A View of the Equations of Meteorological Dynamics and Various Approximations

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1 Introduction

One of the attractions of meteorology is its many-faceted character. It invites study by mathematicians and statisticians as well as by physicists of either practical or theoretical disposition. Amongst other fields, its concerns border or overlap those of oceanography, geophysics, environmental science, biological science, agriculture and human physiology, and impinge on those of economics, politics and psychology. (Climatology, for present purposes, is counted as part of meteorology.) Its breadth can lead to a perception that meteorology is a 'soft' science. This article focuses on part of the subject's 'hard' core: the equations governing atmospheric flow, and the approximate forms used by many numerical modellers and theorists.

A discussion (in section 3) of the basic equations of meteorological dynamics is preceded by a glance at a pre-Newtonian but fundamental subject: fluid kinematics (section 2). Some of the conservation laws which the basic equations express or imply are examined in section 4. Subsequent sections deal with approximate versions of the basic equations. Consistent approximation is one of the mathematical challenges of meteorology, and the sheer range of possible (and permissible?) approximations can be a bewildering feature. The hydrostatic approximation, the hydrostatic primitive equations (HPEs) and the shallow water equations (SWEs) are considered in section 5. The HPEs are the basis of many of the numerical models used worldwide in weather forecasting and for climate simulation, and the SWEs are widely studied as a testbed for further approximations and for numerical schemes.

We pause in section 6 to discuss various vertical coordinate systems, and various approximations of Coriolis effects and the Earth's sphericity beyond those associated with the HPEs. The geostrophic approximation is considered in a diagnostic (non-evolutionary) sense in section 7. Atmospheric wave motion is discussed in linear analytical terms in section 8 – we identify acoustic, gravity (buoyancy) and Rossby (planetary) waves and note the existence of special tropical modes.

Approximations of the HPEs which result in the removal of gravity waves as well as acoustic waves are considered in section 9; the shallow water equations are a convenient vehicle for most of this discussion. The quasi-geostrophic model, QG1, is singled out for particular attention in section 10. QG1 is one of the coarsest of those models that allow time-evolution of synoptic-scale weather systems (the 'Lows' and 'Highs' of the weather forecaster's chart), but it succeeds in representing most of the physical content of more quantitatively accurate models. Its importance in the conceptual development of meteorological dynamics can hardly be over-stated.

In section 11 are discussed various models (other than the HPEs) which allow gravity waves but not acoustic waves. Section 12 gives a brief survey of issues in numerical modelling for weather forecasting and climate simulation, and offers some concluding remarks.

The article is based on three lectures given during various phases of the Isaac Newton Institute programme on *Mathematics of Atmosphere and Ocean Dynamics* (December 1994, July 1996, December 1997). Its approach is elementary in so far as Hamiltonian methods are noted only in brief verbal summary; they are treated at proper length elsewhere in this volume. Much of the material is mainstream, and is covered in greater depth in the texts by Lorenz (1967), Phillips (1973), Haltiner and Williams (1981), Gill (1982), Pedlosky (1987), Lindzen (1990), Carlson (1991), Daley (1991), Holton (1992), Bluestein (1992), James (1994), Dutton (1995) and Green (1999), amongst others. Some new interpretations are presented, however, and later sections deal increasingly with developments which have not yet reverberated outside the research literature. Results that are thought to be new include: a bisection theorem relating the principal directions of curvature of the height field and the dilatation axis in geostrophic flow; a geometric solution of an acoustic/gravity wave dispersion relation; and a fresh perspective on the aptly-named 'omega equation' of QG1. [M.J. Sewell has demonstrated that the first of these results is an example of a general relationship between a certain pair of tensors associated with any 2-dimensional, solenoidal vector field; see section 7.2.] Sections 5.5 and 8.2 contain material covered in unpublished course notes by R.W. Riddaway and J.S.A. Green – notes to which I have been fortunate to have had access both as student and lecturer.

In mathematical respects, meteorological and oceanographic dynamics have much in common, and the atmosphere and oceans are closely-interacting systems, especially on climatological time-scales, but – in the interests of brevity – this article will refer only incidentally to oceanography and the oceans.

2 Fluid kinematics

Deformability is a key feature of a fluid: except in certain very simple flows, particles do not retain the fixed relative spatial relationships that are characteristic of a rigid body in motion. Our discussion in this section draws on the treatments given by Batchelor (1967), Wiin-Nielsen (1973), Ottino (1990) and Bluestein (1992).

