

# A Theory of Economic Growth

## Dynamics and Policy in Overlapping Generations

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**CAMBRIDGE**  
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE  
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS  
The Edinburgh Building, Cambridge CB2 2RU, UK  
40 West 20th Street, New York, NY 10011-4211, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
Ruiz de Alarcón 13, 28014 Madrid, Spain  
Dock House, The Waterfront, Cape Town 8001, South Africa  
<http://www.cambridge.org>

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First published 2002

Printed in the United Kingdom at the University Press, Cambridge

*Typeface* Times Ten 9.75/12 pt.    *System* L<sup>A</sup>T<sub>E</sub>X 2<sub>ε</sub> [TB]

*A catalog record for this book is available from the British Library.*

*Library of Congress Cataloging in Publication Data*  
de la Croix, David

A theory of economic growth : dynamics and policy in overlapping  
generations / David de la Croix, Philippe Michel.

p. cm.

Includes bibliographical references and index.

ISBN 0-521-80642-9 (hc.) – ISBN 0-521-00115-3 (pbk.)

1. Macroeconomics. 2. Economic development. 3. Economic policy.  
4. Generational accounting. I. Michel, Philippe. II. Title.

HB172.5 .L32 2002

339–dc21

2002016579

ISBN 0 521 80642 9 hardback  
ISBN 0 521 00115 3 paperback

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## Competitive Equilibria

The basic overlapping generations model with capital accumulation is due to Allais (1947)<sup>1</sup> and Diamond (1965).<sup>2</sup> Diamond (1965) considers an economy with physical capital and with or without a public sector. It is a framework in which all goods are real, in the sense that they are consumption goods and/or production factors. In this chapter we describe the framework without a public sector, which is the benchmark model to a wide strand of the literature, and we analyze the properties of the competitive equilibrium.

This chapter is organized as follows. Section 1.1 describes the structure of the model, and section 1.2 discusses the main assumptions. The behavior of the agents is analyzed in section 1.3. The notion of temporary equilibrium is introduced and analyzed in section 1.4. Section 1.5 studies the inter-temporal equilibrium with perfect foresight, its existence and uniqueness. Global dynamics are characterized in section 1.6. In section 1.7 we compare the dynamics under perfect foresight with the dynamics resulting from myopic foresight. Finally, some applications and extensions of the model are presented in section 1.8. Examples are provided throughout the chapter.

### 1.1 THE MODEL

Time  $t$  is discrete and goes from 0 to  $\infty$ .  $t$  belongs to the set of integer numbers  $\mathbb{N}$ ,  $t = 0, 1, 2, \dots$ . All decisions are taken at points in time. The current date is called *period*  $t$ , and we study how the economy operates from date  $t = 0$  onwards. At the initial date,  $t = 0$ , there will be initial conditions reflecting the history of the economy.

<sup>1</sup> Malinvaud (1987) has stressed the use of the overlapping generations model in the appendix of the book of Allais (1947).

<sup>2</sup> The basic overlapping generations model for an exchange economy is due to Samuelson (1958).



At each period  $t$ , there exist three goods: capital, labor, and a physical good produced from capital and labor. This physical good is either consumed or invested to build future capital. We take the good produced at each period  $t$  as the numeraire. There is thus a different numeraire in each period.

As there are an infinite number of periods, there are an infinite number of goods.

### 1.1.1 Two-period-lived Individuals

The demographic structure is presented in figure 1.1. In each period  $t$ ,  $N_t$  persons are born, and they live for two periods.<sup>3</sup> From figure 1.1, we understand why this demographic structure is called “overlapping generations”: at each point in time, two generations are alive and overlap.

