1
Introduction

1.1 The Schwarz–Christoffel idea

The idea behind the Schwarz–Christoffel (SC) transformation and its variations is that a conformal transformation f may have a derivative that can be expressed as

$$f' = \prod f_k \tag{1.1}$$

for certain canonical functions f_k. A surprising variety of conformal maps can be fitted into this basic framework. In fact, virtually all conformal transformations whose analytic forms are known are Schwarz–Christoffel maps, albeit sometimes disguised by an additional change of variables.

Geometrically speaking, the significance of (1.1) is that

$$\arg f' = \sum \arg f_k.$$

In the classical transformation, each $\arg f_k$ is designed to be a step function, so the resulting $\arg f'$ is piecewise constant with specific jumps (i.e., f maps the real axis onto a polygon). To be specific, let P be the region in the complex plane $\mathbf{C}$ bounded by a polygon Γ with vertices $w_1, \ldots, w_n$, given in counterclockwise order, and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$. For now, we assume that P is bounded and without cusps or slits, so that $\alpha_k \in (0, 2)$ for each k. Let f be a conformal map of the upper half-plane H^+onto P, and let $z_k = f^{-1}(w_k)$ be the kth **prevertex**.[1] We shall assume $z_n = \infty$ without loss of generality, for if infinity is not already a prevertex, we can simply introduce its image (which lies

[1] The Carathéodory–Osgood theorem [Hen74] guarantees a continuous extension of f to the boundary. Hence the prevertices are well defined.

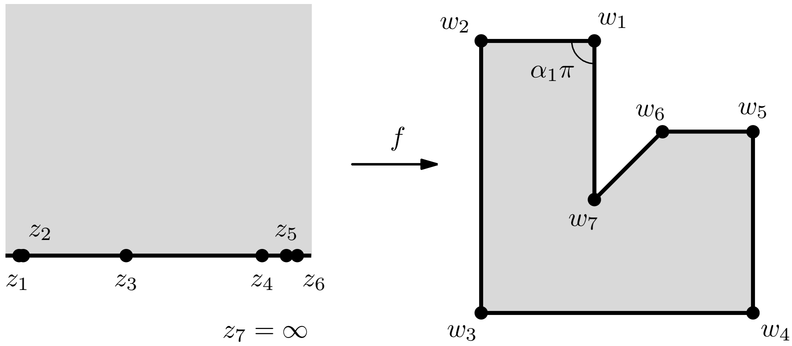

Figure 1.1. Notational conventions for the Schwarz–Christoffel transformation. In this case, z_1 and z_2 are mathematically distinct but graphically difficult to distinguish. As with all figures in this book, everything shown is not just schematic but also quantitatively correct.

on Γ) as a new vertex with interior angle π. The other prevertices $z_1, \ldots, z_{n-1}$ are real. Figure 1.1 illustrates these definitions.

As with all conformal maps, the main effort is in getting the boundary right. By the Schwarz reflection principle, which was invented for this purpose, f can be analytically continued across the segment (z_k, z_{k+1}). In particular, f' exists on this segment, and we see that arg f' must be constant there. Furthermore, arg f' must undergo a specific jump at $z = z_k$, namely

$$\left[\arg f'(z)\right]_{z_k^-}^{z_k^+} = (1 - \alpha_k)\pi = \beta_k \pi. \tag{1.2}$$

The angle $\beta_k \pi$ is the **turning angle** at vertex k. We now identify a function f_k that is analytic in H^+, satisfies (1.2), and otherwise has arg f_k constant on $\mathbf{R}$:

$$f_k = (z - z_k)^{-\beta_k}. \tag{1.3}$$

Any branch consistent with H^+ will work; to be definite, we pick the branch with $f_k(z) > 0$ if $z > z_k$ on $\mathbf{R}$. The action of f_k on the real line is sketched in Figure 1.2.

The preceding argument suggests the form

$$f'(z) = C \prod_{k=1}^{n-1} f_k(z)$$

for some constant C. We will prove the following fundamental theorem of Schwarz–Christoffel mapping in section 2.2.

