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Inverse scattering technique in gravity

The purpose of this chapter is to describe the Inverse Scattering Method (ISM) for the gravitational field. We begin in section 1.1 with a brief overview of the ISM in nonlinear physics. In a nutshell the procedure involves two main steps. The first step consists of finding for a given nonlinear equation a set of linear differential equations (spectral equations) whose integrability conditions are just the nonlinear equation to be solved. The second step consists of finding the class of solutions known as soliton solutions. It turns out that given a particular solution of the nonlinear equation new *soliton* solutions can be generated by purely algebraic operations, after an integration of the linear differential equations for the particular solution. We consider in particular some of the best known equations that admit the ISM such as the Korteweg–de Vries and the sine-Gordon equations. In section 1.2 we write Einstein equations in vacuum for spacetimes that admit an orthogonally transitive two-parameter group of isometries in a convenient way. In section 1.3 we introduce a linear system of equations for which the Einstein equations are the integrability conditions and formulate the ISM in this case. In section 1.4 we explicitly construct the so-called n-soliton solution from a certain background or seed solution by a procedure which involves one integration and a purely algebraic algorithm which involves the so-called pole trajectories. In the last section we discuss the use of the ISM for solving Einstein equations in vacuum with an arbitrary number of dimensions, and the use of the Kaluza–Klein ansatz to find some nonvacuum soliton solutions in four dimensions.

1.1 Outline of the ISM

The ISM is an important tool of mathematical physics by means of which it is possible to solve a certain type of nonlinear partial differential equations using the techniques of linear physics. This book is not about the ISM, its main concern are the so-called soliton solutions, and these only in the context of general relativity. But since such solutions can be obtained by the ISM, it is of course of interest to have some familiarity with the method. However, mastering the ISM is by no means essential for reading this book because, first to find soliton solutions one does not require the full machinery of the ISM, and second the peculiarities of the gravitational case require specific techniques that will be explained in detail in the following sections.

In subsection 1.1.1 we give a brief summary of the ISM including relevant references to the literature. Terms such as Schrödinger equation, scattering data, and transmission and reflection coefficients are borrowed from quantum mechanics, thus readers familiar with that subject may gain some insight from this subsection. Some readers may prefer to have only a quick glance at subsection 1.1.1 and to look in more detail at subsection 1.1.2 where some familiar examples of fluid dynamics and of relativistic physics are discussed. Of particular interest for the purposes of this book is the last example discussed and the method of how to construct solitonic solutions by purely algebraic operations from a given particular solution.

In any case, the key points that should be retained from subsection 1.1.1 are the following. A nonlinear partial differential equation such as (1.1) for the function $u(z,t)$ is integrable by the ISM when the following occur. First, one must be able to associate to the nonlinear equation a linear eigenvalue problem such as (1.2), where the unknown function $u(z,t)$ plays the role of a 'potential' in the linear operator. Given an initial value $u(z,0)$, (1.2) defines scattering data: this is the well known problem in quantum mechanics of scattering of a particle in a potential $u(z,0)$ and includes the transmission and reflection coefficients and the energy eigenvalues. Second, it must be possible to provide an equation such as (1.3) for the time evolution of these data, such that the integrability conditions of the two equations (1.2) and (1.3) implies (1.1). In this case the nonlinear equation is integrable by the ISM and the solution $u(z,t)$ is found by computing the potential corresponding to the time-evolved scattering data. This last step is the inverse scattering problem and requires the solution of a usually nontrivial linear integral equation. Although the whole procedure is generally complicated there is a special class of solutions called *soliton solutions* for which the inverse scattering problem can be solved exactly in analytic form.

1.1.1 The method

Let us consider the nonlinear two-dimensional partial differential equation for the function $u(z,t)$

$$u_{,t} = F(u, u_{,z}, u_{,zz}, \ldots), \tag{1.1}$$

where t is the time variable, z is a space coordinate, and F is a nonlinear function. To integrate this equation, which is first order with respect to time, by the ISM one considers the scattering problem for the following stationary one-dimensional Schrödinger equation,

$$L\psi = \lambda\psi, \quad L = -\frac{d^2}{dz^2} + u(z,t), \tag{1.2}$$

where the unknown function $u(z,t)$ plays the role of the potential. Here the time t in u is an external parameter that should not be confused with the conventional time in quantum mechanics, which appears in the time-dependent Schrödinger equation associated to (1.2). We assume also that $u(t,z)$ vanishes at $z \to \pm\infty$ fast enough (like $z^2 u \to 0$ or faster).

