1

Introduction

In this book, we present a **new, general, and complete theory of observability** and observation, deriving from our papers [18, 19, 32]. This theory is entirely **in the deterministic setting**. Let us mention here that there are several papers preceding these three that exploit the same basic ideas with weaker results. See [16, 17], in collaboration with H. Hammouri.

A list of all main notations is given in an index, page 221.

1. Systems under Consideration

We are concerned with general nonlinear systems of the form:

$$(\Sigma)\quad \begin{aligned} \frac{dx}{dt} &= f(x, u), \\ y &= h(x, u), \end{aligned} \tag{1}$$

typically denoted by Σ, where x, **the state**, belongs to X, an n-dimensional, connected, Hausdorff paracompact differentiable manifold, y, the **output**, takes values in R^{d_y}, and u, the **control variable**, takes values in $U \subset R^{d_u}$. For the sake of simplicity, we take $U = R^{d_u}$ or $U = I^{d_u}$, where $I \subset R$ is a closed interval. But typically U could be any closed submanifold of R^{d_u} with a boundary, a nonempty interior, and possibly with corners. Unless explicitly stated, X has no boundary.

The set of systems will be denoted by $S = F \times H$, where F is the set of u-parametrized vector fields f, and H is the set of functions h. In general, **except when explicitly stated**, f and h are C^∞. However, depending on the context, we will have to consider also analytic systems (C^ω), or C^r systems, for some $r \in \mathcal{N}$. Thus, if necessary, the required degree of differentiability will be stated, but in most cases the notations will remain S, F, H.

The simplest case is when U is empty, the so-called "uncontrolled case." In that situation, we will be able to prove more results than in the general case.

Usually, in practical situations, the output function h of the system does not depend on u. Unfortunately, from the theoretical point of view, this assumption is very awkward and leads to clumsy statements. For that reason, we will currently assume that h depends on the control u.

2. What Is Observability?

The preliminary definition we give here is the oldest one; it comes from the basic theory of linear control systems.

Roughly speaking, "observability" stands for the possibility of reconstructing the full trajectory from the observed data, that is, from the output trajectory in the uncontrolled case, or from the couple (output trajectory, control trajectory) in the controlled case. In other words, observability means that the mapping

$$initial - state \rightarrow output - trajectory$$

is injective, for all fixed control functions. More precise definitions will be given later in the book.

3. Summary of the Book

1. When the number d_y of observations is smaller than or equal to the number d_u of controls, then the relevant observability property is very rigid and is not stable under small perturbations, for germs of systems. Because of that rigidity, this observability property can be given a simple geometric characterization. This is the content of the paper [18] and the purpose of Chapter 3.

2. If, on the contrary, $d_y > d_u$, a remarkable phenomenon happens: The observability becomes generic, in a very strong sense, and for very general classes of control functions.

In Chapter 4, we state and prove a cornucopia of genericity results about observability as we define it. The most important of these results are contained in paper [19]. Some of these results present real technical difficulties.

3. The singular case: in the preceding two cases, the $initial - state \rightarrow output - trajectory$ mapping is regular. What happens if it becomes singular? This problem is too complex. In classical singularity theory, there is a useful and manageable concept of mapping with singularities: that of a "finite mapping." It is interesting that, in the uncontrolled analytic case, this concept can be extended to our $initial - state \rightarrow output - trajectory$ mappings, according to a very original idea of P. Jouan. This idea leads to the very interesting results of paper [32]. The controlled case is very different: If the system is singular, then it is not controllable. In this case, we also have several results,

giving a complete solution of the observation problem. These developments form the content of Chapter 5.

4. Observers: An observer is a device that performs the practical task of state reconstruction. In all cases mentioned above, (1, 2, 3), an asymptotic observer can be constructed explicitly, under the guise of a differential equation that estimates the state of the system asymptotically. The estimation error has an arbitrarily large exponential decay. This is the so-called "high gain construction."