Consider the motion of a fluid in two spatial dimensions relative to Cartesian axes Oxy; see Figure 1(a). Suppose that the velocity field $\mathbf{v} = \mathbf{v}(x, y, t) =$

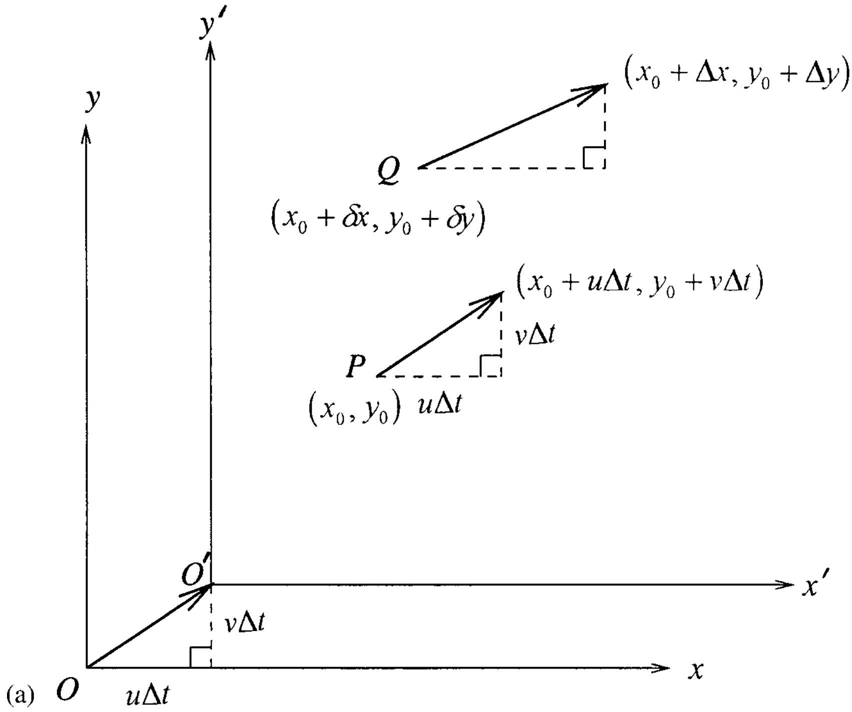

Figure 1: (a) Displacement in time Δt of fluid particles that are in the neighbourhood of the point $P = (x_0, y_0)$ at time $t = t_0$. To leading order, the fluid particle which is at P at $t = t_0$ is displaced to $(x_0 + u\Delta t, y_0 + v\Delta t)$ at $t = t_0 + \Delta t$, where u and v (the components of the flow in the x and y directions) are evaluated at (x_0, y_0, t_0). Also to leading order, a fluid particle which is at $Q = (x_0 + \delta x, y_0 + \delta y)$ at $t = t_0$ is displaced to $(x_0 + \Delta x, y_0 + \Delta y)$ at $t = t_0 + \Delta t$, where Δx and Δy are related to u, v and the spatial derivatives u_x, u_y, v_x, v_y at (x_0, y_0, t_0) according to (2.1). As well as the coordinate system Oxy relative to which u and v are measured, the diagram shows (at $t = t_0 + \Delta t$) the coordinate system $O'x'y'$ which moves with the flow velocity at (x_0, y_0, t_0) and is coincident with the Oxy system at $t = t_0$.

$(u(x, y, t), v(x, y, t))$ varies smoothly in space and time, so that the derivatives u_x, u_y, v_x, v_y are well defined, at least in the neighbourhood of a chosen point $P = (x_0, y_0)$ and time t_0. If a particle which is at point $Q = (x_0 + \delta x, y_0 + \delta y)$ at t_0 is at $(x_0 + \Delta x, y_0 + \Delta y)$ a short time Δt later, then it follows (from the definition of velocity as rate of change of position) that:

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \Delta t + \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \Delta t. \tag{2.1}$$

Here u, v and their first derivatives are evaluated at (x_0, y_0, t_0), and higher-order terms in the Taylor expansion of $\mathbf{v}$ about (x_0, y_0, t_0), have been neglected. The second term on the right side of (2.1) represents translation with the flow at point $P = (x_0, y_0)$. Measuring position $(\delta x', \delta y')$ in a Cartesian system

