1 Finite Difference Equations

In this chapter, the exact analytical solution of linear finite difference equations is discussed. The main purpose is to identify the similarities and differences between solutions of differential equations and finite difference equations. Attention is drawn to the intrinsic problems of using a high-order finite difference equation to approximate a partial differential equation. Since exact analytical solutions are used, the conclusions of this chapter are not subjected to numerical errors.

1.1. Order of Finite Difference Equations: Concept of Solution

Domain: In this chapter the domain considered consists of the set of integers $k = 0$, $\pm 1, \pm 2, \pm 3, \ldots$. The general member of the sequence $\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$ will be denoted by y_k.

An ordinary difference equation is an algorithm relating the values of different members of the sequence y_k. In general, a finite difference equation can be written in the form

$$y_{k+n} = F(y_{k+n-1}, y_{k+n-2}, \ldots, y_k, k), \tag{1.1}$$

where F is a general function.

The order of a difference equation is the difference between the highest and lowest indices appearing in the equation. For linear difference equations, the number of linearly independent solutions is equal to the order of the equation.

A difference equation is linear if it can be put in the following form:

$$y_{k+n} + a_1(k)\, y_{k+n-1} + a_2(k)\, y_{k+n-2} + \cdots + a_{n-1}(k)\, y_{k+1} + a_n(k)\, y_k = R_k, \tag{1.2}$$

where $a_i(k)$, $i = 1, 2, 3, \ldots, n$ and R_k are given functions of k.

EXAMPLES

(a) $y_{k+1} - 3y_k + y_{k-1} = 6e^{-k}$ (second-order, linear)
(b) $y_{k+1} = y^2{}_k$ (first-order, nonlinear)
(c) $y_{k+2} = \sin(y_k)$ (second-order, nonlinear)

The solution of a difference equation is a function $y_k = \phi(k)$ that reduces the equation to an identity.

1.2 Linear Difference Equations with Constant Coefficients

Linear difference equations with constant coefficients can be solved in much the same way as linear differential equations with constant coefficients. The characteristics of the two types of solutions are similar but not identical.

Consider the nth-order homogeneous finite difference equation with constant coefficients:

$$y_{k+n} + a_1 y_{k+n-1} + a_2 y_{k+n-2} + \cdots + a_n y_k = 0, \tag{1.3}$$

where $a_1, a_2, \ldots, a_n$ are constants. The general solution of such an equation has the form:

$$y_k = cr^k, \tag{1.4}$$

where c and r are constants. Substitution of Eq. (1.4) into Eq. (1.3) yields, after factoring out the common factor cr^k,

$$f(r) \equiv r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0. \tag{1.5}$$

Here, $f(r)$ is an nth-order polynomial and thus has n roots r_i, $i = 1, 2, \ldots, n$. For each r_i we have a solution:

$$y_k = c_i r_i^k, \tag{1.6}$$

where c_i is an arbitrary constant. The most general solution may be found by superposition.

1.2.1 Distinct Roots

If the characteristic roots of Eq. (1.5) are distinct, then a fundamental set of solutions is

$$y_k^{(i)} = r_i^k, \qquad i = 1, 2, \ldots, n$$

and the general solution of the homogeneous equation is

$$y_k = c_1 r_1^k + c_2 r_2^k + \cdots + c_n r_n^k, \tag{1.7}$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants.

EXAMPLE. Find the general solution of

$$y_{k+3} - 7y_{k+2} + 14y_{k+1} - 8y_k = 0.$$

Let $y_k = cr^k$. Substitution into the difference equation yields the characteristic equation

$$r^3 - 7r^2 + 14r - 8 = 0$$

or

$$(r-1)(r-2)(r-4) = 0.$$

The characteristic roots are $r = 1, 2$, and 4. Therefore, the general solution is

$$y_k = A + B2^k + C4^k,$$

where A, B, and C are arbitrary constants.