Let $u(z,0)$ be the Cauchy data at time $t = 0$ and consider the so-called direct scattering problem, which consists of finding the full set of scattering data $S(\lambda, 0)$ produced by the potential $u(z,0)$. The scattering data $S(\lambda, 0)$ are the set of quantities that allow us to find the asymptotic values of the eigenfunctions $\psi(\lambda, z, 0)$ at $z \to -\infty$ through the given asymptotic values of $\psi(\lambda, z, 0)$ at $z \to +\infty$ for each value of the spectral parameter λ. This parameter is the energy of the scattered particle and positive values are the continuous spectrum for the problem (1.2). Moreover, a discrete set of negative eigenvalues of λ can also enter into the problem corresponding to the bound states of the particle in the potential u. Thus, the set $S(\lambda, 0)$ should contain the forward and backward scattering amplitudes for the continuous spectrum (in the one-dimensional problem these are the transmission and reflection coefficients, $T(\lambda)$ and $R(\lambda)$, respectively), and the negative eigenvalues λ_n of the discrete spectrum together with some coefficients, C_n, which link the asymptotic values of the eigenfunctions for the bound states $\psi_n(\lambda_n, z, 0)$ at $z \to \pm\infty$.

We can also consider the inverse of the problem just described. The task in this case is to reconstruct the potential $u(z)$ through a given set of scattering data $S(\lambda)$. This is the inverse scattering problem. It has been investigated in detail in the last forty years and the main steps of its solution are now well known. In principle, for any appropriate set of scattering data $S(\lambda)$ it is possible to reconstruct the corresponding potential $u(z)$. It is easy to see that one could solve the Cauchy problem for $u(z,t)$ using this technique. In fact, let us imagine that after constructing the scattering data $S(\lambda, 0)$ corresponding to the potential $u(z,0)$ at $t = 0$ we could know the time evolution of S and are able to get from the initial values $S(\lambda, 0)$ the scattering data $S(\lambda, t)$ at any arbitrary time t. Then we can apply the inverse scattering technique to $S(\lambda, t)$ and reconstruct the potential $u(z,t)$ at any time. This would give the desired solution to the Cauchy problem.

This programme, however, is only attractive if such a 'miracle' can happen which means, for practical purposes, that we need some evolution equations for the scattering data $S(\lambda, t)$ that can be integrated in a simple way. It turns out that for a number of special classes of differential equations of nonlinear physics this is the case. This discovery was made by Gardner, Greene, Kruskal and Miura in 1967 in a famous paper [111] dedicated to the method of solving the Cauchy data problem for the Korteweg–de Vries equation. This was the beginning of a rapid development of the ISM and now we have a vast literature on the subject. One of the more recent books is ref. [231], and readers can also find textbook expositions, including historical reviews, in refs [84, 302]. The review article [259] and the book of collected papers book [247], which includes a good introductory guide through the literature, are also very useful.

Now let us look closer at the remarkable possibility of finding the exact time evolution for the scattering data. The fact is that for integrable cases (in the sense of the ISM) the eigenvalues of the associated spectral problem (1.2) are independent of time t and the eigenfunctions $\psi(\lambda, z, t)$ obey, besides (1.2), another partial differential equation which is of first order in time. This is the key point, since this additional evolution equation for the eigenfunctions allows us to find the exact time dependence of the scattering data. This equation can be written as

