HODGE THEORY AND COMPLEX ALGEBRAIC GEOMETRY I

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1

Holomorphic Functions of Many Variables

In this chapter, we recall the main properties of holomorphic functions of several complex variables. These results will be used freely in the remainder of the text, and will enable us to introduce the notions of a complex manifold, and a holomorphic function defined locally on a complex manifold.

The $\mathbb{C}$-valued holomorphic functions of the complex variables $z_1, \ldots, z_n$ are those whose differential is $\mathbb{C}$-linear, or equivalently, those which are annihilated by the operators $\frac{\partial}{\partial \bar{z}_i}$. It follows immediately from this definition that the set of holomorphic functions forms a ring, and that the composition of two holomorphic functions is holomorphic. The following theorem, however, requires a certain amount of work.

Theorem 1.1 *The holomorphic functions of the complex variables $z_1, \ldots, z_n$ are complex analytic, i.e. they locally admit expansions as power series in the variables z_i.*

This result is an easy consequence of Cauchy's formula in several variables, which is a generalisation of the formula

$$f(z) = \frac{1}{2i\pi} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where f is a holomorphic function defined in a disk of radius > 1, and $|z| < 1$.

Cauchy's formula can also be used to prove Riemann's theorem of analytic continuation:

Theorem 1.2 *Let f be a bounded holomorphic function on the pointed disk. Then f extends to a holomorphic function on the whole disk.*

And also Hartogs' theorem:

Theorem 1.3 *Let f be a holomorphic function defined on the complement of the subset F defined by the equations $z_1 = z_2 = 0$ in a ball B of $\mathbb{C}^n$, $n \geq 2$. Then f extends to a holomorphic function on B.*

(More generally, this theorem remains true if F is an analytic subset of codimension 2, but we only need the present version here.) Hartogs' theorem is used more in complex geometry than Riemann's theorem, because it does not impose any conditions on the function f. More generally, it enables us to show that a holomorphic section of a complex vector bundle over a complex manifold is defined everywhere if it is defined on the complement of an analytic subset of codimension 2. This is classically used to show the invariance of the "plurigenera" under birational transformations.

We conclude this chapter with a proof of an explicit formula for the local solution of the equation

$$\frac{\partial f}{\partial \bar{z}} = g,$$

where g is a differentiable function defined on an open set of $\mathbb{C}$. This will be used in the following chapter, to prove the local exactness of the Dolbeault complex. A good reference for the material in this chapter is Hörmander (1979).

1.1 Holomorphic functions of one variable

1.1.1 Definition and basic properties

Let $U \subset \mathbb{C} \cong \mathbb{R}^2$ be an open set, and $f : U \to \mathbb{C}$ a $\mathcal{C}^1$ map. Let x, y be the linear coordinates on $\mathbb{R}^2$ such that $z = x + iy$ is the canonical linear complex coordinate on $\mathbb{C}$. Consider the complex-valued differential form

$$dz = dx + idy \in \mathrm{Hom}_{\mathbb{R}}(T_U, \mathbb{C}) \cong \Omega_{U,\mathbb{R}} \otimes \mathbb{C}.$$

Clearly dz and its complex conjugate $d\bar{z}$ form a basis of $\Omega_{U,\mathbb{R}} \otimes \mathbb{C}$ over $\mathbb{C}$ at every point of U, since

$$2dx = dz + d\bar{z}, \quad 2idy = dz - d\bar{z}. \tag{1.1}$$

The complex differential form $df \in \mathrm{Hom}_{\mathbb{R}}(T_U, \mathbb{C})$ can thus be uniquely written

$$df_u = f_z(u)dz + f_{\bar{z}}(u)d\bar{z}, \tag{1.2}$$

where the complex-valued functions $u \mapsto f_z(u)$, $u \mapsto f_{\bar{z}}(u)$ are continuous.