1.2.2 Repeated Roots

Now consider the case where one or more of the roots of the characteristic equation are repeated. Suppose the root r_1 has multiplicity m_1, the root r_2 has multiplicity m_2, and the root r_ℓ has multiplicity m_ℓ such that

$$m_1 + m_2 + \cdots + m_\ell = n. \tag{1.8}$$

The characteristic equation can be written as

$$(r - r_1)^{m_1} (r - r_2)^{m_2} \cdots (r - r_\ell)^{m_\ell} = 0. \tag{1.9}$$

Corresponding to a repeated root of the characteristic polynomial (1.9) of multiplicity m, the solution is

$$y_k = \left(A_1 + A_2 k + A_3 k^2 + \cdots A_m k^{m-1}\right) r^k, \tag{1.10}$$

where $A_1, A_2, \ldots, A_m$ are arbitrary constants.

EXAMPLE. Consider the general solution of the equation

$$y_{k+2} - 6y_{k+1} + 9y_k = 0.$$

The characteristic equation is

$$r^2 - 6r + 9 = 0 \quad \text{or} \quad (r-3)^2 = 0.$$

Thus, there is a repeated root $r = 3, 3$. The general solution is

$$y_k = (A + Bk)\, 3^k.$$

1.2.3 Complex Roots

Since the coefficients of the characteristic polynomial are real, complex roots must appear as complex conjugate pairs. Suppose r and r^* ($* =$ complex conjugate) are roots of the characteristic equation; then, corresponding to these roots the solutions may be written as

$$y_{\mathrm{k}}^{(1)} = r^k, \qquad y_{\mathrm{k}}^{(2)} = (r^*)^k.$$

If a real solution is desired, these solutions can be recast into a real form. Let $r = \mathrm{R}e^{i\theta}$, then an alternative set of fundamental solutions is

$$y_k^{(1)} = R^k \cos(k\theta), \qquad y_k^{(2)} = R^k \sin(k\theta).$$

If r and r^* are repeated roots of multiplicity m, then the set of fundamental solutions corresponding to these roots is

$$\begin{aligned} y_k^{(1)} &= R^k \cos(k\theta) & y_k^{(m+1)} &= R^k \sin(k\theta) \\ y_k^{(2)} &= kR^k \cos(k\theta) & y_k^{(m+2)} &= kR^k \sin(k\theta) \\ &\vdots & &\vdots \\ y_k^{(m)} &= k^{m-1} R^k \cos(k\theta) & y_k^{(2m)} &= k^{m-1} R^k \sin(k\theta). \end{aligned} \tag{1.11}$$

EXAMPLE. Find the general solution of

$$y_{k+2} - 4y_{k+1} + 8y_k = 0.$$

The characteristic equation is

$$r^2 - 4r + 8 = 0.$$

The roots are $r = 2 \pm 2i = 2\sqrt{2}\ e^{\pm i(\pi/4)}$. Therefore, the general solution (can be verified by direct substitution) is

$$y_k = A(2\sqrt{2})^k \cos\left(\frac{\pi}{4}k\right) + B(2\sqrt{2})^k \sin\left(\frac{\pi}{4}k\right),$$

where A and B are arbitrary constants.

1.3 Finite Difference Solution as an Approximate Solution of a Boundary Value Problem

A concrete example will now illustrate the inherent difficulties of using the finite difference solution to approximate the solution of a boundary value problem governed by partial differential equations.

Suppose the frequencies of the normal acoustic wave modes of a one-dimensional tube of length L as shown in Figure 1.1 is to be determined. The tube has two closed ends and is filled with air. The governing equations of motion of the air in the tube are the linearized momentum and energy equations, as follows:

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0 \tag{1.12}$$

$$\frac{\partial p}{\partial t} + \gamma p_0 \frac{\partial u}{\partial x} = 0, \tag{1.13}$$

where ρ_0, p_0, and γ are, respectively, the static density, the pressure, and the ratio of specific heats of the air inside the tube; and u is the velocity. The boundary conditions are

$$\text{At} \quad x = 0, L; \qquad u = 0. \tag{1.14}$$

Upon eliminating p from (1.12) and (1.13), the equation for u is

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.15}$$

where $a = (\gamma p_0/\rho_0)^{1/2}$ is the speed of sound.