In their first period of life (when *young*), the individuals are endowed with one unit of labor that they supply inelastically to firms. Their income is equal to the real wage  $w_t$ . They allocate this income between current consumption  $c_t$  and savings  $s_t$ , which are invested in the firms. The budget constraint of period  $t$  is

$$w_t = c_t + s_t. \quad (1.1)$$

In their second period of life  $t + 1$  (when *old*), they are retired. Their income comes from the return on the savings made at time  $t$ . As they do not care

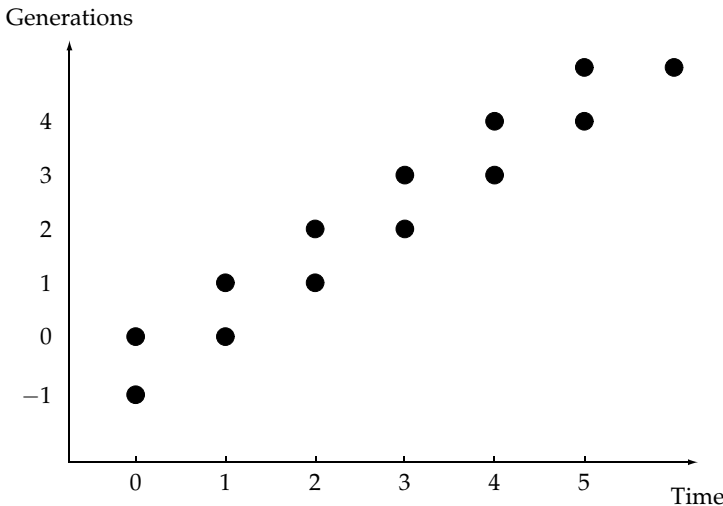


Figure 1.1. Overlapping generations.

<sup>3</sup> Alternatively, we may consider that each person lives three periods, is working in the second period, and retires in the third one. During the first period, he does not take any decisions, and his consumption can be thought of as included in that of his parents.

about events occurring after their death (this assumption will be removed in section 5.1), they consume their income entirely. Denoting by  $R_{t+1}$  ( $= 1 + r_{t+1}$ ) the return factor on savings from time  $t$  to time  $t + 1$ , the income of an old individual is  $R_{t+1}s_t$ , and his consumption is

$$d_{t+1} = R_{t+1}s_t. \quad (1.2)$$

The preferences of the households are defined over their consumption bundle  $(c_t, d_{t+1})$ . We assume that they can be represented by a life-cycle utility function

$$U(c_t, d_{t+1}).$$

At each period  $t \geq 1$ ,  $N_t + N_{t-1}$  individuals are alive, including  $N_t$  young households born in  $t$  and  $N_{t-1}$  old households born in  $t - 1$ . At the first period  $t = 0$ , there are, in addition to the  $N_0$  young households,  $N_{-1}$  old households.<sup>4</sup> Each of these  $N_{-1}$  old persons is the owner of the same fraction  $s_{-1}$  of the installed capital stock  $K_0$ . Productive capital is the only asset in the economy, so that  $s_{-1} = K_0/N_{-1}$ . The income of old persons is equal to  $R_0s_{-1}$ . These people entirely consume their income:

$$d_0 = R_0 s_{-1} = \frac{R_0 K_0}{N_{-1}}.$$

The number of households of each generation grows at a constant rate  $n \in ]-1, +\infty[$ :

$$N_t = (1 + n)N_{t-1}.$$

Consequently, the total population  $N_t + N_{t-1}$  grows also at the rate  $n$ .<sup>5</sup> Note that, since  $n \in ]-1, +\infty[$ , the model may represent economies where population shrinks at a constant rate (negative  $n$ ).

### 1.1.2 Neo-classical Technology

The production technology is the same for all periods.<sup>6</sup> It is represented by the neo-classical production function  $\bar{F}(K, L)$ . The function  $\bar{F}$  is homogeneous of degree one (see appendix A.1.1) with respect to its arguments: capital  $K$  and labor  $L$ .

<sup>4</sup> Nothing is said about the past of these households. It is as if they were born old.

<sup>5</sup> Overlapping generations models can be extended to deal with endogenous fertility as in Becker and Barro (1988), with dynastic altruism, or Galor and Weil (1996), with ad hoc altruism.