This construction is an adaptation to the nonlinear case of the "Luenberger," or of the "extended Kalman filter" method. The last one performs very well in practice. We will present these topics in Chapter 6.

5. Output stabilization: this study can be applied to output stabilization in the preceding cases 1, 2, and 3 above. One of our main results in [32] states that one can stabilize asymptotically a system via an asymptotic observer, using the output observations only, if one can stabilize it asymptotically using smooth state feedback.

This result is "semi-global": One can do this on arbitrarily large compacta. Let us note that, in cases 1 and 2, the $initial - state \rightarrow output - trajectory$ mapping is always immersive. In that case, the stabilizing feedback can be arbitrary. But, if the $initial - state \rightarrow output - trajectory$ mapping is not immersive, then it has to belong to a certain special ring of functions. These results are developed in Chapter 7.

6. In the last chapter, Chapter 8, we give a summary description of two applications in the area of chemical engineering. These represent the fallout from our long cooperation with the Shell company.

The first one, about distillation columns, is of practical interest because distillation columns really are **generic** objects in the petroleum and chemical industries. This application is a perfect illustration of the methods we are proposing for the problems of both observation and dynamic output stabilization.

The second application deals with polymerization reactors, and it constitutes also a very interesting and pertinent illustration. Both applications are the subjects of the articles [64, 65].

The classical notions of observability are inadequate for our purposes. For reasons discussed in the next section, Chapter 2 is devoted to the introduction of new concepts of observability. We hope that our book will vindicate our iconoclastic gesture of discarding the old observability concepts.

4. The New Observability Theory Versus the Old Ones

As we have said, observability is the injectivity of the mapping: $initial - state \rightarrow output - trajectory$. However, the concept of injectivity per se is very hard to handle mathematically because it is unstable. Hence, we have to introduce stronger concepts of observability, for example adding to the injectivity the condition of immersivity (infinitesimal injectivity), as in the classical theory of differentiable mappings.

In this book, we haven't discussed any of the other approaches to observability that have been proposed elsewhere, and we haven't referenced any of them. The reason for this is simple: We have no use for either the concepts nor the results of these other approaches.

In fact, we claim that our approach to observability theory, which is entirely new, is far superior to any of the approaches proposed so far.

Since we cannot discuss all of them, let us focus on the most popular: **the output injection method**.

The **output injection method** is in the spirit of the **feedback linearization** method (popular for the control of nonlinear systems). As for the feedback linearization, one tries to go back to the well-established theory of linear systems. First, one characterizes the systems that can be written as a linear system, plus a perturbation depending on the outputs only (in some coordinates). Second, for these systems only, one applies slight variations of the standard linear constructions of observer systems. This approach suffers from terminal defects.

A. It applies to an extremely small class of systems only. In precise mathematical terms, it means the following. In situation 2 above, where observability is generic, it applies to a class of systems of infinite codimension. In case 1, where the observability is nongeneric, it also applies to an infinite codimension subset of the set of observable systems.
B. Basically, the approach ignores the crucial distinction between the two cases: 1. $d_y \leq d_u$, 2. $d_y > d_u$.
C. The approach does not take into account generic singularities, and it is essentially local in scope.

Of course, these defects have important practical consequences in terms of sensitivity. In particular, in case 2, where the observability property is stable, the method is unstable.

5. A Word about Prerequisites

In this book, we have tried to keep the mathematical prerequisites to a strict minimum. What we need are the following mathematical tools: transversality theory, stratification theory and subanalytic sets, a few facts from several complex variables theory, center manifold theory, and Lyapunov's direct and inverse theorems.

For the benefit of the reader, a summary of the results needed is provided in the Appendix. It is accessible to those with only a modest mathematical background.