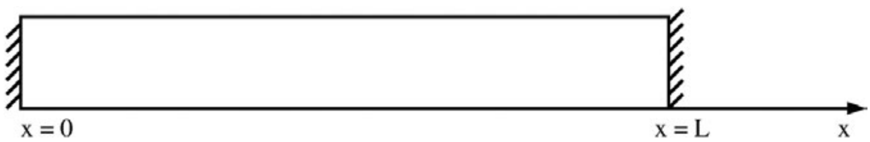

Figure 1.1. A one-dimensional tube with closed ends.

1.3.1 Analytical Solution

To find the normal acoustic modes of the tube, consideration will be given to solutions of the form:

$$u\,(x,t) = \mathrm{Re}[\hat{u}(x)e^{-i\omega t}], \tag{1.16}$$

where Re[ ] is the real part of [ ]. Substitution of Eq. (1.16) into Eqs. (1.15) and (1.14) yields the following eigenvalue problem:

$$\frac{d^2\hat{u}}{dx^2} + \frac{\omega^2}{a^2}\hat{u} = 0 \tag{1.17}$$

$$\hat{u}(0) = \hat{u}(L) = 0. \tag{1.18}$$

The two linearly independent solutions of Eq. (1.17) are

$$\hat{u}\,(x) = A\sin\left(\frac{\omega x}{a}\right) + B\cos\left(\frac{\omega x}{a}\right). \tag{1.19}$$

On imposing boundary conditions (1.18), it is found that

$$B = 0, \quad \text{and} \quad A\sin\left(\frac{\omega L}{a}\right) = 0.$$

For a nontrivial solution A cannot be zero, this leads to,

$$\sin\left(\frac{\omega x}{a}\right) = 0 \qquad \text{or} \qquad \frac{\omega L}{a} = n\pi \quad (n = \text{integer}).$$

Therefore,

$$\omega_n = \frac{n\pi a}{L}, \quad (n = 1, 2, 3, \ldots) \tag{1.20}$$

is the eigenvalue or eigenfrequency. The eigenfunction or mode shape is obtained from Eq. (1.19); i.e.,

$$\hat{u}_n\,(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots. \tag{1.21}$$

1.3.2 Finite Difference Solution

Now consider solving the normal mode problem by finite difference approximation. For this purpose, the tube is divided into M equal intervals with a spacing of $\Delta x = L/M$ as shown in Figure 1.2. ℓ is the spatial index ($\ell = 0$ to M). Both second- and fourth-order standard central difference approximation will be used.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_\ell = \frac{u_{\ell+1} - 2u_\ell + u_{\ell-1}}{(\Delta x)^2} + O(\Delta x^2) \tag{1.22}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_\ell = \frac{-u_{\ell+2} + 16u_{\ell+1} - 30u_\ell + 16u_{\ell-1} - u_{\ell-2}}{12\,(\Delta x)^2} + O(\Delta x^4). \tag{1.23}$$

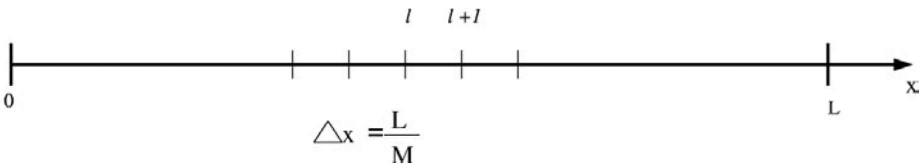

Figure 1.2. The computation grid for finite difference solution.

