressible flows is derived in section 3.1.2 via a Legendre transform of a relevant Hamilton's principle.

2.2 Hamiltonian formulation

The mathematical structure of equations (2.18) gives rise to Poisson systems. Such systems have the form

$$\frac{dF}{dt} = [F, H], \tag{2.23}$$

where H is the Hamiltonian and F is an arbitrary function of the variables z. The Poisson bracket $[\cdot,\cdot]$ is defined by

$$[F, G] = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j}, \tag{2.24}$$

where G is another arbitrary function of z and $\mathbf{J}$ is a tensor (here $i = 1, \ldots, N$ for arbitrary N). The system (2.23), (2.24) is Hamiltonian if and only if bracket (2.24) satisfies the following *conditions* for arbitrary functions F, G, K:

(i) skew-symmetry $[F, G] = -[G, F]$,

(ii) Jacobi's identity $[F, [G, K]] + [K, [F, G]] + [G, [K, F]] = 0$, and

(iii) Leibniz's rule

$$[F\,G, K] = F\,[G, K] + G\,[F, K]. \tag{2.25}$$

By using (2.24) to evaluate (2.25) these conditions imply the following conditions, which define a cosymplectic tensor $\mathbf{J}$:

(i) skew-symmetry $J^{ij} = -J^{ji}$,

(ii) Jacobi's identity

$$J^{im}\frac{\partial J^{jk}}{\partial z^m} + J^{km}\frac{\partial J^{ij}}{\partial z^m} + J^{jm}\frac{\partial J^{ki}}{\partial z^m} = 0, \tag{2.26}$$

(iii) Condition (2.25)(iii) is automatically guaranteed by the form (2.24), because derivatives obey Leibniz's rule (regarding functionals, see Olver [1986]).

Jacobi's identity is often difficult to prove; it is a quadratic identity which means that in perturbation approaches the various orders get mixed. Substitution of $F = z^i$ into (2.23) yields the Hamiltonian equations (2.18). Note that a cosymplectic tensor satisfying conditions (2.26) (i)–(ii) does not need to be invertible. Poisson systems therefore generalize the Hamiltonian systems with invertible $\mathbf{J}$ which were introduced at the end of section 2.1.2. Historically, the theory of Hamiltonian dynamics originated in the realm of classical mechanics, where the following *canonical* Poisson bracket

$$[F, G] = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}, \tag{2.27}$$

for $z = \{q^i, p_i\}$ and with $N = 2K$ even, arises from the canonical equations of motion (2.11) and the corresponding cosymplectic tensor is (2.19). The bracket (2.27) satisfies conditions (2.25) (i)–(iii). The significance of these conditions led to a generalized definition of Hamiltonian systems of the form (2.23), (2.24) for more general, non-canonical, Poisson brackets.

This generalization, however, has important consequences. In contrast to the Poisson bracket (2.27), the bracket (2.24) is neither necessarily canonical nor even-dimensional, and this permits the existence of nontrivial Casimir invariants C, which are solutions of $[C, G] = 0$ for arbitrary G. The invariance of the Casimirs readily follows from this definition since

$$\frac{dC}{dt} = [C, H] = 0. \tag{2.28}$$

Casimir invariants span the kernel of the cosymplectic tensor $\mathbf{J}$ (Littlejohn 1982) since condition $[C, G] = 0$ implies that

$$J^{ij}\frac{\partial C}{\partial z^j} = 0, \tag{2.29}$$

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and vectors with components $\partial C/\partial z^j$ thus span the null space of $\mathbf{J}$.

Other invariants of (continuous) Hamiltonian systems are related to symmetries of the Hamiltonian through Noether's theorem (e.g. Lanczos 1970, Olver 1986, Arnol'd 1989). When a Hamiltonian is invariant under time translation conservation of energy ensues, $dH/dt = [H, H] = 0$, and when a Hamiltonian is invariant under spatial translations conservation of momentum ensues.

When the cosymplectic tensor $\mathbf{J}$ is invertible no nontrivial Casimirs exist and the conditions (2.25) on $\mathbf{J}$ can then be translated into *linear* conditions on the symplectic tensor $\tilde{\mathbf{K}}$

(i) skew-symmetry $\tilde{K}_{ij} = -\tilde{K}_{ji}$,

(ii) Jacobi's identity

$$\frac{\partial \tilde{K}_{ij}}{\partial z^k} + \frac{\partial \tilde{K}_{jk}}{\partial z^i} + \frac{\partial \tilde{K}_{ki}}{\partial z^j} = 0. \tag{2.30}$$

Example 6: The original model derived by Lorenz (1986), which we encountered in various disguises in Examples 2, 4, and 5, reads

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2\,x_3 + b\,x_2\,x_5, \quad \frac{dx_2}{dt} = x_1\,x_3 - b\,x_1\,x_5, \quad \frac{dx_3}{dt} = -x_1\,x_2, \\ \frac{dx_4}{dt} &= -\frac{x_5}{\epsilon}, \quad \frac{dx_5}{dt} = \frac{x_4}{\epsilon} + b\,x_1\,x_2. \end{aligned} \tag{2.31}$$

Its Hamiltonian formulation is

$$\frac{dF}{dt} = [F, H'] \tag{2.32}$$

with Poisson bracket (satisfying conditions (2.25)(i)–(iii))

$$\begin{aligned} [F, G] &= \frac{\partial F}{\partial x_1} x_2 \left(b\,\frac{\partial G}{\partial x_5} - \frac{\partial G}{\partial x_3} \right) + \frac{\partial F}{\partial x_2} x_1 \left(\frac{\partial G}{\partial x_3} - b\,\frac{\partial G}{\partial x_5} \right) + \\ &\frac{\partial F}{\partial x_3}\left(x_2\,\frac{\partial G}{\partial x_1} - x_1\,\frac{\partial G}{\partial x_2} \right) - \frac{1}{\epsilon}\frac{\partial F}{\partial x_4}\frac{\partial G}{\partial x_5} + \frac{\partial F}{\partial x_5}\left(-b\,x_2\,\frac{\partial G}{\partial x_1} + b\,x_1\frac{\partial G}{\partial x_2} + \frac{1}{\epsilon}\frac{\partial G}{\partial x_4} \right) \end{aligned} \tag{2.33}$$

and Hamiltonian

$$H' = H + \frac{3}{2}\,C = \frac{1}{2}\,(x_1^2 + 2\,x_2^2 + x_3^2 + x_4^2 + x_5^2). \tag{2.34}$$

The Casimir invariant $C = \frac{1}{2}\,(x_1^2 + x_2^2)$ shows why the parameter C has been taken constant in previous appearances of Lorenz's model in Examples 2, 4, and 5. Note that variable ϕ in Example 5 follows from the polar transformation $x_1 = \sqrt{2\,C}\,\cos\phi, x_2 = \sqrt{2\,C}\,\sin\phi$.