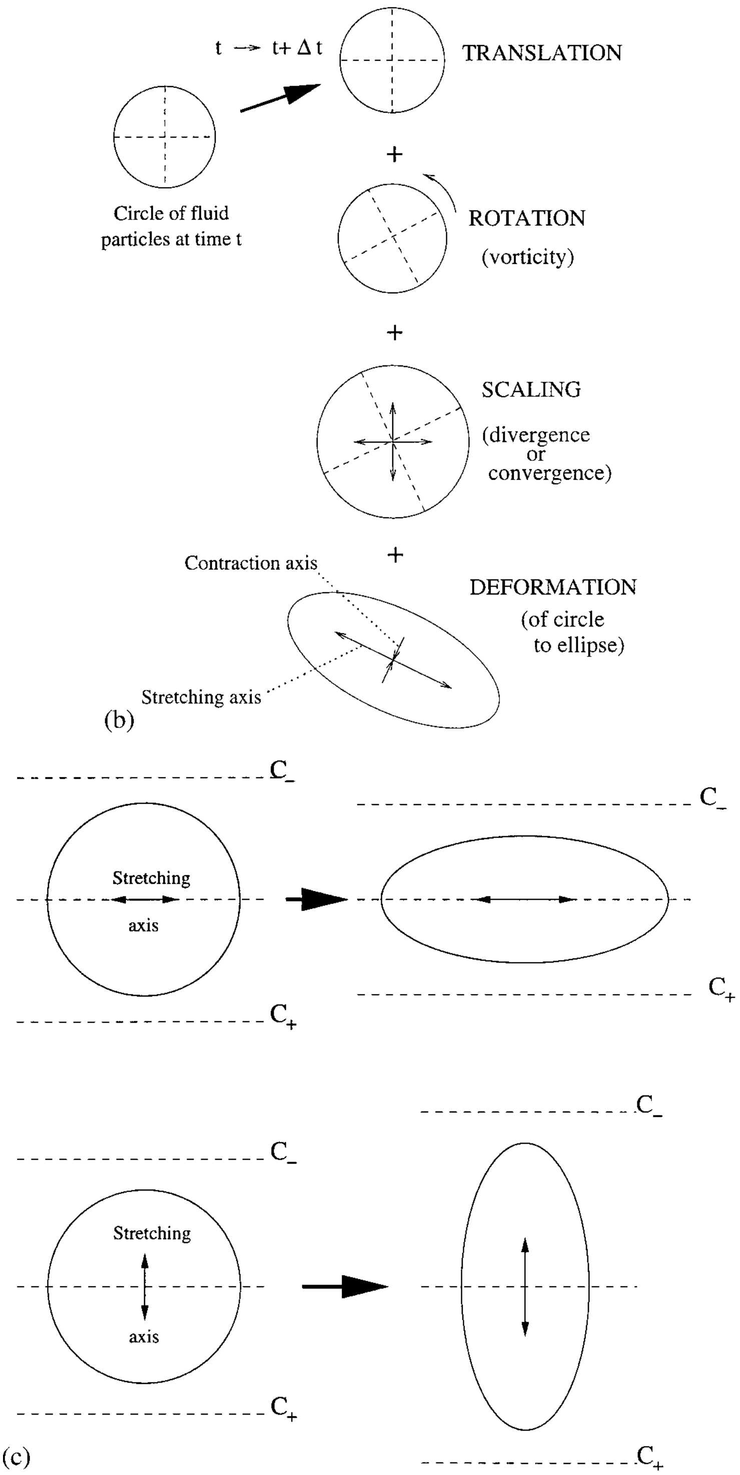

Figure 1: (b) Illustrating that the evolution of an initial circle of fluid particles in a short time Δt is the sum of a translation, a rotation, a scaling and a deformation. (c) Showing the effects of deformation on pre-existing gradients of a conserved scalar C when the stretching axis is respectively perpendicular to and parallel to the gradient of C.