1.3.2.1 Second-Order Approximation

On replacing the spatial derivative of Eq. (1.15) by Eq. (1.22), the finite difference equation to be solved is

$$\frac{d^2 u_\ell}{dt^2} - \frac{a^2}{(\Delta x)^2}(u_{\ell+1} - 2u_\ell + u_{\ell-1}) = 0. \tag{1.24}$$

Eq. (1.24) is a second-order finite difference equation, the same order as the original partial differential equation. For a unique solution, two boundary conditions are required. This is given by the boundary conditions of the physical problem, Eq. (1.14); i.e.,

$$u_0 = 0, \qquad u_M = 0. \tag{1.25}$$

On following Eq. (1.16), a separable solution of a similar form is sought,

$$u_\ell\,(t) = \mathrm{Re}\left[\tilde{u}_\ell e^{-i\omega t}\right]. \tag{1.26}$$

Substitution of Eq. (1.26) into Eqs. (1.24) and (1.25) leads to the following eigenvalue problem:

$$\tilde{u}_{\ell+1} + \left[\frac{\omega^2\,(\Delta x)^2}{a^2} - 2\right]\tilde{u}_\ell + \tilde{u}_{\ell-1} = 0 \tag{1.27}$$

$$\tilde{u}_0 = 0, \qquad \tilde{u}_M = 0. \tag{1.28}$$

Two linearly independent solutions of finite difference equation (1.27) in the form of Eq. (1.4) can easily be found. The characteristic equation is

$$r^2 + \left[\frac{\omega^2\,(\Delta x)^2}{a^2} - 2\right] r + 1 = 0. \tag{1.29}$$

The two roots of Eq. (1.29) are complex conjugates of each other. The absolute value is equal to unity. Thus, the general solution of Eq. (1.27) may be written in the following form:

$$\tilde{u}_\ell = A \sin\,(\Theta\ell) + B\cos\,(\Theta\ell)\,, \tag{1.30}$$

where

$$\Theta = \cos^{-1}\left[1 - \frac{\omega^2\,(\Delta x)^2}{2a^2}\right]. \tag{1.31}$$

Upon imposition of boundary conditions (1.28), it is easy to find

$$B = 0, \quad A\sin\,(\Theta M) = 0.$$

For a nontrivial solution, it is required that $\sin(\theta M) = 0$. Hence,

$$\Theta M = n\pi, \qquad n = 1, 2, 3, \ldots$$

or

$$\cos^{-1}\left[1 - \frac{\omega^2 (\Delta x)^2}{2a^2}\right] = \frac{n\pi}{M}.$$

This yields

$$\omega_n = \frac{2^{\frac{1}{2}} a}{\Delta x}\left(1 - \cos\left(\frac{n\pi}{M}\right)\right)^{\frac{1}{2}}, \qquad n = 1, 2, 3, \ldots \tag{1.32}$$

and from solution (1.30), the eigenfunction or mode shape is

$$\tilde{u}_\ell = \sin(\Theta\ell) = \sin\left(\frac{n\pi\ell}{M}\right). \tag{1.33}$$

Now, it is instructive to compare finite difference solutions (1.32) and (1.33) with the exact solution of the original partial differential equations (1.20) and (1.21). One obvious difference is that the exact solution has infinitely many eigenfrequencies and eigenfunctions, whereas the finite difference solution supports only a finite number ($2M$) of such modes. Furthermore, ω_n of Eq. (1.32) is a good approximation of the exact solution only for $n\pi/M \ll 1$. In other words, a second-order finite difference approximation provides good results only for the low-order long-wave modes. The error increases quickly as n increases.

