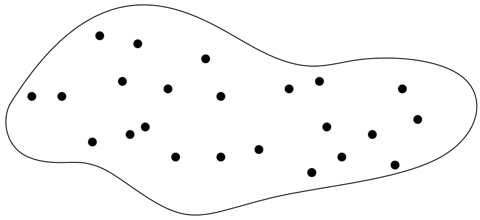
1

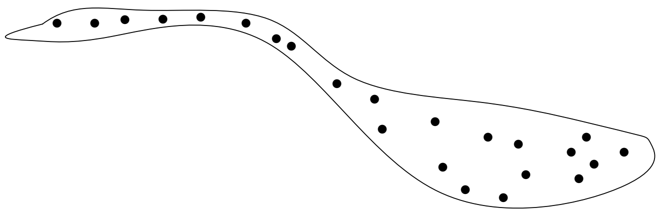
Abstract Sets and Mappings

1.1 Sets, Mappings, and Composition

Let us discuss the idea of abstract constant sets and the mappings between them in order to have a picture of this, our central example, before formalizing a mathematical definition. An abstract set is supposed to have elements, each of which has no structure, and is itself supposed to have no internal structure, except that the elements can be distinguished as equal or unequal, and to have no external structure except for the number of elements. In the category of abstract sets, there occur sets of all possible sizes, including finite and infinite sizes (to be defined later). It has been said that an abstract set is like a mental "bag of dots," except of course that the bag has no shape; thus,



may be a convenient way of picturing a certain abstract set for some considerations, but what is apparently the same abstract set may be pictured as



for other considerations.

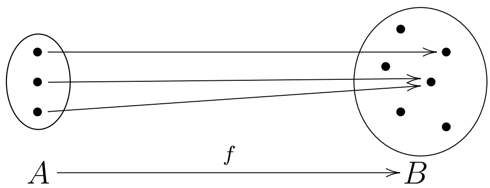
What gives the category of sets its power is the concept of **mapping**. A mapping f from an abstract set A to an abstract set B is often explained through the use of the word *value*. (However, since the elements of B have no structure, it would be misleading to always think of these values as quantities.) Each mapping f from A to B satisfies

for each element x of A
there is exactly one element y of B
such that y is a value of f at x

This justifies the phrase "*the* value"; the value of f at x is usually denoted by $f(x)$; it is an element of B. Thus, a mapping is single-valued and everywhere defined (everywhere on its domain) as in analysis, but it also has a *definite codomain* (usually bigger than its set of actual values). *Any* f at all that satisfies this one property is considered to be a mapping from A to B in the category of abstract constant sets; that is why these mappings are referred to as "arbitrary". An important and suggestive notation is the following:

Notation 1.1: *The arrow notation* $A \xrightarrow{f} B$ *just means the* **domain** *of* f *is* A *and the* **codomain** *of* f *is* B, *and we write* $dom(f) = A$ *and* $cod(f) = B$. (We will usually use capital letters for sets and lowercase letters for mappings.) For printing convenience, in simple cases this is also written with a colon $f : A \longrightarrow B$. We can regard the notation $f : A \longrightarrow B$ as expressing the statement $dom(f) = A$ & $cod(f) = B$, where & is the logical symbol for *and*.

For small A and B, a mapping from A to B can be pictured using its cograph or internal diagram by

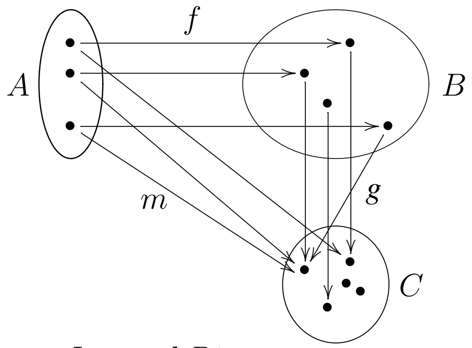


where $f(x)$ is the dot at the right end of the line that has x at its left end for each of the three possible elements x.