Definition 1.4 *We write $f_z = \frac{\partial f}{\partial z}$ and $f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}}$.*

By (1.1) we obviously have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{1.3}$$

We can also consider the decomposition (1.2) as the decomposition of $df \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts:

Lemma 1.5 *We have* $\frac{\partial f}{\partial \overline{z}}(u) = 0$ *if and only if the* $\mathbb{R}$*-linear map*

$$df_u : T_{U,u} \cong \mathbb{C} \to \mathbb{C}$$

is $\mathbb{C}$*-linear, i.e. is equal to multiplication by a complex number, which is then equal to* $\frac{\partial f}{\partial z}(u)$*.*

Proof Because $\frac{\partial}{\partial y} = i\frac{\partial}{\partial x}$ for the natural complex structure on $T_{U,u}$, the morphism $df_u : T_{U,u} \to \mathbb{C}$ is $\mathbb{C}$-linear if and only if we have

$$\frac{\partial f}{\partial y}(u) = i\frac{\partial f}{\partial x}(u),$$

and by (1.3), this is equivalent to $\frac{\partial f}{\partial \overline{z}}(u) = 0$. Moreover, we then have $df_u = \frac{\partial f}{\partial z}(u)dz$, i.e.

$$df_u\left(\frac{\partial}{\partial x}\right) = \frac{\partial f}{\partial z}(u), \quad df_u\left(\frac{\partial}{\partial y}\right) = i\frac{\partial f}{\partial z}(u),$$

which proves the second assertion, since the natural isomorphism $T_{U,u} \cong \mathbb{C}$ sends $\frac{\partial}{\partial x}$ to 1. □

Definition 1.6 *The function* f *is said to be holomorphic if it satisfies one of the equivalent conditions of lemma 1.5 at every point of* U*.*

Lemma 1.7 *If* f *is holomorphic and does not vanish on* U*, then* $\frac{1}{f}$ *is holomorphic. Similarly, if* f*,* g *are holomorphic,* fg *and* $f+g$ *and* $g \circ f$ *(when* g *is defined on the image of* f*) are all holomorphic.*

Proof The map $z \mapsto \frac{1}{z}$ is holomorphic on $\mathbb{C}^*$, so that the first assertion follows from the last one. Furthermore, if g and f are $\mathcal{C}^1$ and g is defined on the image of f, then $g \circ f$ is $\mathcal{C}^1$ and we have

$$d(g \circ f)_u = dg_{f(u)} \circ df_u.$$

If $dg_{f(u)}$ and df_u are both $\mathbb{C}$-linear for the natural identifications of $T_{\mathbb{C},u}$, $T_{\mathbb{C},f(u)}$ and $T_{\mathbb{C},g\circ f(u)}$ with $\mathbb{C}$, then $d(g \circ f)_u$ is also $\mathbb{C}$-linear, and the last assertion is proved. The other properties are proved similarly. □

In particular, we will use the following corollary.

Corollary 1.8 *If f is holomorphic on U, the map g defined by*

$$g(z) = \frac{f(z)}{z-a}$$

is holomorphic on $U - \{a\}$.

1.1.2 Background on Stokes' formula

Let α be a $\mathcal{C}^1$ differential k-form on an n-dimensional manifold U (cf. definition 2.3 and section 2.1.2 in the following chapter). If $x_1, \ldots, x_n$ are local coordinates on U, we can write

$$\alpha = \sum_I \alpha_I dx_I,$$

where the indices I parametrise the totally ordered subsets $i_1 < \cdots < i_k$ of $\{1, \ldots, n\}$, with $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. We can then define the continuous $(k+1)$-form

$$d\alpha = \sum_{I,i} \frac{\partial \alpha_I}{\partial x_i} dx_i \wedge dx_I; \tag{1.4}$$

we check that it is independent of the choice of coordinates. This follows from the more general fact that if V is an m-dimensional manifold and $\phi : V \to U$ is a $\mathcal{C}^1$ map given in local coordinates by $\phi^* x_i := x_i \circ \phi = \phi_i(y_1, \ldots, y_m)$, then for every differential form $\alpha = \sum_I \alpha_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, we can define its inverse image

$$\phi^* \alpha = \sum_I \alpha_I \circ \phi \; d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

Moreover, if ϕ is $\mathcal{C}^2$, this image inverse satisfies

$$d(\phi^* \alpha) = \phi^*(d\alpha),$$

where the coordinates y_i (and the formulae (1.4)) are used on the left, while the coordinates x_i are used on the right.

A $\mathcal{C}^0$ differential k-form α can be integrated over the compact oriented k-dimensional submanifolds of U with boundary, or over the image of such manifolds under differentiable maps.