1.3.2.2 Fourth-Order Approximation

If the fourth-order approximation of Eq. (1.23) is used instead of Eq. (1.22), it is easy to show that the governing finite difference equation for $\tilde{u}_\ell$ is

$$\tilde{u}_{\ell+2} - 16\tilde{u}_{\ell+1} + \left(30 - \frac{12\omega^2 (\Delta x)^2}{a^2}\right)\tilde{u}_\ell - 16\tilde{u}_{\ell-1} + \tilde{u}_{\ell-2} = 0. \tag{1.34}$$

The two physical boundary conditions of Eq. (1.15) are

$$\tilde{u}_0 = 0, \qquad \tilde{u}_M = 0. \tag{1.35}$$

Now, Eq. (1.34) is a fourth-order finite difference equation. There are four linearly independent solutions. In order to have a unique solution, four boundary conditions are necessary. However, only two physical boundary conditions are available. To ensure a unique solution of the fourth-order finite difference equation, two extra (nonphysical) boundary conditions need to be created. Also, two of the four solutions of Eq. (1.34) are spurious solutions unrelated to the physical problem. Therefore, the use of high-order approximation will result in

(A) Possible generation of spurious numerical solutions.
(B) A need for extra boundary conditions or special boundary treatment.

These are definite disadvantages in the use of a high-order scheme to approximate partial differential equations. Are there any advantages? To show that there could be an advantage, note that the eigenfunction of the finite difference equation (1.33) is identical to the exact eigenfunction (1.21). As it turns out, the eigenfunction (1.33)

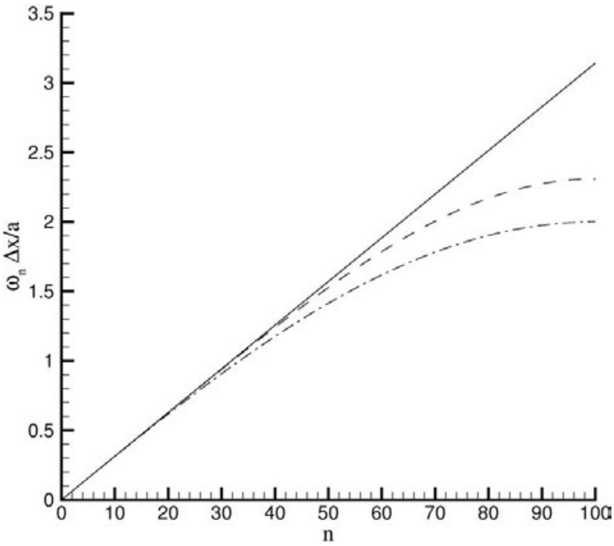

Figure 1.3. Comparison of normal mode frequencies. ___________, exact, Eq. (1.20); — — — —, fourth-order, Eq. (1.37); ······· , second-order, Eq. (1.32).

of the second-order approximation is also the eigenfunction of the fourth-order approximation, namely, the solution of Eqs. (1.34) and (1.35) is

$$\tilde{u}_\ell = \sin\left(\frac{n\pi\ell}{M}\right), \qquad n = 1, 2, 3, \ldots. \tag{1.36}$$

These eigenfunctions satisfy boundary conditions (1.35). On the substitution of solution (1.36) into Eq. (1.34), it is easy to find that the corresponding eigenfrequency is given by

$$\omega_n = \frac{a}{(\Delta x)\, 6^{\frac{1}{2}}}\left[15 - 16\cos\left(\frac{n\pi}{M}\right) + \cos\left(\frac{2n\pi}{M}\right)\right]^{\frac{1}{2}}. \tag{1.37}$$

It is straightforward to find that frequency formula (1.37) is a much improved approximation to the exact eigenfrequency of formula (1.20) than formula (1.32) of the second-order method. Figure 1.3 shows a comparison for the case $M = 100$. This result illustrates the fact that, when the problems of spurious waves and extra boundary conditions are adequately taken care of, a high-order method does give more accurate numerical results.

1.4 Finite Difference Model for a Surface of Discontinuity

How best to transform a boundary value problem governed by partial differential equations into a computation problem governed by difference equations is not always obvious. The task is even more difficult if the original problem involves a surface of discontinuity. There is a lack of discussion in the literature about how to model a discontinuity in the context of finite difference. The purpose of this section is to show how one such model may be set up. At the same time, this model will demonstrate that the finite difference formulation of boundary value problems may support spurious boundary modes. These modes might have no counterpart in the original problem.