Abstract sets and mappings are a **category**, which means above all that there is a **composition** of mappings, i.e., given any pair $f : A \longrightarrow B$ and $g : B \longrightarrow C$ there is a specified way of combining them to give a resulting mapping $g \circ f : A \longrightarrow C$. Note that the codomain set of the first mapping f must be *exactly the same set* as the domain set of the second mapping g. It is common to use the notation $\circ$ for composition and to read it as "following," but we will also, and much more often, denote the composite "g following f" just by gf. A particular instance of composition can be pictured by an external diagram or by an internal diagram as below. First consider any three mappings f, g, and m with domains and codomains as indicated:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "m"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

External Diagram



Internal Diagram

The internal cograph diagrams express the full information about particular maps, which is often more than we need; thus, we will use simple, external diagrams wherever possible.

Since any mapping satisfies restrictions of the kind "for each . . . there is exactly one . . . ," in the diagram above, we observe that

- for each element a of A there is exactly one element b of B for which b is a value of f at a (briefly $f(a) = b$);
- for each element b of B there is exactly one element c of C for which c is a value of g at b (briefly $g(b) = c$);
- for each element a of A there is exactly one element c of C for which c is a value of m at a (briefly $m(a) = c$).

The external diagram above is said to be a "commutative diagram", if and only if m is actually the composite of g *following* f; then, notationally, we write simply $m = gf$.

More precisely, for the triangular diagram to be considered commutative, the relation between f, g, m must have the following property:

> For each element a of A we can find the value of $m(a)$ by proceeding in two steps: first find $f(a)$ and then find $g(f(a))$; the latter is the *same* as $m(a)$.

(Examining the internal diagram shows that $m = gf$ in the figure above.)

A familiar example, when $A = B = C$ is a set of numbers equipped with structural mappings providing addition and multiplication, involves $f(x) = x^2$ and $g(x) = x + 2$ so that $(g \circ f)(x) = x^2 + 2$. The value of the composite mapping at x is the result of taking the value of g at the value of f at x. In contexts such as this where both multiplication and composition are present, it is necessary to use distinct notations for them.

Exercise 1.2
Express the mapping that associates to a number x the value $\sqrt{x^2+2}$ as a composite of *three* mappings. ♢

We need to be more precise about the concept of category. The ideas of set, mapping, and composition will guide our definition, but we need one more ingredient. For each set A there is the **identity mapping** $1_A : A \longrightarrow A$ whose values are determined by $1_A(x) = x$. For any set A, this definition determines a particular mapping among the (possibly many) mappings whose domain and codomain are both A.

On the basis of the preceding considerations we have part of the information required to define the general notion of "category". The first two items listed correspond to abstract sets and arbitrary mappings in the example of the category of sets.

A category $\mathcal{C}$ has the following data:

- Objects: denoted $A, B, C, \ldots$
- Arrows: denoted $f, g, h, \ldots$ (arrows are also often called *morphisms* or *maps*)
- To each arrow f is assigned an object called its *domain* and an object called its *codomain* (if f has domain A and codomain B, this is denoted $f : A \longrightarrow B$)
- Composition: To each $f : A \longrightarrow B$ and $g : B \longrightarrow C$ there is assigned an arrow $gf : A \longrightarrow C$ called *"the composite of f and g"* (or "g following f")
- Identities: To each object A is assigned an arrow $1_A : A \longrightarrow A$ called *"the identity on* A".

1.2 Listings, Properties, and Elements

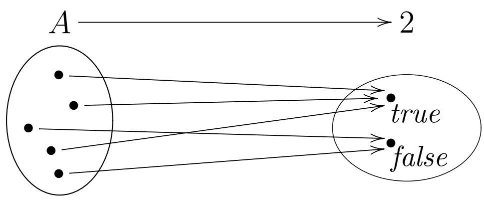
We have not finished defining *category* because the preceding data must be constrained by some general requirements. We first continue with the discussion of elements. Indeed, we can immediately simplify things a little: an idea of element is not necessary as a *separate* idea because we may always identify the elements themselves as special mappings. That will be an extreme case of the *parameterizing* of elements of sets. Let us start with a more intermediate case, for example, the set of mathematicians, *together with* the indication of two examples, say Sir Isaac Newton and Gottfried Wilhelm Leibniz. Mathematically, the model will consist not only of an abstract set A, (to stand for the set of all mathematicians) but also of another abstract set of two elements 1 and 2 to act as labels *and* the specified mapping with codomain A whose value at 1 is "Newton" and whose value at 2 is "Leibniz". The two-element set is the domain of the parameterization.