To begin with, let us recall that a k-dimensional manifold with boundary is a topological space covered by open sets U_i which are homeomorphic, via certain maps ϕ_i, to open subsets of $\mathbb{R}^k$ or to $]0, 1] \times V$, where V is an open set of $\mathbb{R}^{k-1}$. We require the transition functions $\phi_i \circ \phi_j^{-1}$ to be differentiable on $\phi_j(U_i \cap U_j)$. When $\phi_j(U_i \cap U_j)$ contains points on the boundary of U_j, i.e. is locally isomorphic to $]0, 1] \times V$, where V is an open set of $\mathbb{R}^{k-1}$, $\phi_i(U_i \cap U_j)$ must also be locally isomorphic to $]0, 1] \times V'$, where V' is an open set of $\mathbb{R}^{k-1}$, and the differentiable map $\phi_i \circ \phi_j^{-1}$ must locally extend to a diffeomorphism of a neighbourhood in $\mathbb{R}^k$ of $]0, 1] \times V$ to a neighbourhood of $]0, 1] \times V'$, inducing a diffeomorphism from $1 \times V$ to $1 \times V'$. In particular, the boundary of M, which we denote by ∂M and which is defined, with the preceding notation, as the union of the $\phi_i^{-1}(1 \times V)$, is a closed set of M which possesses an induced differentiable manifold structure.

The manifold with boundary M is said to be oriented if the diffeomorphisms $\phi_i \circ \phi_j^{-1}$ have positive Jacobian. The boundary of M is then also naturally oriented by the charts $1 \times V$, where V is an open set of $\mathbb{R}^{k-1}$ as above, since the induced transition diffeomorphisms

$$\phi_i \circ \phi_j^{-1}{}_{|1\times V} : V \to V'$$

also have positive Jacobian.

If M is a k-dimensional manifold with boundary and $\phi : M \to U$ is a $\mathcal{C}^1$ differentiable map (along the boundary of M, which is locally isomorphic to $]0, 1] \times V$, we require ϕ to extend locally to a $\mathcal{C}^1$ map on a neighbourhood $]0, 1 + \epsilon[\times V$ of $\{1\} \times V$), then for every continuous k-form α, we have the inverse image $\beta = \phi^*\alpha$ defined above, which is a continuous k-form on M. If moreover M is oriented and compact, such a form can be integrated over M as follows. Let f_i be a partition of unity subordinate to a covering of M by open sets U_i as above, which we may assume to be diffeomorphic to $]0, 1] \times]0, 1[^{k-1}$ or to $]0, 1[^k$. Then $\beta = \sum_i f_i\beta$, and the form $f_i\beta$ on U_i extends to a continuous form on $[0, 1]^k$. Setting $f_i\beta = g_i(x_1, \ldots, x_k)dx_1 \wedge \cdots \wedge dx_k$, we then define

$$\int_M \beta = \sum_i \int_{U_i} f_i\beta$$

$$\int_{U_i} f_i\beta = \int_0^1 \cdots \int_0^1 g_i(x_1, \ldots, x_k)dx_1 \ldots dx_k.$$

The change of variables formula for multiple integrals and the fact that the authorised variable changes have positive Jacobians ensure that $\int_M \beta$ is well-defined independently of the choice of oriented charts, i.e. of local orientation-preserving coordinates.

Remark 1.9 *If we change the orientation of M, i.e. if we compose all the charts with local diffeomorphisms of $\mathbb{R}^k$ with negative Jacobians, the integrals $\int_M \phi^*\alpha$ change sign. This follows from the change of variables formula for multiple integrals, which uses only the absolute value of the Jacobian, whereas the change of variables formula for differential forms of maximal degree uses the Jacobian itself.*

Suppose now that α is a $\mathcal{C}^1$ $(k-1)$-form on U. Then, as $\phi_{|\partial M}$ is differentiable and ∂M is a compact oriented manifold of dimension $k-1$, we can compute the integral $\int_{\partial M} \phi^*\alpha$. Moreover, we can integrate the differential $\phi^* d\alpha = d\phi^*\alpha$ over M. We then have

Theorem 1.10 *(Stokes' formula) The following equality holds:*

$$\int_M \phi^* d\alpha = \int_{\partial M} \phi^*\alpha. \tag{1.5}$$

In particular, if $d\alpha = 0$, we have $\int_{\partial M} \phi^*\alpha = 0$.

Proof Using a partition of unity, we are reduced to showing (1.5) when $\phi^*\alpha$ has compact support in an open set U_i of M as above. This follows immediately from the formula (1.4) for the differential, and the equality

$$\int_0^1 f'(t)dt = f(1) - f(0),$$

which holds for any $\mathcal{C}^1$ function f. □

We will use Stokes' formula very frequently throughout this text. In particular, it will enable us to pair the de Rham cohomology with the singular homology. The following consequence will be particularly useful.