6. Comments

6.1. Comment about the Dynamic Output Stabilization Problem

At several places in the book, we make the assumption that the state space X, is just the Euclidean space R^n. If one wants only to estimate the state, this is not a reasonable assumption: the state space can be anything. However, for the dynamic output stabilization of systems that are state-feedback stabilizable, it is a reasonable assumption because the basin of attraction of an asymptotically stable equilibrium point of a vector field is diffeomorphic to R^n (see [51]).

6.2. Historical Comments

6.2.1. About "Observability"

The observability notion was introduced first in the context of linear systems theory. In this context, the **Luenberger observer**, and the **Kalman filter** were introduced, in the deterministic and stochastic settings, respectively.

For linear systems, **the observability notion is independent of the control function** (either the $initial - state \rightarrow output - trajectory$ mapping is injective for all control functions, or it is not injective for each control function). **This is no longer true for nonlinear systems.** Moreover, as we show in this book, in the general case where $d_y \leq d_u$, observability (for all inputs) is not at all a generic property. For these reasons (and certainly also just for tractability), several weaker different notions of observability have been introduced, which are generic and which agree with the old observability notion in the special case of linear systems. In this setting, there is the pioneer work [24]. As we said, these notions are totally inadequate for our purposes, and we just forget about them.

6.2.2. About Universal Inputs

Let us say that a control function **separates** two states, if the corresponding output trajectories, from these two initial states, do not coincide.

For a nonlinear system, a **universal input** is a control function that separates all the couples of states that can be separated by some control.

We want to mention a pioneer work by H. J. Sussmann [47], in which it is proved, roughly speaking, that "universal inputs do exist." For this purpose, the author made use of the properties of subanalytic sets, in a spirit very similar to the one in this book.

6.2.3. *About the Applications*

In Chapter 8, we present two applications from chemical engineering science. There are already several other applications of our theory in many fields, but we had to choose.

The two applications we have chosen look rather convincing, because they are not "academic," and some refinements of the theory are really used. Moreover, these two applications, besides their illustrative character, are very important in practice and have been addressed by research workers in control theory, using other techniques, for many years. It is hard to give an exhaustive list of other studies (related to control and observation theory) on distillation columns and polymerization reactors. However, let us give a few references that are significant:

For distillation columns: [58], [61], [62].
For polymerization reactors: [56], [57], [60], [59].

Regarding distillation columns, it would be very interesting (and probably very difficult) to study the case of **azeotropic distillations**, which is not addressed in this volume. It seems that all the theory collapses in this case of azeotropic distillations.

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Excerpt
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Part I

Observability and Observers

2

Observability Concepts

In this chapter, we will state and explain the various definitions of observability that will be used in this book (see Section 4 in Chapter 1).

1. Infinitesimal and Uniform Infinitesimal Observability

The space of control functions under consideration will just be the space $L^\infty[U]$ of all measurable bounded, U-valued functions $u : [0, T_u[\rightarrow U$, defined on semi-open intervals $[0, T_u[$ depending on u. The space of our output functions will be the space $L[R^{d_y}]$ of all measurable functions $y : [0, T_y[\rightarrow R^{d_y}$, defined on the semi-open intervals $[0, T_y[$. Usually, input and output functions are defined on closed intervals. However, this is irrelevant. The following considerations led us to work with semi-open intervals. For any input $\hat{u} \in L^\infty[U]$ and any initial state x_0, the maximal solution of the Cauchy problem for positive times

$$\frac{d\hat{x}}{dt} = f(\hat{x}(t), \hat{u}(t)), \hat{x}(0) = x_0$$

is defined on a semi-open interval $[0, e(\hat{u}, x_0)[$, where $0 < e(\hat{u}, x_0) \leq T_{\hat{u}}$. If $e(\hat{u}, x_0) < T_{\hat{u}}$, then, $e(\hat{u}, x_0)$ is the positive escape time of x_0 for the time dependent vector field $f(., \hat{u}(t))$. It is well known that, for all $\hat{u} \in L^\infty[U]$, the function $x_0 \rightarrow e(\hat{u}, x_0) \in \bar{R}^*_+$ is lower semi-continuous ($\bar{R}^*_+ = \{a | 0 < a \leq \infty\}$).