$O'x'y'$ (Figure 1(a)) moving with this translation velocity (i.e. $\delta x' = \Delta x - u\Delta t$, $\delta y' = \Delta y - v\Delta t$)) gives

$$\delta\mathbf{x}' = \mathbf{A}\delta\mathbf{x},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 + u_x\Delta t & u_y\Delta t \\ v_x\Delta t & 1 + v_y\Delta t \end{pmatrix} \tag{2.2}$$

and

$$\delta\mathbf{x} = (\delta x, \delta y)\,; \quad \delta\mathbf{x}' = (\delta x', \delta y')\,.$$

Define divergence δ, vorticity ζ and deformation components as

$$\begin{aligned} \delta &= u_x + v_y; & \zeta &= v_x - u_y; \\ D_1 &= u_x - v_y; & D_2 &= v_x + u_y. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we get

$$\mathbf{A} = \mathbf{I} + (\mathbf{R} + \mathbf{S} + \mathbf{D})\Delta t, \tag{2.4}$$

where $\mathbf{I}$ is the unit diagonal matrix, and

$$2\mathbf{R} = \begin{pmatrix} 0 & -\zeta \\ \zeta & 0 \end{pmatrix}, \quad 2\mathbf{S} = \mathbf{I}\delta, \quad 2\mathbf{D} = \begin{pmatrix} D_1 & D_2 \\ D_2 & -D_1 \end{pmatrix}. \tag{2.5}$$

Also, to first order in Δt, $\mathbf{A}$ can be expressed as the product of three matrices:

$$\mathbf{A} = (\mathbf{I} + \mathbf{R}\Delta t)(\mathbf{I} + \mathbf{S}\Delta t)(\mathbf{I} + \mathbf{D}\Delta t) + O(\Delta t^2) = \hat{\mathbf{R}}\hat{\mathbf{S}}\hat{\mathbf{D}} + O(\Delta t^2)$$

with

$$\begin{aligned} \hat{\mathbf{R}} &= \begin{pmatrix} 1 & -\frac{1}{2}\zeta\Delta t \\ \frac{1}{2}\zeta\Delta t & 1 \end{pmatrix}; \hat{\mathbf{S}} = \left(1 + \tfrac{1}{2}\delta\Delta t\right)\mathbf{I}; \\ \hat{\mathbf{D}} &= \begin{pmatrix} 1 + \frac{1}{2}D_1\Delta t & \frac{1}{2}D_2\Delta t \\ \frac{1}{2}D_2\Delta t & 1 - \frac{1}{2}D_1\Delta t \end{pmatrix}. \end{aligned} \tag{2.6}$$

Consider particles which formed a *circle* centred on (x_0, y_0) at time t_0; see Figure 1(b). It is readily shown that the matrices $\hat{\mathbf{R}}$, $\hat{\mathbf{S}}$ and $\hat{\mathbf{D}}$ correspond respectively to (infinitesimal) *rotation*, *scaling* and *deformation* of the circle of particles over the time interval $[t_0, t_0 + \Delta t]$.

The rotation ($\hat{\mathbf{R}}$) is associated with vorticity (ζ), and corresponds to a turning of the initial circle through an angle $\frac{1}{2}\zeta\Delta t$ counterclockwise. The scaling ($\hat{\mathbf{S}}$) is associated with divergence (δ); it represents an isotropic change of size (a uniform magnification or minification) in which the radius of the circle changes by a factor $(1 + \frac{1}{2}\delta\Delta t)$.

The deformation ($\hat{\mathbf{D}}$) corresponds to a change of shape: the initial circle becomes an ellipse. The major axis of the ellipse (the stretching or dilatation axis) is inclined to the x axis at an angle $\frac{1}{2}\tan^{-1}(D_2/D_1)$. If the initial radius of the circle is chosen as the unit of distance, the semi-major axis of the ellipse is $1 + \frac{1}{2}D\Delta t$, where $D^2 = D_1^2 + D_2^2$ is the square of the total deformation; and the semi-minor axis of the ellipse, the contraction axis of the initial circle, is $(1-\frac{1}{2}D\Delta t)$. The magnitudes and directions of the major and minor axes are given by the eigenvalues and eigenvectors of $\hat{\mathbf{D}}$. (The eigenvalues of $\mathbf{D}$ are $\pm\frac{1}{2}D\Delta t$. Its eigenvectors too are parallel to the axes of stretching and contraction.) Area is preserved, to order Δt, during the deformation.

As illustrated in Figure 1(b), the evolution of the initial circle of particles is (for small Δt) a combination of (i) translation, (ii) rotation, (iii) scaling, and (iv) deformation (to an ellipse).

In analytical terms, particle locations in the neighbourhood of (x_0, y_0) are transformed into locations in the neighbourhood of $(x_0 + u\Delta t, y_0 + v\Delta t)$ according to an infinitesimal general (non-conformal, non-isometric) mapping; see Klein (1938), p. 105. The details of the mapping are determined by the first derivatives of u and v in the neighbourhood of (x_0, y_0).