Corollary 1.11 *If α is a differential form of degree $n-1$ on a compact n-dimensional manifold without boundary, then $\int_M d\alpha = 0$.*

1.1.3 Cauchy's formula

We propose to apply Stokes' formula (1.5), using the following lemma.

Lemma 1.12 *Let $f : U \to \mathbb{C}$ be a holomorphic map. Then the complex differential form $f dz$ is closed.*

Proof We have $d(f dz) = d(f dx + i f dy) = \frac{\partial f}{\partial y} dy \wedge dx + i \frac{\partial f}{\partial x} dx \wedge dy$. Thus $\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}$ implies that $d(f dz) = 0$. □

By corollary 1.8, we thus also have the following.

Corollary 1.13 *If f is holomorphic on U, the differential form $\frac{f}{z - z_0} dz$ is closed on $U - \{z_0\}$.*

Suppose now that U contains a closed disk D. For every $z_0 \in D$, let D_ϵ be the open disk of radius ϵ centred at z_0 which is contained in D for sufficiently small ϵ. Then $D - D_\epsilon$ is a manifold with boundary, whose boundary is the union of the circle ∂D and the circle of centre z_0 and radius ϵ, the first with its natural orientation, the second with the opposite orientation. For holomorphic f, Stokes' formula and corollary 1.1.3 then give the equality

$$\frac{1}{2i\pi} \int_{\partial D} \frac{f(z)}{z - z_0} dz = \frac{1}{2i\pi} \int_{|z - z_0| = \epsilon} \frac{f(z)}{z - z_0} dz. \tag{1.6}$$

Furthermore, we have the following.

Lemma 1.14 *If f is a function which is continuous at z_0, then*

$$\lim_{\epsilon \to 0} \frac{1}{2i\pi} \int_{|z - z_0| = \epsilon} \frac{f(z)}{z - z_0} dz = f(z_0).$$

Proof The circle of radius ϵ and centre z_0 is parametrised by the map $\gamma : t \mapsto z_0 + \epsilon e^{2i\pi t}$ on the segment $[0, 1]$. We have $\gamma^* \left(\frac{1}{2i\pi} \frac{f(z)}{z - z_0} dz \right) = f(z_0 + \epsilon e^{2i\pi t}) dt$. Thus,

$$\frac{1}{2i\pi} \int_{|z - z_0| = \epsilon} \frac{f(z)}{z - z_0} dz = \int_0^1 f(z_0 + \epsilon e^{2i\pi t}) dt. \tag{1.7}$$

But as f is continuous at z_0, the functions $f_\epsilon(t) = f(z_0 + \epsilon e^{2i\pi t})$ converge uniformly, as ϵ tends to 0, to the constant function equal to $f(z_0)$. We thus have

$$\lim_{\epsilon\to 0}\int_0^1 f(z_0+\epsilon e^{2i\pi t})dt = \int_0^1 f(z_0)dt = f(z_0).$$

□

Combining lemma 1.14 and equality (1.6), we now have

Theorem 1.15 *(Cauchy's formula) Let f be a holomorphic function on U and D a closed disk contained in U. Then for every point z_0 in the interior of D, we have the equality*

$$f(z_0) = \frac{1}{2i\pi}\int_{\partial D}\frac{f(z)}{z-z_0}dz. \tag{1.8}$$

1.2 Holomorphic functions of several variables

1.2.1 Cauchy's formula and analyticity

Let U be an open set of $\mathbb{C}^n$, and let $f : U \to \mathbb{C}$ be a $\mathcal{C}^1$ map. For $u \in U$, we have a canonical identification $T_{U,u} \cong \mathbb{C}^n$. We can thus generalise the notion of a holomorphic function to higher dimensions.

Definition 1.16 *The function f is said to be holomorphic if for every $u \in U$, the differential*

$$df_u \in \mathrm{Hom}(T_{U,u}, \mathbb{C}) \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$$

is $\mathbb{C}$-linear.

It is easy to prove that lemma 1.7 remains true in higher dimensions. Furthermore, we have the three following characterisations of holomorphic functions.