Such a specific *parameterization* of elements is one of two kinds of features of a set ignored or held in abeyance when we form the *abstract* set. Essentially, all of the terms – **parameterization, listing, family** – have abstractly the same meaning: simply looking at one *mapping* into a set A of interest, rather than just at the one set A all by itself.

Whenever we need to insist upon the abstractness of the sets, such a preferred listing is one of the two kinds of features we are abstracting away.

The other of the two aspects of the elements of an actual concrete aggregation (which are to be ignored upon abstraction) involves the **properties** that the elements might have. For example, consider the set of all the mathematicians and the property "was born during the seventeenth century" that some of the mathematicians have and some do not. One might think that this is an important property of mathematicians as such, but nonetheless one might momentarily just be interested in how many mathematicians there are.

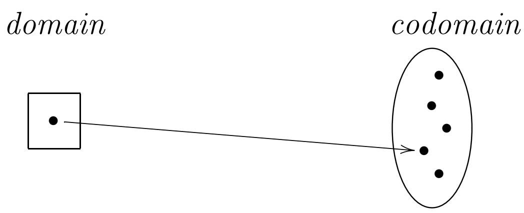
Certain properties are interpreted as particular mappings by using the two-element set of "truth values" – true, false – from which we *also* arrive (by the abstraction) at the abstract set of two elements within which "true" could be taken as exemplary. If we consider a particular mapping such as



we see that all those elements of A that go to "true" will constitute one portion of A, and so f determines a property "true" for some elements, and "not true," or "false," for others. There are properties for which the codomain of f will need more than two elements, for example, age of people: the codomain will need at least as many elements as there are different ages.

As far as listing or parameterizing is concerned, an extreme case is to imagine that *all* the elements have been listed by the given procedure. The opposite extreme case is one in which *no* examples of elements are being offered even though the actual set A under discussion has some arbitrary size. That is, in this extreme case the index set is an *empty* set. Of course, the whole listing or parameterization in this extreme case amounts really to nothing more than the one abstract set A itself.

Just short of the extreme of not listing any is listing just one element. We can do this using a one-element set as parameter set.

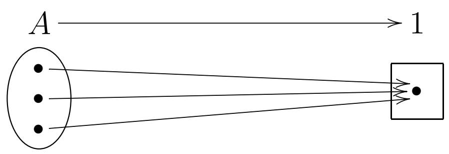


To characterize mathematically what the one-element set is, we will consider it in terms of the property that does not distinguish. The following is the first axiom we require of the category of sets and mappings.

AXIOM: TERMINAL SET
There is a set 1 *such that for any set* A *there is exactly one mapping* $A \longrightarrow 1$. *This unique mapping is given the same name* A *as the set that is its domain.*

We call 1 a **terminal object** of the category of sets and mappings. There may or may not be more than one terminal object; it will make no difference to the mathematical content. In a given discussion the symbol 1 will denote a chosen terminal object; as we will see, which terminal object is chosen will also have no effect on the mathematical content.

Several axioms will be stated as we proceed. The axiom just stated is part of the stronger requirement that the category of sets and mappings has finite inverse limits (see Section 3.6). A typical cograph picture is



Only a one-element set $V = 1$ can have the extreme feature that one cannot detect any distinctions between the elements of A by using only "properties" $A \longrightarrow V$. Having understood what a one-element set is in terms of mapping *to* it, we can now use mappings *from* it to get more information about arbitrary A.