<sup>6</sup> The generalization of this to include deterministic labor-savings technical progress is performed in section 1.8.6. Notice however that, in standard growth models, technological improvement can coexist with a balance path only with a special type of utility function (see e.g., King, Plosser, and Rebelo (1990)).

During the production process, the capital stock depreciates physically at a rate  $\delta \in [0, 1]$ .<sup>7</sup>

For simplicity we also assume that, after the production process, the part of capital that is not depreciated is identical to the good produced, so that we may define a total production function:

$$F(K, L) = \bar{F}(K, L) + (1 - \delta)K, \quad (1.3)$$

which is also homogeneous of the first degree, implying that the technology exhibits constant returns to scale:

$$F(\lambda K, \lambda L) = \lambda F(K, L) \quad \forall \lambda > 0.$$

### 1.1.3 Firms

We assume a representative firm producing at period  $t$ . This assumption is not restrictive, as, with constant return to scale, the number of firms does not matter and production is independent of the number of firms which use the same technology. At time  $t = 0$  the capital stock  $K_0$  is already installed in the firm producing at  $t = 0$ . For all  $t \geq 1$ , capital  $K_t$  is productive at time  $t$  and is built from the savings of time  $t - 1$  (there is a one-period time-to-build). The representative firm that produces at time  $t$  exists during two periods,  $t - 1$  and  $t$ .<sup>8</sup> At time  $t - 1$  it “receives” the deposits  $I_{t-1}$  from the young households. This deposit of goods produced at time  $t - 1$  becomes the productive capital used in the production process at time  $t$ :

$$K_t = I_{t-1} = N_{t-1}s_{t-1}.$$

The households remain the owners of the stock of capital and will receive the profits of the firm when old.

## 1.2 MAIN ASSUMPTIONS

### 1.2.1 The Assumptions on the Utility Function

The life-cycle utility function is assumed to be additively separable:

$$U(c, d) = u(c) + \beta u(d), \quad (1.4)$$

where  $\beta$  is the psychological discount factor:  $\beta = 1/(1 + \rho)$ , where  $\rho$  is the rate of time preference, which varies inversely with  $\beta$ . The non-separable case is treated in section 1.8.3.

<sup>7</sup> As one period represents 20 or 30 years, it is often assumed that the depreciation rate is 1.

<sup>8</sup> Alternatively, we may assume that firms live forever. This would not change the results, as the firms' program is in any case a static one.

We assume that the instantaneous utility function  $u$  is twice continuously differentiable on the set of strictly positive real numbers  $\mathbb{R}_{++}$ , strictly increasing (no satiation), and concave (decreasing marginal utility):

**Assumption H1.**

For all  $c > 0$ , one has  $u'(c) > 0$ ,  $u''(c) < 0$ , and  $\lim_{c \rightarrow 0} u'(c) = +\infty$ .

The hypothesis of an infinite marginal utility of zero consumption implies that the household always chooses a positive consumption level  $c$  when it maximizes its life-cycle utility (as long as its disposable income is positive).

The two assumptions of additive separability and concavity imply that  $c$  and  $d$  are normal commodities, i.e., that their demands are non-decreasing in wealth.

**Example:** The CIES<sup>9</sup> (constant inter-temporal elasticity of substitution) utility function,

$$u(c) = \left[1 - \frac{1}{\sigma}\right]^{-1} c^{1-\frac{1}{\sigma}}, \quad \sigma > 0, \quad \sigma \neq 1,$$

satisfies the hypothesis **H1**. Indeed,

$$u'(c) = c^{-\frac{1}{\sigma}} > 0, \quad u''(c) = -\frac{1}{\sigma} c^{-\frac{1}{\sigma}-1} < 0,$$

and

$$\lim_{c \rightarrow 0} c^{-1/\sigma} = +\infty.$$

The parameter  $-\frac{1}{\sigma}$  is the elasticity of marginal utility:

$$\frac{u''(c)c}{u'(c)} = -\frac{1}{\sigma}.$$

We show below that the elasticity of marginal utility is also the reciprocal of the inter-temporal elasticity of substitution.<sup>10</sup>

The case of a logarithmic utility function,

$$u(c) = \ln(c), \quad u'(c) = \frac{1}{c}, \quad u''(c) = -\frac{1}{c^2},$$

gives an elasticity of marginal utility equal to  $-1$ . The CIES utility function is plotted in figure 1.2 for the three possible cases:  $\sigma > 1$ ,  $\sigma = 1$  (logarithmic utility), and  $\sigma < 1$ .