Theorem 1.17 *The following three properties are equivalent for a $\mathcal{C}^1$ function f:*
(i) f is holomorphic.
(ii) In the neighbourhood of each point $z_0 \in U$, f admits an expansion as a power series of the form

$$f(z_0+z) = \sum_I \alpha_I z^I, \tag{1.9}$$

where I runs through the set of the n-tuples of integers $(i_1, \ldots, i_n)$ with $i_k \geq 0$, and $z^I := z_1^{i_1}\cdots z_n^{i_n}$. The coefficients of the series (1.9) satisfy the following *property: there exist* $R_1 > 0, \ldots, R_n > 0$ *such that the power series*

$$\sum_I |\alpha_I| r^I$$

converges for every $r_1 < R_1, \ldots, r_n < R_n$.
(iii) If $D = \{(\zeta_1, \ldots, \zeta_n) \mid |\zeta_i - a_i| \leq \alpha_i\}$ *is a polydisk contained in* U*, then for every* $z = (z_1, \ldots, z_n) \in D^0$*, we have the equality*

$$f(z) = \left(\frac{1}{2i\pi}\right)^n \int_{|\zeta_i - a_i| = \alpha_i} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}. \qquad (1.10)$$

In the preceding formula, the integral is taken over a product of circles, equipped with the orientation which is the product of the natural orientations.

Remark 1.18 *Because of property (ii), holomorphic functions are also known as complex analytic functions.*

Proof The implications (iii)⇒(ii)⇒(i) are obvious: indeed, (iii)⇒(ii) is obtained, for ζ in the product of circles $\{\zeta \mid |\zeta_i - a_i| = \alpha_i\}$ and z in the interior of D, by expanding the functions

$$\frac{f(\zeta)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \frac{f(\zeta)}{((\zeta_1 - a_1) - (z_1 - a_1)) \cdots ((\zeta_n - a_n) - (z_n - a_n))}$$

as a power series in $z_1 - a_1, \ldots, z_n - a_n$ whose coefficients are continuous functions of ζ. The uniform convergence in ζ of this expansion then shows that the integral (1.10) admits the corresponding power series expansion in $z_1 - a_1, \ldots, z_n - a_n$, so that (ii) holds for the function defined by (1.10).

(ii)⇒(i) follows from the fact that every polynomial function of $z_1, \ldots, z_n$ is holomorphic, and that as the series (1.9) is a uniform limit of polynomials whose derivatives also converge uniformly, its differential is the limit of the differentials of these polynomials. As the differential of each polynomial is $\mathbb{C}$-linear, the same holds for the series, which is thus also holomorphic.

It remains to see that (i)⇒(iii), which is Cauchy's formula in several variables. We can prove it by induction on the dimension, using Cauchy's formula (1.8). We can also directly apply Stokes' formula, using the following analogue of lemma 1.12.

Lemma 1.19 *If* f *is holomorphic, then the differential form* $f(z) dz_1 \wedge \cdots \wedge dz_n$ *is closed.*

The product of circles $\prod_i \{|\zeta_i - z_i| = \epsilon\}$ is contained in D for sufficiently small ϵ, and homotopic in $D - \bigcup_i \{\zeta \mid \zeta_i = z_i\}$ to the product of circles $\prod_i \{|\zeta_i - a_i| = \alpha_i\}$, which means that there exists an oriented compact manifold M of dimension n and a differentiable map

$$\phi : [0,1] \times M \to D - \cup_i \{\zeta \mid \zeta_i = z_i\}$$

such that $\phi_{|0\times M}$ is a diffeomorphism from M to the first product of circles, and $\phi_{|1\times M}$ is a diffeomorphism from M to the second product of circles, the first isomorphism being compatible with the orientations, and the second changing the orientation. We then deduce from lemma 1.19 that if $\beta = f(\zeta)\frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}$, the differential form $\phi^*\beta$ is closed on $[0,1] \times M$, and thus, by Stokes' formula, satisfies

$$\int_{\partial[0,1]\times M} \phi^*\beta = 0.$$

For ϵ sufficiently small, this gives the equality

$$\left(\frac{1}{2i\pi}\right)^n \int_{|\zeta_i - a_i| = \alpha_i} f(\zeta)\frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}$$
$$= \left(\frac{1}{2i\pi}\right)^n \int_{|\zeta_i - z_i| = \epsilon} f(\zeta)\frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}.$$

But the limit of the right-hand term as ϵ tends to 0 is equal to $f(z)$ by the same argument as above.