The matrices $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{D}$ defined by (2.3) and (2.5) together constitute a decomposition of the 2D velocity gradient tensor $\mathbf{T}$:

$$\mathbf{T} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \quad (= \mathrm{grad}_2\, \mathbf{v}). \tag{2.7}$$

The quantity $\mathbf{R}$ (sometimes called the *body spin matrix*) is the skew-symmetric part of $\mathbf{T}$; $\mathbf{S} + \mathbf{D}$ is the symmetric part of $\mathbf{T}$ (the *Eulerian rate of strain matrix*). Vorticity (associated with $\mathbf{R}$) is seen to be essentially a rigid body property. Deformation (associated with $\mathbf{D}$) is essentially a non-rigid body property; the same statement could be made about the divergence (associated with $\mathbf{S}$), and some authors treat divergence as a special kind of deformation.

For 3D flow $\mathbf{u} = \mathbf{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$, the treatment may be repeated for an initial *sphere* of particles and 3D velocity gradient tensor $\mathbf{T}$:

$$\mathbf{T} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} \quad (= \mathrm{grad}\, \mathbf{u}). \tag{2.8}$$

The results are similar to those of the 2D case, though more complicated in analytical terms. The sphere undergoes a translation, a rotation, a scaling and a deformation to an ellipsoid. The rotation is through an angle $\frac{1}{2}|\mathbf{Z}|\Delta t$ about the direction of $\mathbf{Z} = \mathrm{curl}\, \mathbf{u}$ (the vorticity vector), and the scaling is $1 + \frac{1}{2}\Delta t\, \mathrm{div}\, \mathbf{u}$. The deformation is specified by the orientation and magnitude of the principal axes of the ellipsoid. In general, there is a stretching axis, a contraction axis and an intermediate axis, which may be an axis of contraction or stretching; degenerate cases may occur. The components of the deformation (not given here) determine the orientation and size of the principal axes and the extents of the stretching and contraction. The spatial relationship of the velocity and vorticity vectors to the principal axes of the deformation ellipsoid will be of general kinematic and dynamic importance.

Tensor considerations obviously enter fluid dynamics at a pre-Newtonian level. The tensorial character of flow kinematics is evident also on direct physical grounds from a consideration of the effect of a deformation on a pre-existing gradient of some conserved scalar field. Figure 1(c) (representing a 2D case) shows that a pre-existing gradient perpendicular to the stretching axis increases as a consequence of the deformation, whereas a gradient parallel to the stretching axis decreases. A pre-existing gradient at 45° to the stretching axis remains unchanged in magnitude. These effects are important in the formation of *fronts* – regions of large horizontal gradients of temperature and other properties – in the atmosphere and oceans [see Hoskins (1982) and Hewson (1998)]. Tensor considerations also play an important role in the proper representation of viscous effects, in the analysis of interactions between eddies and mean flows, and in the parametrization of subgridscale Reynolds stresses in numerical models; see Williams (1972), Hoskins *et al.* (1983) and Adcroft and Marshall (1998). It turns out, however, that vorticity – a vector quantity – figures more prominently than deformation in the dynamics of meteorological flows. Although we shall refer again in this article to deformation, the bulk of the treatment will involve nothing more complicated than vector analysis and the manipulation of vector differential operators.

3 Fluid dynamics and thermodynamics

This section gives an elementary account of those equations of thermodynamics and fluid dynamics from which the future state of the atmosphere may be forecast, given its present state.

3.1 Local and total time derivatives; advection

Consider some meteorological field $\mathfrak{I}$. This field might be a scalar quantity, such as temperature; or a vector, such as the flow velocity $\mathbf{u}$. Assume that $\mathfrak{I}$ is a function of time t and position $\mathbf{r}$ in some chosen coordinate frame:

$$\mathfrak{I} = \mathfrak{I}(\mathbf{r}, t).$$

Assume also that $\mathfrak{I}(\mathbf{r}, t)$ is differentiable with respect to each argument. Then first-order Taylor expansion of $\mathfrak{I}$ about $\mathfrak{I}(\mathbf{r}, t)$ gives

$$\delta\mathfrak{I} \equiv \mathfrak{I}(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - \mathfrak{I}(\mathbf{r}, t) = (\delta\mathbf{r}.\,\mathrm{grad})\mathfrak{I} + (\partial\mathfrak{I}/\partial t)\delta t. \tag{3.1}$$