Definition 1.1. The input-output mapping P of Σ is defined as follows:

$$P : L^\infty[U] \times X \rightarrow L[R^{d_y}], (\hat{u}, x_0) \rightarrow P(\hat{u}, x_0),$$

where $P(\hat{u}, x_0)$ is the function $\hat{y} : [0, e(\hat{u}, x_0)[\rightarrow R^{d_y}$ defined by

$$\hat{y}(t) = h(\hat{x}(t), \hat{u}(t)).$$

The mapping $P_{\hat{u}} : X \rightarrow L[R^{d_y}]$ is $P_{\hat{u}}(x_0) = P(\hat{u}, x_0)$.

Definition 1.2.[1] A system is called **observable** if for any triple $(\hat{u}, x_1, x_2) \in L^\infty[U] \times X \times X$, $x_1 \neq x_2$, the set of all $t \in [0, \min(e(\hat{u}, x_1), e(\hat{u}, x_2))[$ such that $P(\hat{u}, x_1)(t) \neq P(\hat{u}, x_2)(t)$ has positive measure.

Now, we define the "**first variation**" of Σ, or the "**lift of Σ on *TX*.**" The mapping $f : X \times U \to TX$ induces the partial tangent mapping $T_X f : TX \times U \to TTX$ (tangent bundle of TX). Then, if ω denotes the canonical involution of TTX (see [1]), $\omega \circ T_X f$ defines a parametrized vector field on TX, also denoted by $T_X f$. Similarly, the function $h : X \times U \to R^{d_y}$ has a differential $d_X h : TX \times U \to R^{d_y}$. The first variation of Σ is the input–output system:

$$(T\Sigma) \begin{cases} \frac{d\xi}{dt} = T_X f(\xi, u) = T_X f_u(\xi), \\ \eta = d_X h(\xi, u) = d_X h_u(\xi). \end{cases} \quad (2)$$

Its input–output mapping is denoted by dP, and the trajectories of (1) and (2) are related as follows:

If ξ: $[0, T_\xi[\to TX$ is a trajectory of (2) associated with the input $\hat{u}$, the projection $\pi(\xi) : [0, T_\xi[\to X$ is a trajectory of Σ associated with the same input. Conversely, if $\varphi_t(x_0, \hat{u}) : [0, e(\hat{u}, x_0)[\to X$ is the trajectory of Σ starting from x_0 for the input $\hat{u}$, the map $x \to \varphi_\tau(x, \hat{u})$ is a diffeomorphism from a neighborhood of x_0 onto its image, for all $\tau \in [0, e(\hat{u}, x_0)[$. Let $T_X\varphi_\tau : T_{x_0}X \to T_z X$, $z = \varphi_\tau(x_0, \hat{u})$ be its tangent mapping. Then, for all $\xi_0 \in T_{x_0}X$:

$$e_{T\Sigma}(\hat{u}, \xi_0) = e_\Sigma(\hat{u}, \pi(\xi_0)) = e_\Sigma(\hat{u}, x_0),$$

and, for almost all $\tau \in [0, e(\hat{u}, x_0)[$:

$$dP(\hat{u}, \xi_0)(\tau) = d_X h(T_X\varphi_\tau(\hat{u}, \xi_0), \hat{u}(\tau)) = d_X\left(P^\tau_{\Sigma,\hat{u}}\right)(\xi_0). \quad (3)$$

The right-hand side of these equalities (3) is the differential of the function $P^\tau_{\Sigma,\hat{u}} : V \to R^{d_y}$, where V is the open set:

$$V = \{x \in X | 0 < \tau < e(\hat{u}, x)\}, \text{ and } P^\tau_{\Sigma,\hat{u}}(x) = P(\hat{u}, x)(\tau).$$