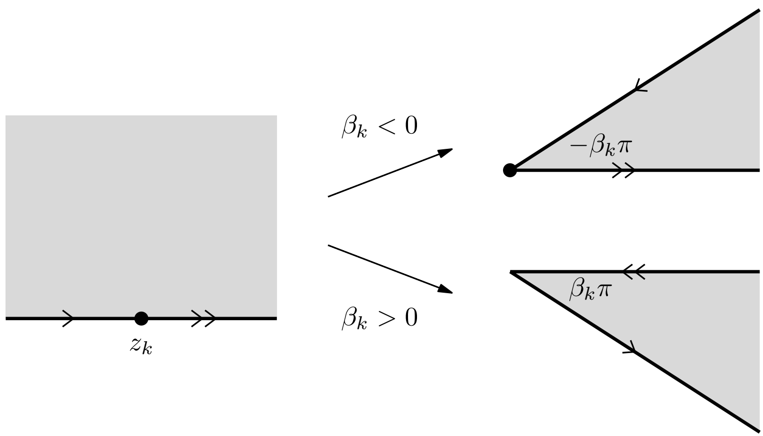

Figure 1.2. Action of a term (1.3) in the SC product. In either case, the argument of the image jumps by $\beta_k\pi$ at z_k.

Theorem 1.1. *Let P be the interior of a polygon Γ having vertices $w_1, \ldots, w_n$ and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$ in counterclockwise order. Let f be any conformal map from the upper half-plane H^+ to P with $f(\infty) = w_n$. Then*

$$f(z) = A + C \int^{z} \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1}\, d\zeta \tag{1.4}$$

for some complex constants A and C, where $w_k = f(z_k)$ for $k = 1, \ldots, n-1$.

The lower integration limit is left unspecified, as it affects only the value of A.

The formula also applies to polygons that have slits ($\alpha = 2$) or vertices at infinity ($-2 \le \alpha \le 0$). Indeed, arbitrary real exponents can meaningfully appear in (1.4), although the resulting region may overlap itself and not be bounded by a polygon in the usual sense of the term; see section 4.7.

Formula (1.4) can be adapted to maps from different regions (such as the unit disk), to exterior maps, to maps with branch points, to doubly connected regions, to regions bounded by circular arcs, and even to piecewise analytic boundaries. These and other variations are the subject of Chapter 4.

But there is a major difficulty we have not yet mentioned: without knowledge of the prevertices z_k, we cannot use (1.4) to compute values of the map. In view of how we arrived at (1.4), the image $f(\mathbf{R} \cup \{\infty\})$ of the extended real line will necessarily be *some* polygon whose interior angles match those of P, no matter what real values of z_k are used; that much is forced by the parameters α_k. (Here we are broadening the usual idea of "polygon" to allow

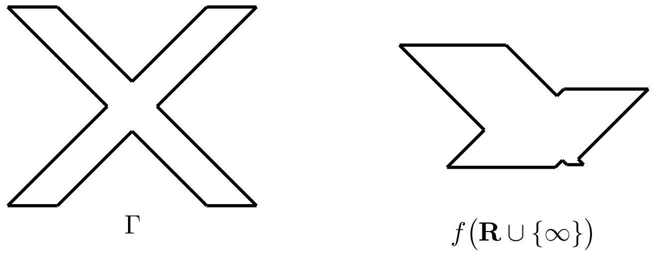

Figure 1.3. The effect of prevertices on side lengths. The region on the left is the "target," whereas the region on the right illustrates the type of distortion that may occur if the prevertices are chosen incorrectly.

self-intersections.) The prevertices, however, influence the side lengths of $f(\mathbf{R} \cup \{\infty\})$, as illustrated in Figure 1.3. Determining their correct values for a given polygon is the Schwarz–Christoffel **parameter problem**, and its solution is the first step in using the SC formula.[2] In sections 2.3–2.5 we will consider some of the classical cases for which the parameter problem can be solved explicitly.

In the majority of practical problems, there is no analytic solution for the prevertices, which depend nonlinearly on the side lengths of Γ. Numerical computation is also usually needed to evaluate the integral in (1.4) and to invert the map. Thus, much of the potential of SC mapping went unrealized until computers became readily available in the last quarter of the twentieth century. Numerical issues are discussed in Chapter 3.