<sup>9</sup> The standard name CRRA (for constant relative risk aversion) does not seem suited to a framework in which there is no uncertainty.

<sup>10</sup> In a framework with uncertainty, the coefficient of relative risk aversion is equal to the elasticity of marginal utility.

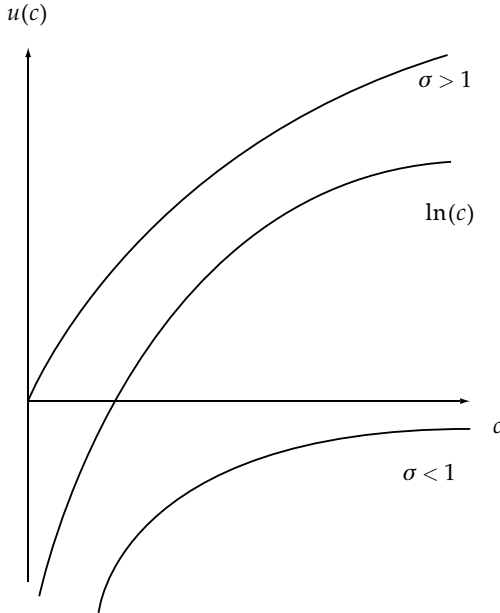


Figure 1.2. The CIES utility function. When  $\sigma > 1$ , the utility function is positive valued. When  $\sigma = 1$ , the utility function is simply the  $\ln$  function. When  $\sigma < 1$ , the utility function is negative valued.

### 1.2.2 The Assumptions on the Production Function

As the production function is homogeneous of degree one, it can be expressed by the mean of a function of one variable  $k = K/L$ :

$$F(K, L) = LF\left(\frac{K}{L}, 1\right) = Lf(k),$$

where  $f(k) = F(k, 1)$  is the production function in its intensive form. We make the following hypothesis on the function  $f(\cdot)$ : it is defined on the set of (strictly) positive real numbers  $\mathbb{R}_{++}$  and is twice continuously differentiable. It satisfies:

**Assumption H2.**

For all  $k > 0$ , one has  $f(k) > 0$ ,  $f'(k) > 0$ , and  $f''(k) < 0$ .

This hypothesis amounts to assuming that the function  $F$  is positive valued, increasing, and strictly concave with respect to capital  $K$ : indeed,  $f'(k) = F'_K(k, 1) = F'_K(K, L)$  because the derivative  $F'_K$  is homogeneous of degree 0, and  $f''(k) = F''_{KK}(k, 1) = LF''_{KK}(K, L)$  because  $F''_{KK}$  is homogeneous of degree  $-1$  (appendix A.1.1).

A consequence of **H2** is the following (appendix A.1.2):

$$\text{For all } k > 0, \quad \omega(k) = f(k) - kf'(k) = F'_L(K, L) > 0.$$

The hypothesis **H2** implies thus that the marginal productivity of labor is strictly positive.

In order to include the popular CES production function, we make no assumption on the limits of the function and its derivatives when  $k \rightarrow 0$  and  $k \rightarrow +\infty$ . Nevertheless, **H2** implies that the function  $f(k)$  and  $\omega(k)$  admits non-negative limits when  $k$  goes to zero. Thus we may assume that these functions are continuous on the set of non-negative real numbers  $\mathbb{R}_+$  with values in  $\mathbb{R}_+$ .