$$\dot{\psi} = A\psi, \tag{1.3}$$

where the differential operator A depends on $u(z, t)$ and contains only derivatives with respect to the space coordinate z. This remarkable set of equations, namely, (1.2) and (1.3), is often called a *Lax pair*, or *Lax representation* of the integrable system, or *L–A pair* [186]. The existence of two equations for the eigenfunction ψ means that a selfconsistency condition must be satisfied. In each case it is easy to show that this condition coincides exactly with the original equation of interest, (1.1). Consequently, the problem can now be put into a slightly different form: all integrable nonlinear two-dimensional equations are the selfconsistency conditions for the existence of a joint spectrum and a joint set of eigenfunctions for two differential operators whose coefficients (which play the role of potentials) depend on $u(z, t)$ and, in general, on its derivatives. This was the basic point for a further generalization of the ISM to multicomponent fields $u(z, t)$ and to several families of differential operators. This work was largely due to Zakharov and Shabat (see ref. [231], chapter 3, and ref. [84], chapter 6, and references therein). Of course, only very special classes of nonlinear differential equations admit L–A pairs and still today there is no general approach on how to find these classes. Despite the existence of a number of powerful techniques each differential system needs individual and, often, sophisticated consideration.

Let us return to our problem (1.1). From what we have just said we know that this equation is integrable by the ISM if the time evolution of the scattering data can be found. However, it is important to understand the restricted sense of this integrability. In order to perform an actual integration we need to be able to solve the inverse scattering problem for the data $S(\lambda, t)$. In general this cannot be done in analytic form, because the inverse problem $S(\lambda, t) \rightarrow u(z, t)$ is based on complicated integral equations of the Gelfand, Levitan and Marchenko [231]. Also there is no possibility, in general, for analytic solutions of the direct scattering problem $u(z, 0) \to S(\lambda, 0)$. What can really be done in general is to find the explicit expression for the asymptotic values of the field $u(z, t)$ at $t \to +\infty$ directly through the initial Cauchy data. Of course, the possibility of even this restricted use of the ISM is very valuable because in many physical problems all we need to know is the late time asymptotic values of the field.

Soliton solutions. Another great advantage of the ISM is really remarkable: for each integrable equation (1.1) (or system of equations) there are special classes of solutions $u(z, t)$ for which the direct and inverse scattering problems can be solved exactly in analytic form! These are the so-called *soliton solutions*. We mentioned before that for the continuous spectrum of positive λs the scattering data consist of the backward and forward scattering amplitudes or the reflection and the transmission coefficients, $R(\lambda)$ and $T(\lambda)$ respectively. The reflection coefficient is identically zero for solitons, and this property is independent of time. It can be shown that if for some initial potential $u(z, 0)$ all the coefficients $R(\lambda, 0)$ vanish, then they will vanish at any time t due to the evolution equations of the scattering data. The solutions $u(z, t)$ of that kind are often called 'reflectionless potentials'. In such cases the values λ_n of the discrete spectrum and the coefficients $C_n(\lambda_n, t)$, the time evolution of which can be also easily found, determine all the structure of the ISM. It is well known that the values λ_n coincide with the simple poles of the transmission amplitude $T(\lambda)$, and the positions of these poles completely determine the analytical structure of the scattering data and the eigenfunctions of the spectral problem (1.2) in the complex λ-plane. The transmission amplitude and the behaviour of the eigenfunctions of (1.2) and (1.3) as functions of the spectral parameter λ in the complex λ-plane are completely determined by this simple pole structure. In this case even a first look at the equation of the ISM suffices to see that the main steps of the ISM for the solitonic case are purely algebraic. This is integrability in its simplest direct sense.

1.1.2 Generalization and examples

Although we have discussed the idea of the ISM with the example of the first-order differential equation with respect to time for a single function $u(z, t)$, the qualitative character of our previous statements also remains valid in any extended integrable case. The generalization to second order equations and to multicomponent fields $u(z, t)$ is straightforward. In these cases instead of (1.2) and (1.3) we have two systems of equations and the multicomponent analogue of the spectral problem (1.2) presents no difficulties [231]. For such extended versions of the ISM we need only a change in the terminology. The generalized version of (1.2) is no longer a Schrödinger equation, but some Schrödinger-type system, and the same for the inverse scattering transformation of Gelfand, Levitan and Marchenko. In addition the parameter λ can no longer be the energy but is instead some spectral parameter, etc.