1.2 History

The roots of conformal mapping lie early in the nineteenth century. Gauss considered such problems in the 1820s. The Riemann mapping theorem was first stated in Riemann's celebrated doctoral dissertation of 1851: any simply connected region in the complex plane can be conformally mapped onto any other, provided that neither is the entire plane.[3] The Schwarz–Christoffel formula was discovered soon afterwards, independently by Christoffel in 1867 and Schwarz in 1869.

[2] Sometimes the constants A and C are included as unknowns in the parameter problem. However, they can be found easily once the prevertices are known, for they just describe a scaling, rotation, and translation of the image.

[3] Riemann's proof, based on the Dirichlet principle, was later pointed out by Weierstrass to be incomplete. Rigorous proofs did not appear until the work of Koebe, Osgood, Carathéodory, and Hilbert early in the twentieth century.

Elwin Bruno Christoffel (1829–1900) was born in the German town of Montjoie (now Monschau) and was studying mathematics in Berlin under Dirichlet and others when Riemann's dissertation appeared.[4] Christoffel completed his doctoral degree in 1856 and in 1862 succeeded Dedekind as a professor of mathematics at the Swiss Federal Institute of Technology in Zurich. It was in Zurich that he published the first paper on the Schwarz–Christoffel formula, with the Italian title, "Sul problema delle temperature stazonarie e la rappresentazione di una data superficie" [Chr67]. Christoffel's motivation was the problem of heat conduction, which he approached by means of the Green's function. This paper presented the discovery that, in the case of a polygonal domain, the Green's function could be obtained via a conformal map from the half-plane, as in (1.4). In subsequent papers he extended these ideas to exteriors of polygons and to curved boundaries [Chr70a, Chr70b, Chr71].

Hermann Amandus Schwarz (1843–1921) grew up nearly a generation after Christoffel but also very much under the influence of Riemann. In the late 1860s he was living in Halle, where his discovery of the Schwarz–Christoffel formula apparently came independently of Christoffel's. His three papers on the subject [Sch69a, Sch69b, Sch90] cover much of the same territory as Christoffel's, including the generalizations to curved boundaries (section 4.11) and to circular polygons (section 4.10), but the emphasis is quite different—more numerical and more concerned with particular cases such as triangles in [Sch69b] and quadrilaterals in [Sch69a].[5] Schwarz even published the world's first plot of a Schwarz–Christoffel map, reproduced in Figure 1.4. Schwarz's papers included his famous reflection principle: if an analytic function f, extended continuously to a straight or circular boundary arc, maps the boundary arc to another straight or circular arc, then f can be analytically continued across the arc by reflection.

In 1869 Christoffel moved briefly to the Gewerbeakademie in Berlin, and Schwarz succeeded him in Zurich. By this time the two were well aware of each other's work; the phrase *Schwarz–Christoffel transformation* is now nearly universal (although the order of the names is reversed in some of the literature of the former Soviet Union).

In the 130 years since its discovery, the Schwarz–Christoffel formula has had an extensive impact in theoretical complex analysis, especially as a constructive

[4] For extensive biographical information on Christoffel, the reader is referred to the sesquicentennial volume [BF81], particularly Pfluger's paper therein on Christoffel's work on the SC formula.

[5] Schwarz also credits Weierstrass for proving the existence of a solution for the unknown parameters (which Schwarz proved for $n = 4$) in the general case.

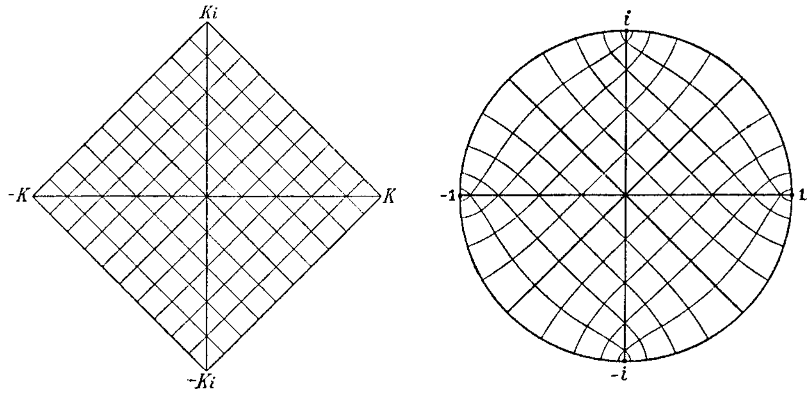

Figure 1.4. Schwarz's 1869 plot of the conformal map of a square onto a disk, reproduced from [Sch69b].

tool for proving the Riemann mapping theorem and related results. Its practical implementation—the main subject of this book—lagged far behind. Schwarz himself was the first to point out the importance of the parameter problem (discussed in the preceding section). This problem limited practical use to simple special cases, until the invention of computers.