Notice that additional hypotheses are often made to describe the properties of  $f(\cdot)$  on the boundary:

**Assumption A1.**

$$f(0) = 0.$$

**Assumption A2.**

$$\begin{aligned} \lim_{k \rightarrow 0} f'(k) &= +\infty, \\ \lim_{k \rightarrow +\infty} f'(k) &= 0. \end{aligned}$$

**Assumption A3.**

$$\begin{aligned} \lim_{k \rightarrow 0} f'(k) &= +\infty, \\ \lim_{k \rightarrow +\infty} f'(k) &< 1. \end{aligned}$$

The assumption **A1** states that capital is essential for production. The assumption **A2** is called the *Inada conditions*. One of these conditions is violated by the CES production function<sup>11</sup> except in the limit case of the Cobb–Douglas function (see appendix A.1.2). The assumption **A3** is less restrictive than **A2**

<sup>11</sup> This production function was first introduced by Arrow, Chenery, Minhas, and Solow (1961). The justification given at that time can be used here to avoid imposing Inada conditions: “Two competing alternative [production functions] hold the field at present: the Walras–Leontief–Harrod–Domar of constant input coefficients; and the Cobb–Douglas function, which implies a unitary elasticity of substitution between labor and capital. From a mathematical point of view, zero and one are perhaps the most convenient alternatives for this elasticity. Economic analysis based on these assumptions, however, often leads to conclusions that are unduly restrictive.”

and allows us to include the case of a depreciation rate  $\delta < 1$ . In the sequel of this chapter, we work without making these assumptions.

**Example:** The CES (constant elasticity of substitution) production function,

$$\bar{F}(K, L) = A[\alpha K^{-\rho} + (1 - \alpha)L^{-\rho}]^{-1/\rho},$$

$$A > 0, \quad 0 < \alpha < 1, \quad \rho > -1, \quad \rho \neq 0,$$

is homogeneous of the first degree. The elasticity of substitution between  $K$  and  $L$  is equal to

$$\frac{1}{1 + \rho}.$$

In at the limit when  $\rho$  tends to  $-1$ , the function is linear and the two factors of production are perfect substitutes (the corresponding isoquants are plotted in figure 1.3). This case is excluded by assumption **H2**. They become less and less substitutable as  $\rho$  increases. In the limit when  $\rho \rightarrow +\infty$  the factors of production are perfect complements. The Cobb–Douglas case<sup>12</sup> is obtained as a special

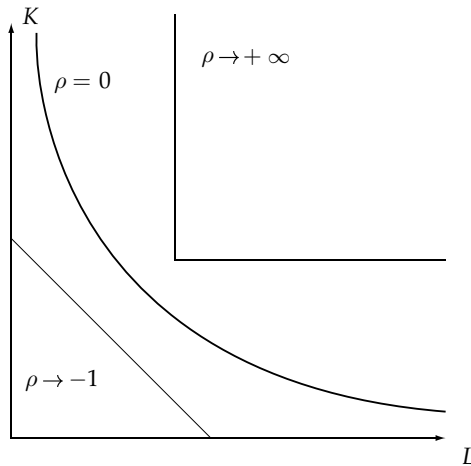


Figure 1.3. The CES production function. The isoquant when  $\rho \rightarrow +\infty$  shows that capital and labor are complements (no substitution possibilities) in the production process (Leontief technology). For  $\rho = 0$  we obtain the isoquant of the Cobb–Douglas function. When  $\rho \rightarrow -1$ , the isoquant becomes linear, and capital and labor can be substituted perfectly.

<sup>12</sup> This function was introduced in Douglas (1934) to study American production over the period 1899–1922. The striking agreement between the actual production series and the one generated by the Cobb–Douglas function is at the basis of the success of this production function.