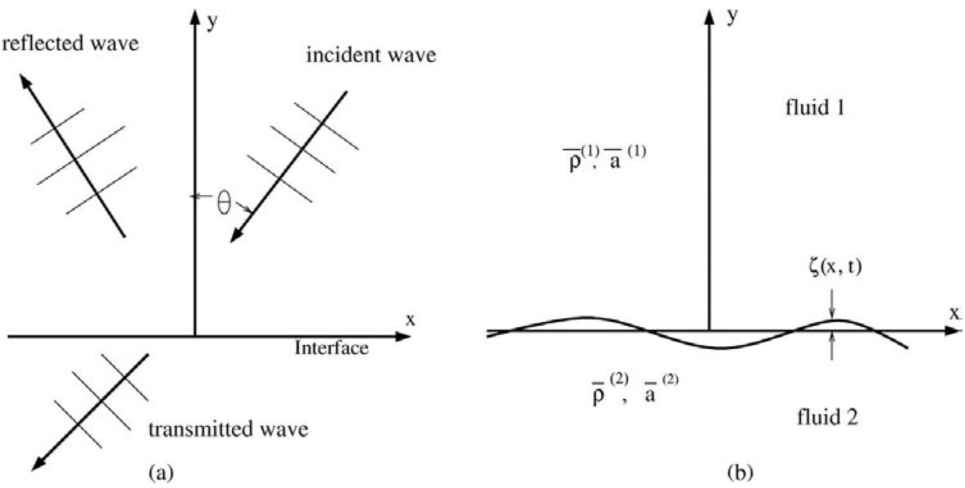

Figure 1.4. Schematic diagram showing (a) the incident, reflected, and transmitted sound waves at a fluid interface, (b) the slightly deformed fluid interface.

They are not generally known or expected. If one of these spurious boundary modes grows in time, then this could lead to numerical instability or a divergent solution.

Consider the transmission of sound through the interface of two fluids of densities $\overline{\rho}^{(1)}$ and $\overline{\rho}^{(2)}$ and sound speeds $\overline{a}^{(1)}$ and $\overline{a}^{(2)}$, respectively, as shown in Figure 1.4a. It is known that refraction takes place at such an interface.

Superscripts (1) and (2) will be used to denote the flow variables above and below the interface. For small-amplitude incident sound waves, it is sufficient to use the linearized Euler equations and interface boundary conditions. Let $y = \varsigma(x, t)$ be the location of the interface. The governing equations are

$$y \geq 0, \qquad \frac{\partial u^{(1)}}{\partial t} = -\frac{1}{\overline{\rho}^{(1)}} \frac{\partial p^{(1)}}{\partial x} \tag{1.38}$$

$$\frac{\partial v^{(1)}}{\partial t} = -\frac{1}{\overline{\rho}^{(1)}} \frac{\partial p^{(1)}}{\partial y} \tag{1.39}$$

$$\frac{\partial p^{(1)}}{\partial t} + \gamma \overline{p}^{(1)} \left(\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} \right) = 0 \tag{1.40}$$

$$y \leq 0 \qquad \frac{\partial u^{(2)}}{\partial t} = -\frac{1}{\overline{\rho}^{(2)}} \frac{\partial p^{(2)}}{\partial x} \tag{1.41}$$

$$\frac{\partial v^{(2)}}{\partial t} = -\frac{1}{\overline{\rho}^{(2)}} \frac{\partial p^{(2)}}{\partial y} \tag{1.42}$$

$$\frac{\partial p^{(2)}}{\partial t} + \gamma \overline{p}^{(2)} \left(\frac{\partial u^{(2)}}{\partial x} + \frac{\partial v^{(2)}}{\partial y} \right) = 0. \tag{1.43}$$