Further development of the ISM [312] showed that most of the known two-dimensional equations and their possible integrable generalizations can be represented as selfconsistency conditions for two matrix equations,

$$\psi_{,z} = U^{(1)}\psi, \quad \psi_{,t} = V^{(1)}\psi, \tag{1.4}$$

where the matrices $U^{(1)}$ and $V^{(1)}$ depend rationally on the complex spectral parameter λ and on two real spacetime coordinates z and t. The column matrix ψ is a function of these three independent variables also. Differentiating the first of these two equations with respect to t and the second one with respect to z we obtain, after equating the results, the consistency condition for system (1.4):

$$U^{(1)}_{,t} - V^{(1)}_{,z} + U^{(1)}V^{(1)} - V^{(1)}U^{(1)} = 0. \tag{1.5}$$

This condition should be satisfied for all values of λ and this requirement coincides explicitly with the integrable differential equation (or system) of interest. Let us see a few examples [231], which will be of special interest.

Korteweg–de Vries equation. If we choose

$$U^{(1)} = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \tag{1.6}$$

$$V^{(1)} = 4i\lambda^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 4\lambda^2 \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} + 2i\lambda \begin{pmatrix} u & 0 \\ u_{,z} & -u \end{pmatrix} + \begin{pmatrix} -u_{,z} & 2u \\ 2u^2 - u_{,zz} & u_{,z} \end{pmatrix}, \tag{1.7}$$

then the left hand side of (1.5) becomes a fourth order polynomial in λ. All the coefficients of this polynomial, except one, vanish identically and we get from (1.5):

$$\begin{pmatrix} 0 & 0 \\ u_{,t} - 6uu_{,z} - u_{,zzz} & 0 \end{pmatrix} = 0, \tag{1.8}$$

which is the Korteweg–de Vries equation:

$$u_{,t} - 6uu_{,z} - u_{,zzz} = 0, \tag{1.9}$$

an equation of the form of (1.1). The function ψ in this case is the column

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.10}$$

and from the first equation of (1.4) we have the following spectral problem:

$$\psi_{1,z} = i\lambda\psi_1 + \psi_2, \tag{1.11}$$

$$\psi_{2,z} = -i\lambda\psi_2 + u\psi_1, \tag{1.12}$$

which is equivalent to the Schrödinger equation (1.2). In fact, from the first equation (1.11) we can express ψ_2 in terms of ψ_1, and then substituting into the second, we get

$$-\psi_{1,zz} + u\psi_1 = \lambda^2\psi_1, \tag{1.13}$$

which coincides with (1.2) after a redefinition of the spectral parameter ($\lambda^2 \to \lambda$).

A second example appears when one is dealing with relativistic invariant second order field equations. From the mathematical point of view the physical nature of the variables z and t in (1.4) is irrelevant and we can interpret them as null (light-like) coordinates. But in order to avoid notational confusion, here and in the following, the variables t and z are always, respectively, time-like and space-like coordinates, and we introduce a pair of null coordinates ζ and η as

$$\zeta = \frac{1}{2}(z+t), \quad \eta = \frac{1}{2}(z-t). \tag{1.14}$$

Now, instead of (1.4) and (1.5) we have, in these new coordinates,

$$\psi_{,\zeta} = U^{(2)}\psi, \quad \psi_{,\eta} = V^{(2)}\psi, \tag{1.15}$$

$$U^{(2)}_{,\eta} - V^{(2)}_{,\zeta} + U^{(2)}V^{(2)} - V^{(2)}U^{(2)} = 0, \tag{1.16}$$

where $U^{(2)} = U^{(1)} + V^{(1)}$ and $V^{(2)} = U^{(1)} - V^{(1)}$.

Sine-Gordon equation. If we choose

$$\left.\begin{aligned} U^{(2)} &= i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & u_{,\zeta} \\ u_{,\zeta} & 0 \end{pmatrix}, \\ V^{(2)} &= \frac{1}{4i\lambda} \begin{pmatrix} \cos u & -i\sin u \\ i\sin u & -\cos u \end{pmatrix}, \end{aligned}\right\} \tag{1.17}$$

we get, from (1.16),

$$\begin{pmatrix} 0 & u_{,\zeta\eta} - \sin u \\ u_{,\zeta\eta} - \sin u & 0 \end{pmatrix} = 0, \tag{1.18}$$

which is the sine-Gordon equation:

$$u_{,\zeta\eta} = \sin u. \tag{1.19}$$

The function ψ is still the column (1.10) and the spectral problem that follows from the first of equations (1.15) gives

$$\psi_{1,\zeta} = i\lambda\psi_1 + \frac{i}{2}u_{,\zeta}\psi_2, \tag{1.20}$$

$$\psi_{2,\zeta} = -i\lambda\psi_2 + \frac{i}{2}u_{,\zeta}\psi_1. \tag{1.21}$$

After solving the direct scattering problem for this 'stationary' system it is easy to find the evolution of scattering data in the 'time' η. The inverse scattering transform then gives the solution for $u(\zeta, \eta)$ (see the details in ref. [231]).