Algorithmic discussions of the computation of Schwarz–Christoffel maps to prescribed polygons appear in several books, including those of Kantorovich and Krylov [KK64] and Gaier [Gai64]. Algorithms and in some cases computer programs have also appeared in numerous technical articles over the years, but in most of the earlier cases the authors were unaware of each other's work, and the quality of the result was wanting. Crucial issues that were often neglected included efficient evaluation of the SC integral and the need to impose necessary ordering conditions on the prevertices while solving the parameter problem. The most generally applicable computer programs for the classical problem are those of Trefethen [Tre80] (SCPACK) and Driscoll [Dri96] (SC Toolbox). The former was developed around 1980, and the latter began development in 1993. Both have been widely disseminated in the public domain.

Here is a list, more or less chronological, of contributors to constructive SC mapping of whom we are aware.

Gauss (1820s): Idea of conformal mapping
Riemann (1851): Riemann mapping theorem
Christoffel [Chr67, Chr70a, Chr70b, Chr71]: Discovery of SC formula and variants
Schwarz [Sch69a, Sch69b, Sch90]: Discovery of SC formula and variants
Kantorovich & Krylov [KK64] (first published 1936)

Polozkii (1955)
Filchakov ([Fil61], 1968, 1969, 1975)
Binns [Bin61, Bin62, Bin64]
Pisacane & Malvern [PM63]
Savenkov (1963, 1964)
Gaier [Gai64]: Book on numerical conformal mapping
Haeusler (1966)
Lawrenson & Gupta [LG68]: Adaptive quadrature, equations solver for parameters
Beigel (1969)
Hoffman (1971, 1974)
Gaier [Gai72]: "Crowding" phenomenon
Howe [How73]
Vecheslavov, Tolstobrova & Kokoulin [VT73, VK74]: Doubly connected regions
Foster & Anderson [FA74, And75]
Cherednichenko & Zhelankina [CZ75]
Squire [Squ75]
Meyer [Mey79]: Comparison of algorithms
Nicolaide [Nic77]
Prochazka [HP78, Pro83]: FORTRAN package KABBAV
Davis et al. [Dav79, ADHE82, SD85]: Curved boundaries
Hopkins & Roberts [HR79]: Solution by Kufarev's method
Reppe [Rep79]: First fully robust algorithm
Binns, Rees & Kahan [BRK79]
Volkov [Vol79, Vol87, Vol88]
Trefethen [Tre80, Tre84, Tre89, Tre93]: Robust algorithm, SCPACK, generalized parameter problems
Brown [Bro81]
Tozoni [Toz83]
Hoekstra (1983, [Hoe86]): Curved boundaries, doubly connected regions
Sridhar & Davis [SD85]: Strip maps
Floryan & Zemach [Flo85, Flo86, FZ87]: Channel flows, periodic regions
Bjørstad & Grosse [BG87]: Software for circular-arc polygons
Dias, Elcrat & Trefethen [ET86, DET87, DE92]: Free-streamline flows
Däppen [Däp87, Däp88]: Doubly connected regions
Costamagna ([Cos87, Cos01]): Applications in electricity and magnetism
Howell & Trefethen [How90, HT90, How93, How94]: Integration methods, elongated regions, circular-arc polygons
Pearce [Pea91]: Gearlike domains

Chaudhry [Cha92, CS92]: Piecewise smooth boundaries
Gutlyanskii & Zaidan [GZ94]: Kufarev's method
Driscoll [Dri96]: SC Toolbox for MATLAB
Hu [Hu95, Hu98]: Doubly connected regions (FORTRAN package DSCPACK)
Driscoll & Vavasis [DV98]: CRDT algorithm based on cross-ratios
Jamili (1999): Doubly connected regions

For more background information on conformal mapping in general and Schwarz–Christoffel mapping in particular, see [AF97, BF81, Hen74, Neh52, SL91, TD98, vS59, Wal64].