Definition 1.3: *An* **element** *of a set* A *is any mapping whose codomain is* A *and whose domain is* 1 *(or abbreviated* ... $1 \xrightarrow{a} A$*).*

(Why does 1 itself have exactly one element according to this definition?)
The first consequence of our definition is that

element is a special case of mapping.

A second expression of the role of 1 is that

evaluation is a special case of composition.

In other words, if we consider any mapping f from A to B and then consider any element a of A, the codomain of a and the domain of f are the same; thus, we can form the composite fa,

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ 1 \arrow[ur, "a"] & & B \end{tikzcd}$$

which will be a mapping $1 \longrightarrow B$. But since the domain is 1, this means that fa is an *element* of B. Which element is it? It can only be, and clearly is, the *value* of f at a:

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ 1 \arrow[ur, "a"] \arrow[rr, dashed, "fa"'] & & B \end{tikzcd}$$

That is, if a is an element, $fa = f(a)$.
Finally, a third important expression of the role of 1 is that

evaluation of a composite is a special case of the Associative law

of composition (which will be one of the clauses in the definition of *category*). In order to see this, suppose $m = gf$ and consider

$$\begin{tikzcd} & A \arrow[rr, dashed, "m=gf"] \arrow[dr, "f"] & & C \\ 1 \arrow[ur, "a"] \arrow[rr, dashed, "fa"'] \arrow[urrr, bend right=60, "g(fa)"'] & & B \arrow[ur, "g"] & \end{tikzcd}$$

The formula (in which we introduce the symbols $\forall$ to mean "for all" and $\Rightarrow$ to mean "implies")

$$m = gf \Longrightarrow [\forall\, a[1 \xrightarrow{a} A \Rightarrow m(a) = g(fa)]]$$

expresses our idea of evaluation of the composition of two mappings; i. e. if m is the composite of f and g, then for any element a of the domain of f the value of m at a is equal to the value of g at $f(a)$. More briefly, $(gf)a = g(fa)$, which is a case of the associative law.

The three points emphasized here mean that our internal pictures can be (when necessary or useful) completely interpreted in terms of external pictures by also using the set 1.

Notice that the axiom of the terminal set and the definition of element imply immediately that the set 1 whose existence is guaranteed by the axiom has *exactly*

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one element, namely, the unique mapping from 1 to 1. There is always an identity mapping from a set to itself, so this unique mapping from 1 to 1 must be the identity mapping on 1.

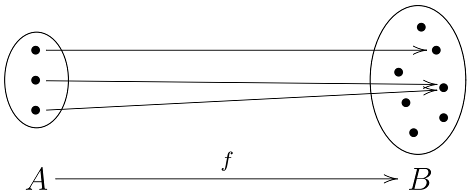
We want to introduce two more logical symbols: the symbol ∃ is read "there exists," and ∃! is read "there exists exactly one". Thus, we can repeat the characteristic feature of every $f : A \longrightarrow B$ as follows:

$$\forall a : 1 \longrightarrow A \quad \exists!\, b : 1 \longrightarrow B [b \text{ is a value of } f \text{ at } a]$$

But this is a special case of the fact that composition in general is uniquely defined.

1.3 Surjective and Injective Mappings

Recall the first internal diagram (cograph) of a mapping that we considered:



Note that it is *not* the case for the f in our picture that

for each element b of B
there is an element x of A
for which b is the value of f at x. ($f(x) = b$)

Definition 1.4: *A mapping* $f : A \longrightarrow B$ *that has the existence property* "for each element b of B there is an element x of A for which $b = f(x)$" *is called a* **surjective mapping**.

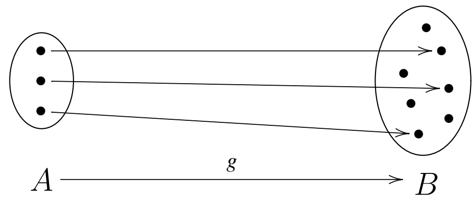
Neither is it the case that the f in our picture has the property

for each element b of B
there is at most one element x of A
for which $f(x) = b$

Definition 1.5: *A mapping* $f : A \longrightarrow B$ *that has the uniqueness property* "given any element b of B there is at most one element x of A for which $f(x) = b$" *is called an* **injective mapping**. *In other words, if f is an injective mapping, then for all elements x, x' of A, if $f(x) = f(x')$, then $x = x'$.*

Definition 1.6: *A mapping that is both surjective and injective is called* **bijective**.