Equation (3.1) applies to any (infinitesimal) choice of $\delta\mathbf{r}$, δt. Choose $\delta\mathbf{r}$ to be the displacement in time δt corresponding to the velocity $\mathbf{u}$ of the air currently at position $\mathbf{r}$. Then $\delta\mathbf{r}/\delta t = \mathbf{u}$, and (3.1) becomes

$$\frac{D\mathfrak{I}}{Dt} \equiv \frac{\delta\mathfrak{I}}{\delta t} = (\mathbf{u}.\,\mathrm{grad})\mathfrak{I} + \frac{\partial\mathfrak{I}}{\partial t}. \tag{3.2}$$

The term $D\Im/Dt$ is the rate of change of $\Im$ following a parcel of air; it is known as the *total* (or material, or substantial, or individual, or Lagrangian) time derivative of $\Im$. The term $\partial\Im/\partial t$ is the *local* (or Eulerian) time derivative of $\Im$; it is the rate of change of $\Im$ at a point fixed in the chosen coordinate frame.

Some important physical laws (such as Newton's second law of motion) give information about material time derivatives. The users of weather forecasts are usually – not always – interested in the consequences of the local rate of change of $\Im$. A Grampian farmer in the highlands of Scotland may wish to know what the temperature of the air in the neighbourhood of the farm will be tomorrow, but is unlikely to want to know what the temperature of the air which is at the farm now will be tomorrow; that body of air may be over the North Sea by then. Hence the trivial re-expression of (3.2) as

$$\frac{\partial\Im}{\partial t} = \frac{D\Im}{Dt} - (\mathbf{u}.\,\mathrm{grad})\Im \tag{3.3}$$

is of fundamental importance in meteorology. Within its generality, (3.3) expresses the key physical notion that when $\Im$ is conserved on fluid particles ($D\Im/Dt = 0$) the value of $\Im$ at a fixed point in our coordinate frame will nevertheless be changing ($\partial\Im/\partial t \neq 0$) if fluid having a different value of $\Im$ is being brought in, or advected, by the flow ($-(\mathbf{u}.\,\mathrm{grad})\Im \neq 0$). The term $-(\mathbf{u}.\,\mathrm{grad})\Im$ represents the (rate of) advection of $\Im$. A vexed issue of terminology will be side-stepped in this article by using the expression 'advection term' to describe both $-(\mathbf{u}.\,\mathrm{grad})\Im$ (as in (3.3)) and $+(\mathbf{u}.\,\mathrm{grad})\Im$ (as in (3.2)).

We now consider how various choices of $\Im$, and the application of various physical laws, lead to expressions for the local rates of change of meteorological fields. With the needs of our Grampian farmer in mind, we begin by choosing $\Im = T$, the temperature.

3.2 First law of thermodynamics

Suppose that a parcel of air having unit mass, temperature T and (specific) volume α undergoes a change of (specific) entropy δs. According to the first law of thermodynamics, the concomitant changes δT and $\delta\alpha$ of T and α are related by

$$c_v\delta T + p\delta\alpha = T\delta s. \tag{3.4}$$

Here c_v is the specific heat at constant volume and p is the pressure of the parcel of air. Since the first law of thermodynamics applies to the parcel of air as it moves, it follows from (3.4) that

$$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = T\frac{Ds}{Dt} \equiv Q. \tag{3.5}$$

In meteorology, Q is usually thought of as the total heating rate per unit mass; strictly, it is the heating rate that would achieve, by reversible processes, the same rates of change of T and α as those occurring in the actual irreversible system (Lorenz 1967, p.14). Equation (3.5) can be written in terms of density $\rho = 1/\alpha$ as

$$c_v \frac{DT}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = Q. \tag{3.6}$$

In either form, however, the first law of thermodynamics gives only a relationship between the material derivatives of T and a density variable.