2

Essentials of Schwarz–Christoffel mapping

2.1 Polygons

For the rest of this book, a (generalized) **polygon** Γ is defined by a collection of vertices $w_1, \dots, w_n$ and real interior angles $\alpha_1\pi, \dots, \alpha_n\pi$. It is convenient for indexing purposes to define $w_{n+1} = w_1$ and $w_0 = w_n$. The vertices, which lie in the extended complex plane $\mathbf{C} \cup \{\infty\}$, are given in counterclockwise order with respect to the interior of the polygon (i.e., locally the polygon is "to the left" as one traverses the side from w_k to w_{k+1}).

The interior angle at vertex k is defined as the angle swept from the outgoing side at w_k to the incoming side. If $|w_k| < \infty$, we have $\alpha_k \in (0, 2]$. If $\alpha_k = 2$, the sides incident on w_k are collinear, and w_k is the tip of a slit. The definition of the interior angle is applied on the Riemann sphere if $w_k = \infty$. In this case, $\alpha_k \in [-2, 0]$. See Figure 2.1. Specifying α_k is redundant if w_k and its neighbors are finite, but otherwise α_k is needed to determine the polygon uniquely.

In addition to the preceding restrictions on the angles α_k, we require that the polygon make a total turn of 2π. That is,

$$\sum_{k=1}^{n}(1-\alpha_k) = 2, \tag{2.1}$$

or, equivalently,

$$\sum_{k=1}^{n}\alpha_k = n-2.$$

We shall also, unless explicitly stated otherwise, require the polygon to be **simple** (forbid it from intersecting itself and thus covering part of the plane more than once). This condition has no elementary expression in terms of the

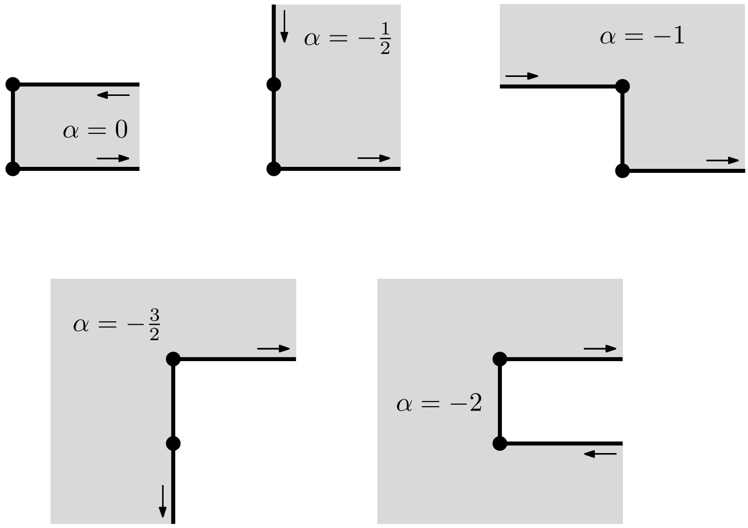

Figure 2.1. Examples of interior angles corresponding to a vertex at infinity.

vertices and angles—in a sense, it is artificial. We may occasionally use the term *polygon* to refer to a region bounded by a polygon. Context should keep the meaning clear.

2.2 The Schwarz–Christoffel formula

We now complete the proof of the half-plane formula of Theorem 1.1.

Theorem 2.1. *Let P be the interior of a polygon Γ having vertices $w_1, \ldots, w_n$ and interior angles $\alpha_1\pi, \ldots, \alpha_n\pi$ in counterclockwise order. Let f be any conformal map from the upper half-plane H^+ to P with $f(\infty) = w_n$. Then*

Schwarz–Christoffel formula for a half-plane

$$\boxed{f(z) = A + C \int^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1}\, d\zeta} \tag{2.2}$$

for some complex constants A and C, where $w_k = f(z_k)$ for $k = 1, \ldots, n-1$.

Proof. For simplicity, we treat just the case where all prevertices are finite and the product ranges over indices 1 to n. By the Schwarz reflection principle, the mapping function f can be analytically continued into the lower half-plane; the