Thus, the f pictured above is neither surjective nor injective, but in the figure below $g : A \longrightarrow B$ is an *injective* mapping from the same A and to the same B.



Exercise 1.7
Is the pictured g surjective? ◊

Exercise 1.8
Are there any surjective mappings $A \longrightarrow B$ for the pictured A, B? ◊

Exercise 1.9
How many mappings from the three-element set A to the seven-element set B are there? Can we picture them all? ◊

Exercise 1.10
Same as 1.9, but for mappings $B \longrightarrow A$ from a seven-element to a three-element set. ◊

Exercise 1.11
Are there any surjective $B \longrightarrow A$? Are there any injective ones? ◊

Exercise 1.12
What definition of "$f_1 \neq f_2$" is presupposed by the idea "number of" mappings we used in 1.9 and 1.10? ◊

Exercises 1.9 and 1.12 illustrate that the feature "external number/internal inequality of instances" characteristic of an abstract set is also associated with the notion "mapping from A to B," except that the elements (the mappings) are not free of structure. But abstractness of the sets really means that the elements are for the moment considered without internal structure. By considering the mappings from A to B with their internal structure ignored, we obtain a new abstract set B^A. Conversely, we will see in Chapter 5 how any abstract set F of the right size can act as mappings between given abstract sets. (For example, in computers variable programs are just a particular kind of variable data.)

1.4 Associativity and Categories

Recall that we saw in Section 1.2 that an "associative law" in a special case expresses the evaluation of composition. Indeed, whenever we have

$$1 \xrightarrow{a} A \xrightarrow{f} B \xrightarrow{g} C$$

then we have the equation $(gf)(a) = g(fa)$.

If we replace a by a general mapping $u : T \longrightarrow A$ whose domain is not necessarily 1, we obtain the **Associative law**

$$(gf)u = g(fu)$$

which actually turns out to be true for any three mappings that can be composed; i.e., that from the commutativity of the two triangles below we can conclude that moreover the outer two composite paths from T to C have equal composites (it is said that the whole diagram is therefore "commutative").

$$(gf)u = g(fu) \qquad \begin{tikzcd} & A \arrow[rr, "gf"] \arrow[dr, "f"] & & C \\ T \arrow[ur, "u"] \arrow[rr, "fu"'] & & B \arrow[ur, "g"'] & \end{tikzcd}$$

Since the 1 among abstract sets has the special feature (which we discuss in Section 1.5) that it can *separate* mappings, in abstract sets the general associative law follows from the special case in which $T = 1$.

An important property of identity mappings is that they not only "do nothing" to an element but that they have this same property with respect to composition. Thus, if $1_A : A \longrightarrow A$ and $1_B : B \longrightarrow B$ are identity mappings, then for any $f : A \longrightarrow B$ we have the equations

$$f 1_A = f = 1_B f$$

With these ideas in hand we are ready to give the completed definition of category. The beginning of our specification repeats what we had before:

Definition 1.13: *A* **category** $\mathcal{C}$ has the following data:

- Objects: denoted $A, B, C, \ldots$
- Arrows: denoted $f, g, h, \ldots$ (arrows are also often called **morphisms** or **maps**)
- To each arrow f is assigned an object called its **domain** and an object called its **codomain** (if f has domain A and codomain B, this is denoted $f : A \longrightarrow B$ or $A \xrightarrow{f} B$)
- Composition: To each $f : A \longrightarrow B$ and $g : B \longrightarrow C$, there is assigned an arrow $gf : A \longrightarrow C$ called **"the composite** g **following** f**"**
- Identities: To each object A is assigned an arrow $1_A : A \longrightarrow A$ called **"the identity on A"**.