In general the matrices U and V can have an arbitrary size $N \times N$ (the same follows for the column matrix ψ) as well as a more complicated dependence on the parameter λ. Each choice will give some complicated (in general) integrable system of differential equations. Most of them do not yet have a physical interpretation but a number of interesting possibilities arise.

Principal chiral field equation. Let us consider, first of all, the case when U and V are regular at infinity in the λ-plane and have simple poles only at finite values of the spectral parameter (we should not confuse these poles with the poles of the scattering data in the same plane). As was shown in ref. [312] in this case we can construct matrices U and V which vanish at $|\lambda| \to \infty$, due to the gauge freedom in the system (1.15)–(1.16). We shall restrict ourselves to the simplest case in which U and V have only one pole each. Without loss of generality we can choose the positions of these poles to be at $\lambda = \lambda_0$ and $\lambda = -\lambda_0$, where λ_0 is an arbitrary constant. Now for $U^{(2)}$ and $V^{(2)}$ we have

$$U^{(2)} = \frac{K}{\lambda - \lambda_0}, \qquad V^{(2)} = \frac{L}{\lambda + \lambda_0}, \tag{1.22}$$

where the matrices K and L are independent of λ. Substitution of (1.22) into (1.16) shows that the left hand side of (1.16) vanishes if and only if the following relations hold:

$$K_{,\eta} - L_{,\zeta} = 0, \tag{1.23}$$

$$K_{,\eta} + L_{,\zeta} + \frac{1}{\lambda_0}(KL - LK) = 0. \tag{1.24}$$

Equation (1.24) suggests that we can represent K and L in terms of 'logarithmic derivatives' of some matrix g as

$$K = -\lambda_0 g_{,\zeta} g^{-1}, \qquad L = \lambda_0 g_{,\eta} g^{-1}. \tag{1.25}$$

Then, (1.24) is simply the integrability condition of (1.25) for the matrix g, and (1.23) is the field equation for some integrable relativistic invariant model:

$$\left(g_{,\zeta} g^{-1}\right)_{,\eta} + \left(g_{,\eta} g^{-1}\right)_{,\zeta} = 0. \tag{1.26}$$

This matrix equation is associated with the model of the so-called principal chiral field and received much attention in the 1980s and 1990s. The first description of the integrability of this model in the language of the commutative representation (1.16) was given in ref. [312], but a more detailed description can be found in ref. [311] or in ref. [231]. The exact solution of the corresponding quantum chiral field model was investigated in refs [244] and [95].

From any solution $\psi(\zeta, \eta, \lambda)$ of the 'L–A pair' (1.15) one immediately gets a solution of the field equation (1.26) for g. In fact, from (1.15), (1.22) and (1.25) it follows that

$$\psi_{,\zeta}\psi^{-1} = \frac{K}{\lambda - \lambda_0} = \frac{-\lambda_0 g_{,\zeta} g^{-1}}{\lambda - \lambda_0} \longrightarrow g_{,\zeta} g^{-1}, \tag{1.27}$$

$$\psi_{,\eta}\psi^{-1} = \frac{L}{\lambda + \lambda_0} = \frac{\lambda_0 g_{,\eta} g^{-1}}{\lambda + \lambda_0} \longrightarrow g_{,\eta} g^{-1}, \tag{1.28}$$

when $\lambda \to 0$, which means that the matrix of interest equals the matrix eigenfunction $\psi(\zeta, \eta, \lambda)$ at the point $\lambda = 0$,

$$g(\zeta, \eta) = \psi(\zeta, \eta, 0). \tag{1.29}$$

The solution of the general Cauchy problem for (1.26) can be obtained in the framework of the classical ISM in the form we have explained. We can also use a more elegant and modern method, based on the Riemann problem in the theory of functions of complex variables, which was proposed by Zakharov and Shabat [231, 312]. Of course any method will lead us to integral equations of the Gelfand, Levitan and Marchenko type and the Zakharov and Shabat method is no exception. But what is important for us here is that the previous approach is the best suited for practical calculations in the solitonic case. In this book we will deal only with solitons and we will follow the commutative representation (1.15) and (1.16) of the ISM.