The dynamic and kinematic boundary conditions at the interface are

$$y = 0, \qquad p^{(1)} = p^{(2)} \tag{1.44}$$

$$\frac{\partial \zeta}{\partial t} = v^{(1)} = v^{(2)}. \tag{1.45}$$

For static equilibrium, the pressure balance condition is

$$\overline{p}^{(1)} = \overline{p}^{(2)} \quad \text{or} \quad \overline{\rho}^{(1)}(\overline{a}^{(1)})^2 = \overline{\rho}^{(2)}(\overline{a}^{(2)})^2. \tag{1.46}$$

1.4.1 The Transmission Problem

Consider a plane acoustic wave of angular frequency ω incident on the interface at an angle of incidence θ as shown in Figure 1.4. The appropriate solution of Eqs. (1.38) to (1.40) may be written in the following form:

$$\begin{bmatrix} u^{(1)} \\ v^{(1)} \\ p^{(1)} \end{bmatrix}_{\text{incident}} = \text{Re}\left\{ A \begin{bmatrix} -\frac{\sin\theta}{\overline{\rho}^{(1)}\overline{a}^{(1)}} \\ -\frac{\cos\theta}{\overline{\rho}^{(1)}\overline{a}^{(1)}} \\ 1 \end{bmatrix} e^{-i\omega(\sin\theta x + \cos\theta y + \overline{a}^{(1)}t)/\overline{a}^{(1)}} \right\}, \tag{1.47}$$

where A is the amplitude and Re{} is the real part of. The reflected wave in region (1) has a form similar to Eq. (1.47), which may be written as

$$\begin{bmatrix} u^{(1)} \\ v^{(1)} \\ p^{(1)} \end{bmatrix}_{\text{reflected}} = \text{Re}\left\{ R \begin{bmatrix} -\frac{\sin\theta}{\overline{\rho}^{(1)}\overline{a}^{(1)}} \\ \frac{\cos\theta}{\overline{\rho}^{(1)}\overline{a}^{(1)}} \\ 1 \end{bmatrix} e^{-i\omega(\sin\theta x - \cos\theta y + \overline{a}^{(1)}t)/\overline{a}^{(1)}} \right\}, \tag{1.48}$$

where R is the amplitude of the reflected wave. The transmitted wave in region (2) must have the same dependence on x and t as the incidence wave. Let

$$\begin{bmatrix} u^{(2)} \\ v^{(2)} \\ p^{(2)} \end{bmatrix}_{\text{transmitted}} = \text{Re}\left\{ \begin{bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{p}(y) \end{bmatrix} e^{-i\omega(\sin\theta x + \overline{a}^{(1)}t)/\overline{a}^{(1)}} \right\}. \tag{1.49}$$

By substituting Eq. (1.49) into Eqs. (1.41) to (1.43), it is easy to find after some simple elimination,

$$\frac{d^2\hat{p}}{\partial y^2} + \frac{\omega^2}{\left(\overline{a}^2\right)^2}\left[1 - \left(\frac{\overline{a}^{(2)}}{\overline{a}^{(1)}}\right)^2 \sin^2\theta\right]\hat{p} = 0. \tag{1.50}$$

On solving Eq. (1.50), the transmitted wave with an amplitude T may be written as

$$\begin{bmatrix} u^{(2)} \\ v^{(2)} \\ p^{(2)} \end{bmatrix}_{\text{transmitted}}$$
$$= \text{Re}\left\{ T \begin{bmatrix} -\frac{\sin\theta}{\overline{\rho}^{(2)}\overline{a}^{(1)}} \\ -\frac{[1-(\overline{a}^{(2)}/\overline{a}^{(1)})^2\sin^2\theta]^{1/2}}{\overline{\rho}^{(2)}\overline{a}^{(2)}} \\ 1 \end{bmatrix} e^{-i\omega\{\sin\theta x + (\overline{a}^{(1)}/\overline{a}^{(2)})[1-(\overline{a}^{(2)}/\overline{a}^{(1)})^2\sin^2\theta]^{1/2}y + \overline{a}^{(1)}t\}/\overline{a}^{(1)}} \right\}. \tag{1.51}$$