For any $a > 0$, let $L^\infty_{loc}([0, a[; R^{d_y})$ denote the space of measurable functions $v : [0, a[\to R^{d_y}$ which are locally in L^∞. For all $\hat{u} \in L^\infty(U)$, $x_0 \in X$, the restriction of dP to $\{\hat{u}\} \times T_{x_0}X$ defines a linear mapping:

$$\begin{aligned} dP_{\hat{u},x_0} &: T_{x_0}X \to L^\infty_{loc}([0, e(\hat{u}, x_0)[; R^{d_y}), \\ dP_{\hat{u},x_0}&(\xi_0)(t) = dP(\hat{u}, \xi_0)(t). \end{aligned} \quad (4)$$

[1] In nonlinear control theory, the notion of observability defined here is usually referred to as "uniform observability." Let us stress that it is just the old basic observability notion used for linear systems.

Definition 1.3. The system Σ is called infinitesimally observable at $(\hat{u}, x) \in L^\infty[U] \times X$ if the linear mapping $dP_{\hat{u},x}$ is injective. It is called infinitesimally observable at $\hat{u} \in L^\infty[U]$ if it is infinitesimally observable at all pairs $(\hat{u}, x)$, $x \in X$, and called **uniformly infinitesimally observable** if it is infinitesimally observable at all $\hat{u} \in L^\infty[U]$.

Remark 1.1. In view of the relation 3 above, the fact that the system is infinitesimally observable at $\hat{u} \in L^\infty[U]$ means that the mapping $P_{\hat{u}} : X \to L[R^{d_y}]$ is an immersion of X into $L[R^{d_y}]$ (as was stated, $P_{\hat{u}}$ is differentiable in the following sense: we know that $e(\hat{u}, x) \geq e(\hat{u}, x_0) - \varepsilon$ in a neighborhood U_ε of x_0. Then $P_{\hat{u}}$ is differentiable in the classical sense from U_ε into $L^\infty([0, e(\hat{u}, x_0) - \varepsilon]; R^{d_y})$. $P_{\hat{u}}$ is an immersion in the sense that these differential maps are injective).

This notion of uniform infinitesimal observability is the one which makes sense in practice, when $d_y \leq d_u$. In most of the examples from real life we know of, when $d_y \leq d_u$, the system is uniformly infinitesimally observable.

A very frequent situation in practice is the following: The physical state space for x is an open subset $\check{X}$ of X, and $\check{X}$ is positively invariant under the dynamics of Σ. The trajectories $\hat{x}(t)$ that are unobservable take their values in the boundary $\partial \check{X}$, and the corresponding controls $\hat{u}(t)$ take their values in ∂U. In particular, this will be the case for the first example we show in Chapter 8.

2. The Canonical Flag of Distributions

In this section, we assume that $d_y = 1$. As above, set: $h_u(x) = h(x, u)$, $f_u(x) = f(x, u)$. Associated with the system Σ, there is a family of flags $\{D_0(u) \supset D_1(u) \supset \ldots \supset D_{n-1}(u)\}$ of distributions on X (parametrized by **the value** $u \in U$ of the control). $D_0(u) = \mathrm{Ker}(d_X h_u(x))$, where d_X denotes again the differential with respect to the x variable only. For $0 \leq k < n - 1$:

$$D_{k+1}(u) = D_k(u) \cap Ker\Big(d_X\Big(L_{f_u}^{k+1}(h_u)\Big)\Big),$$

where L_{f_u} is the Lie derivative operator on X, w.r.t. the vector field f_u. Let us set:

$$D(u) = \{D_0(u) \supset D_1(u) \supset \ldots \supset D_{n-1}(u)\}. \tag{5}$$

This u-dependent flag of distributions is not **regular** in general (i.e., $D_i(u)$ does not have the constant rank $n - i - 1$).