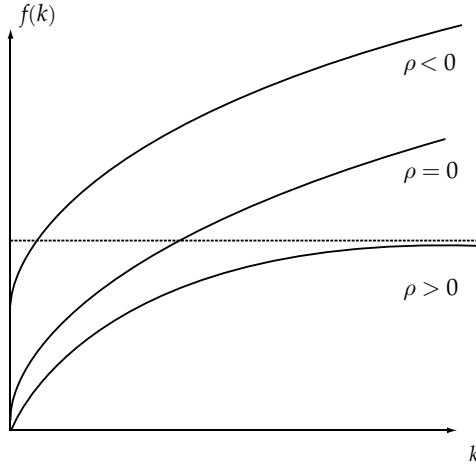


Figure 1.4. The CES production function in intensive form. When  $\rho < 0$ , each production factor is not essential to production and  $f(0) > 0$ . When  $\rho \geq 0$ ,  $f(0) = 0$ . The limit of  $f'(k)$  when  $k \rightarrow 0$  is infinite when  $\rho \leq 0$  and finite when  $\rho > 0$ . The limit of  $f'(k)$  when  $k \rightarrow +\infty$  is positive when  $\rho < 0$  and is 0 when  $\rho \geq 0$ . Only the Cobb–Douglas case  $\rho = 0$  satisfies the Inada conditions.

case when  $\rho \rightarrow 0$  (see appendix A.1.5):

$$\bar{F}(K, L) = AK^\alpha L^{1-\alpha}.$$

The three cases are plotted in figure 1.4. With complete depreciation of capital, the function  $F = \bar{F}$ , and can be written in intensive form:

$$f(k) = A[\alpha k^{-\rho} + (1 - \alpha)]^{-1/\rho}.$$

It is easy to check that  $f(k)$  satisfies **H2**:

$$f'(k) = \alpha A[\alpha + (1 - \alpha)k^\rho]^{-\frac{1+\rho}{\rho}} = \frac{\alpha}{A^\rho} \left( \frac{f(k)}{k} \right)^{1+\rho} > 0,$$

and

$$f''(k) = -\alpha(1 - \alpha)A(1 + \rho)[\alpha + (1 - \alpha)k^\rho]^{-\frac{1+2\rho}{\rho}} k^{\rho-1} < 0.$$

Finally, the marginal productivity of labor is

$$\omega(k) = \frac{1 - \alpha}{A^\rho} f(k)^{1+\rho} > 0.$$



1.3 THE BEHAVIOR OF THE AGENTS AT PERIOD  $t$ 

All agents in this economy are price-takers, and all markets are competitive.<sup>13</sup> At a given date, young individuals decide how much to consume and to save, old individuals consume, producing firms hire labor and produce, and investing firms collect the savings from the young and build up the capital stock for the next period. We devote special attention to the savings behavior of the young persons, as it is the engine of capital accumulation.

## 1.3.1 The Young Individuals

At time  $t$  each young individual receives  $w_t$  units of the produced good as a wage. He allocates this income between consumption and savings in invested goods (1.1). He anticipates a return  $R_{t+1}^e$  for his savings and thus, according to (1.2), a future consumption  $d_{t+1}^e = R_{t+1}^e s_t$ .

Each young individual maximizes

$$\begin{aligned} & u(c_t) + \beta u(d_{t+1}^e) \\ \text{s.t. } & w_t = c_t + s_t, \\ & d_{t+1}^e = R_{t+1}^e s_t, \\ & c_t \geq 0, \quad d_{t+1}^e \geq 0. \end{aligned} \tag{1.5}$$

There are two ways to solve the problem. We may first substitute  $c_t$  and  $d_{t+1}^e$  in the objective function, which leads to

$$u(w_t - s_t) + \beta u(R_{t+1}^e s_t),$$

which is, according to **H1**, strictly concave with respect to  $s_t$ . The solution,

$$s_t = s(w_t, R_{t+1}^e),$$

is interior as a consequence of **H1** and is characterized by the first-order condition