3.3 Mass continuity

Information about the material derivative of density, $D\rho/Dt$ (see (3.6)), may be obtained from mass conservation. The mass within a volume τ (fixed relative to the chosen coordinate frame) changes only to the extent that there is net inflow or outflow of mass at the boundary S of the volume. Hence

$$\frac{\partial}{\partial t}\int_\tau \rho\, d\tau = -\int_S \rho\mathbf{u}.\,\mathbf{dS} = -\int_\tau \operatorname{div}\rho\mathbf{u}\, d\tau \tag{3.7}$$

by the divergence theorem. Equation (3.7) applies to any volume τ, so the local equality

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{u} = 0 \tag{3.8}$$

must hold. Equation (3.8) is a form of the (mass) continuity equation. By using (3.3), we may deduce an alternative form:

$$\frac{D\rho}{Dt} + \rho\operatorname{div}\mathbf{u} = 0. \tag{3.9}$$

3.4 Perfect gas law

Equations (3.6) and (3.9), taken together with (3.3) in the form

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - (\mathbf{u}.\,\mathrm{grad})T,$$

enable us to evaluate the local rate of change $\partial T/\partial t$ so long as we know the current values of Q, p, ρ and the flow vector $\mathbf{u}$. The current value of ρ can be found from observations of p and T by using the perfect gas law in the form

$$p = \rho RT \tag{3.10}$$

where R is the gas constant per unit mass. Equation (3.10) has no time derivatives. In meteorological parlance, it is a *diagnostic* equation; equations involving time derivatives are called *prognostic*. We have now set up the apparatus to evaluate, and hence (knowing the current values of T, p and $\mathbf{u}$) to calculate T at our chosen location at a later time $t + \delta t$. If we were content to take δt

to equal 24 hours, we could calculate an expected value of T at the chosen location tomorrow. The calculated value would probably be very inaccurate, and for this reason (and others) the calculation of a 24-hour temperature forecast proceeds in practice by performing a number of *time steps* δt which are much shorter than 24 hours. (Typically, the time steps are of the order of 10 minutes.) This process requires values of Q, p, ρ and $\mathbf{u}$ at each time step. Hence we require a prognostic equation for the flow $\mathbf{u}$; in general, we cannot forecast the temperature accurately for more than (say) an hour ahead without forecasting the flow too. In any case, many users of weather forecasts – including our Scottish farmer, if there are new lambs on the hill – will want to know what tomorrow's wind speed and direction are likely to be.

3.5 Newton's second law

Newton's second law of motion relates the inertial acceleration of an element of air to the net force acting on it. Contributory forces include the pressure gradient force, gravity, and friction. If (as is usually convenient) velocities and accelerations are measured relative to the rotating frame of the solid Earth, Coriolis and centrifugal 'forces' must be introduced to allow for the transformation from inertial to accelerating (rotating) frame; see Stommel and Moore (1989) and Persson (1998) for discussion.

The Lagrangian rate of change of the velocity $\mathbf{u}$ of an element of air, relative to the rotating Earth, is then given by

$$\frac{D\mathbf{u}}{DT} = \underbrace{-2\boldsymbol{\Omega}\times\mathbf{u}}_{\text{Coriolis}} - \underbrace{\alpha\,\mathrm{grad}\,p}_{\substack{\text{Pressure}\\ \text{gradient}}} - \underbrace{\mathrm{grad}\,\Phi}_{\substack{\text{Apparent}\\ \text{gravity}}} + \underbrace{\mathbf{F}}_{\substack{\text{Friction and all}\\ \text{other forces}}} . \tag{3.11}$$

Equation (3.11) is the Navier–Stokes equation for motion and acceleration relative to the Earth, whose rotation vector is $\boldsymbol{\Omega}$. 'Apparent gravity', with potential function Φ, consists of the contribution (dominant in the atmosphere) of true Newtonian gravity and the contribution of the centrifugal force $-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$; here $\mathbf{r}$ is position vector relative to a frame rotating with the Earth, and having its origin at the centre of the Earth – see Figure 2. In (3.11) all forces are expressed per unit mass of air.

Equation (3.11) may be used in conjunction with

$$\frac{\partial\mathbf{u}}{\partial t} = \frac{D\mathbf{u}}{Dt} - (\mathbf{u}.\,\mathrm{grad})\mathbf{u}, \tag{3.12}$$

(the appropriate form of (3.3)) to give an expression for the local rate of change of $\mathbf{u}$, i.e. $\partial\mathbf{u}/\partial t$. The advection term is nonlinear in $\mathbf{u}$; the pressure gradient term, $\alpha\,\mathrm{grad}\,p$, is also in a certain sense nonlinear (as are the advection terms which arise from the first law of thermodynamics and the continuity equation).