If we are interested only in the solitonic solutions of (1.26) we do not need to study the Riemann problem, the spectrum and the direct and inverse scattering transforms. All we need to know is one particular exact solution (g_0, ψ_0) of (1.26) and (1.15), which we will call the background solution or the seed solution, together with the number of solitons we wish to introduce on this background. We know already that in the solitonic case the poles of the transmission amplitude completely determine the problem. Since the transmission amplitude is just a part of the eigenfunction $\psi(\zeta, \eta, \lambda)$, such a function exhibits the same simple pole structure in some arbitrarily large, but finite, part of the λ-plane. Simple inspection shows that in this case $\psi(\zeta, \eta, \lambda)$ can be represented in the form

$$\psi = \chi \psi_0, \tag{1.30}$$

where $\psi_0(\zeta, \eta, \lambda)$ is the particular solution mentioned before and χ is a new matrix, called the *dressing matrix*, which can be normalized in such a way that it tends to the unit matrix, I, when $|\lambda| \to \infty$. Then the λ dependence of the χ matrix for the solitonic case is very simple:

$$\chi = I + \sum_{n=1}^{N} \frac{\chi_n}{\lambda - \lambda_n}, \tag{1.31}$$

where λ_n are arbitrary constants and the χ_n matrices are independent of λ. The number of poles in (1.31) corresponds to the number of solitons which we have added to the background (g_0,ψ_0). Of course the set of λ_n constitutes the discrete spectrum of the spectral problem (1.15), but this need not concern us here. After choosing any set of parameters λ_n and a background solution (g_0,ψ_0), we should substitute (1.30) and (1.31) into (1.15), and the matrices χ_n will be obtained by purely algebraic operations. After that, from (1.31), (1.30) and (1.29) we obtain the solution for $g(\zeta,\eta)$ in terms of the background solution g_0:

$$g = \chi(\zeta,\eta,0)g_0 = g_0 - \left(\sum_{n=1}^{N} \lambda_n^{-1}\chi_n\right) g_0. \tag{1.32}$$

This is an example of the so-called *dressing technique* developed by Zakharov and Shabat. For the pure solitonic case it is straightforward to compute the new solutions from a given background solution.

1.2 The integrable ansatz in general relativity

If we wish to apply the two-dimensional ISM to the Einstein equations in vacuum

$$R_{\mu\nu} = 0, \tag{1.33}$$

where $R_{\mu\nu}$ is the Ricci tensor, we need to examine the particular case in which the metric tensor $g_{\mu\nu}$ depends on two variables only, which correspond to spacetimes that admit two commuting Killing vector fields, i.e. an Abelian two-parameter group of isometries. In this chapter we take these variables to be the time-like and the space-like coordinates $x^0 = t$ and $x^3 = z$ respectively. This corresponds to nonstationary gravitational fields, i.e. to wave-like and cosmological solutions of Einstein equations (1.33), and the two Killing vectors are both space-like. In any spacetime using the coordinate transformation freedom, $x^\mu = x^\mu(x'^\nu)$, we can fix the following constraints on the metric tensor

$$g_{00} = -g_{33}, \quad g_{03} = 0, \quad g_{0a} = 0. \tag{1.34}$$

Here, and in the following the Latin indices $a, b, c, \ldots$ take the values 1, 2. In these coordinates the spacetime interval becomes

$$ds^2 = f(dz^2 - dt^2) + g_{ab}dx^a dx^b + 2g_{a3}dx^a dz, \tag{1.35}$$

where $f = -g_{00} = g_{33}$. If we now restrict ourselves to the case in which all metric components in (1.35) depend on t and z only, the Einstein equations for such a metric are still too complicated for the ISM or, more precisely, it is unknown at present whether the ISM can be applied in this case. The situation is different in the particular case in which $g_{a3} = 0$. Since it is not possible to eliminate the metric coefficients g_{a3} by any further coordinate transformation