$$u'(w_t - s_t) = \beta R_{t+1}^e u'(R_{t+1}^e s_t). \tag{1.6}$$

<sup>13</sup> Departures from this assumption can be found in the overlapping generations literature. Devereux and Lockwood (1991) and de la Croix and Licandro (1995) analyze capital accumulation under trade-unionism; in such an overlapping generations framework, the wage bargaining process takes place between the young workers and the old capitalists. Weddepohl and Yildirim (1993) study the fixed price temporary equilibria and rationing in an overlapping generations model with capital accumulation. de la Croix and Licandro (2000) also study underemployment of resources, but rationing comes from technological rigidities and idiosyncratic uncertainty instead of fixed prices. Cournot competition on the goods market is introduced in a model with capital by d'Aspremont, Gérard-Varet, and Ferreira (2000), and monopolistic competition is studied in Jacobsen (2000).

The function  $s(\cdot)$  is called the *savings function*, and its properties will be analyzed later on.

The second method to solve the problem is to eliminate  $s_t$  to obtain the inter-temporal budget constraint of the household:

$$c_t + \frac{1}{R_{t+1}^e} d_{t+1}^e = w_t.$$

We next build the following Lagrangian:

$$u(c_t) + \beta u(d_{t+1}^e) + \lambda_t \left( w_t - c_t - \frac{d_{t+1}^e}{R_{t+1}^e} \right),$$

where  $\lambda_t$  is a Lagrange multiplier. The first-order conditions for a maximum are

$$u'(c_t) = \lambda_t \quad \text{and} \quad \beta u'(d_{t+1}^e) = \frac{\lambda_t}{R_{t+1}^e}.$$

Eliminating  $\lambda_t$ , we obtain

$$u'(c_t) = \beta R_{t+1}^e u'(d_{t+1}^e), \quad (1.7)$$

which is the same as equation (1.6).

Note that the above inter-temporal problem is similar to a static problem where the individual chooses the consumption of two different contemporaneous goods. Here the two goods are distinguished by the date at which they are produced. The price of the good of the second period is the reciprocal of the rate of return,  $1/R_{t+1}^e$ .

### 1.3.2 The Inter-temporal Elasticity of Substitution

Equation (1.7) can be used to compute the change in the consumption plan in the face of a shift in the expected rate of return. Indeed, (1.7) can be rewritten

$$\frac{u'(c_t)}{u'(x_{t+1}c_t)} = \beta R_{t+1}^e, \quad (1.8)$$

where  $x_{t+1} = d_{t+1}^e/c_t$ . The inter-temporal elasticity of substitution measures the effect of a change in  $R_{t+1}^e$  on  $x_{t+1}$ . We differentiate with respect to  $R_{t+1}^e$  and  $x_{t+1}$ :

$$u'(c_t) \frac{-1}{[u'(x_{t+1}c_t)]^2} u''(x_{t+1}c_t) c_t dx_{t+1} = \beta dR_{t+1}^e.$$

Combining this expression with equation (1.8) yields

$$\frac{-1}{u'(x_{t+1}c_t)} u''(x_{t+1}c_t) c_t dx_{t+1} = \frac{dR_{t+1}^e}{R_{t+1}^e}.$$

Using the definition of  $x_{t+1}$  and rearranging, we have

$$\frac{dx_{t+1}}{x_{t+1}} = \frac{-u'(d_{t+1}^e)}{u''(d_{t+1}^e)d_{t+1}^e} \frac{dR_{t+1}^e}{R_{t+1}^e} = \sigma(d_{t+1}^e) \frac{dR_{t+1}^e}{R_{t+1}^e}.$$

Hence, the size of the effect of  $dR_{t+1}^e/R_{t+1}^e$  on  $dx_{t+1}/x_{t+1}$  is given by  $u'(d_{t+1}^e)/[-u''(d_{t+1}^e)d_{t+1}^e]$ .

The quantity

$$-\frac{u'(d)}{du''(d)} \equiv \sigma(d) > 0$$

is the reciprocal of the elasticity of marginal utility evaluated at  $d$  in absolute value. The effect of a change in the expected rate of return on consumption is captured by  $\sigma(d)$ . In the literature,  $\sigma(d)$  is referred as the inter-temporal elasticity of substitution.<sup>14</sup>  $\sigma(d)$  measures the percentage change in the ratio  $d_{t+1}/c_t$  associated with a one percent change in the rate of return. It measures the willingness of the consumer to shift consumption across time in response to changes in the expected rate of return.

**Example:** With the CES utility function

$$u(c) = \left(1 - \frac{1}{\sigma}\right)^{-1} c^{1-\frac{1}{\sigma}},$$

the optimality condition (1.7) is

$$\frac{d_{t+1}^e}{c_t} = (\beta R_{t+1}^e)^\sigma,$$

and the parameter  $\sigma$  is the inter-temporal elasticity of substitution, which is independent of  $d$ .

In the logarithmic case,  $\sigma = 1$ , and

$$\frac{d_{t+1}^e}{c_t} = \beta R_{t+1}^e.$$

### 1.3.3 The Properties of the Savings Function

The savings function

$$s(w, R) = \arg \max[u(w - s) + \beta u(Rs)]$$

will be central in the subsequent analysis. It is thus useful to analyze its properties. It is characterized by the marginal condition

$$\phi(s, w, R) \equiv -u'(w - s) + \beta R u'(Rs) = 0. \quad (1.9)$$

<sup>14</sup> In models with uncertainty this coefficient  $\sigma(d)$  has another interpretation. It is the reciprocal of the coefficient of relative risk aversion.

Following the assumptions **H1**, the function  $s(w, R)$  is defined and continuously differentiable on the set of pairs  $(w, R) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ , i.e. for  $w > 0$  and  $R > 0$ .

To compute its partial derivatives, we use the implicit function theorem (see appendix A.2.2) and differentiate  $\phi(s, w, R) = 0$ :

$$\phi'_s ds + \phi'_w dw + \phi'_R dR = 0,$$

in which

$$\phi'_s = u''(w - s) + \beta R^2 u''(Rs) < 0,$$

$$\phi'_w = -u''(w - s) > 0,$$

$$\phi'_R = \beta u'(Rs) + \beta R s u''(Rs) = \beta u'(Rs) \left(1 - \frac{1}{\sigma(Rs)}\right),$$

where

$$\sigma(Rs) = \frac{u'(Rs)}{-Rs u''(Rs)}$$

is the inter-temporal elasticity of substitution evaluated at  $d = Rs$ . The partial derivatives of  $s(w, R)$  with respect to  $w$  and  $R$  are

$$s'_w(w, R) = -\frac{\phi'_w}{\phi'_s} = \frac{1}{1 + \frac{\beta R^2 u''(Rs)}{u''(w-s)}},$$

$$s'_R(w, R) = -\frac{\phi'_R}{\phi'_s} = \frac{-\beta u'(Rs) \left(1 - \frac{1}{\sigma(Rs)}\right)}{u''(w-s) + \beta R^2 u''(Rs)}.$$

We thus have that the marginal propensity to save out of income is between 0 and 1:

$$0 < s'_w < 1,$$

which reflects the fact that consumption goods are normal goods. The effect of the rate of return on savings is ambiguous. We have that

$$s'_R \begin{cases} \leq 0 \\ \geq 0 \end{cases} \quad \text{if } \sigma(Rs) \begin{cases} \leq 1 \\ \geq 1 \end{cases}.$$

A rise in the return on savings has two effects for the consumer: (1) an income effect, as the revenue from savings will be higher, all other things being equal; (2) a substitution effect, making it profitable to substitute consumption today for consumption tomorrow. When the inter-temporal elasticity of substitution is lower than 1, the substitution effect is dominated by the income effect. In that case, a rise in the rate of return has a negative effect on savings. When the inter-temporal elasticity of substitution is higher than 1, the households are ready to exploit the rise in the remuneration of savings by consuming relatively less today. The effect of a rise in the rate of return is in this case to boost savings. When the inter-temporal elasticity of substitution is equal