□

Remark 1.20 *The homotopy must have values in $D - \bigcup_i \{\zeta \mid \zeta_i = z_i\}$ and not only in D, in order to guarantee that the form $\phi^* f(\zeta)\frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}$ is $\mathcal{C}^1$ in $[0,1] \times M$ and to be able to apply Stokes' formula.*

1.2.2 Applications of Cauchy's formula

Let us give some applications of theorem 1.17. To begin with, we have

Theorem 1.21 *(The maximum principle) Let f be a holomorphic function on an open subset U of $\mathbb{C}^n$. If $|f|$ admits a local maximum at a point $u \in U$, then f is constant in the neighbourhood of this point.*

Proof Let $R_1, \ldots, R_n$ be positive real numbers such that for every $\epsilon_i \le R_i$, the polydisk $D_{\epsilon_\cdot} = \{\zeta \in \mathbb{C}^n \mid |\zeta_i - u_i| \le \epsilon_i\}$ is contained in U. Then we have

Cauchy's formula

$$f(u) = \left(\frac{1}{2i\pi}\right)^n \int_{|\zeta_i - u_i| = \epsilon_i} f(\zeta) \frac{d\zeta_1}{\zeta_1 - u_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - u_n}.$$

Parametrising the circles $|\zeta_i - u_i| = \epsilon_i$ by $\gamma_i(t) = u_i + \epsilon_i e^{2i\pi t}$, $t \in [0, 1]$, this can be written as

$$f(u) = \int_0^1 \cdots \int_0^1 f(u_1 + \epsilon_1 e^{2i\pi t_1}, \dots, u_n + \epsilon_n e^{2i\pi t_n}) dt_1 \cdots dt_n. \quad (1.11)$$

But we have the inequality

$$\left| \int_0^1 \cdots \int_0^1 f(u_1 + \epsilon_1 e^{2i\pi t_1}, \dots, u_n + \epsilon_n e^{2i\pi t_n}) dt_1 \cdots dt_n \right| \\ \leq \int_0^1 \cdots \int_0^1 |f(u_1 + \epsilon_1 e^{2i\pi t_1}, \dots, u_n + \epsilon_n e^{2i\pi t_n})| \, dt_1 \cdots dt_n, \quad (1.12)$$

and equality holds if and only if the argument of $f(u_1 + \epsilon_1 e^{2i\pi t_1}, \dots, u_n + \epsilon_n e^{2i\pi t_n})$ is constant, necessarily equal to that of $f(u)$ by (1.11).

Now, for sufficiently small ϵ_i, we have by hypothesis

$$|f(u)| \geq |f(u_1 + \epsilon_1 e^{2i\pi t_1}, \dots, u_n + \epsilon_n e^{2i\pi t_n})| \, .$$

Combining this inequality with (1.11) and (1.12), we obtain

$$|f(u)| \leq \int_0^1 \cdots \int_0^1 |f(u_1 + \epsilon_1 e^{2i\pi t_1}, \dots, u_n + \epsilon_n e^{2i\pi t_n})| \, dt_1 \cdots dt_n \\ \leq \int_0^1 \cdots \int_0^1 |f(u)| \, dt_1 \cdots dt_n = |f(u)| \, .$$

The equality of the two extreme terms then implies equality at every step; the first equality implies that the argument of f is constant, equal to that of $f(u)$ on each product of circles as above, and the second equality shows that the function f must have constant modulus equal to $|f(u)|$, for sufficiently small ϵ_i. Letting the multiradius of the polydisks $D_{\epsilon.}$ vary, we have thus shown that f is constant, equal to $f(u)$ on a neighbourhood of u possibly minus the hyperplanes $\{\zeta_i = u_i\}$, i.e. in fact constant in the neighbourhood of u by continuity. □

Another essential application is the principle of analytic continuation.

Theorem 1.22 *Let U be a connected open set of $\mathbb{C}^n$, and f a holomorphic function on U. If f vanishes on an open set of U, then f is identically zero.*

Proof This follows from the fact that by the characterisation (ii), f is in particular analytic (i.e. locally equal to the sum of its Taylor series). We can thus apply the principle of analytic continuation to f. We recall that the latter is shown by noting that if f is analytic, the open set consisting of the points in whose neighbourhood f vanishes is equal to the closed set consisting of the points where f and all its derivatives vanish. □

Let us now give some subtler applications of Cauchy's formula (1.10) or its generalisations. These theorems show that the possible singularities of a holomorphic function cannot exist unless the function is not bounded (Riemann), and is not defined on the complement of an analytic subset of codimension 2 (Hartogs).

Theorem 1.23 *(Riemann) Let f be a holomorphic function defined on $U - \{z \mid z_1 = 0\}$, where U is an open set of $\mathbb{C}^n$. Then if f is locally bounded on U, f extends to a holomorphic map on U.*

Proof Since this is a local statement, it suffices to show that if U contains a polydisk $D = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| \leq r_i\}$ on which f is bounded, then we can extend f to the points in the interior of D. We propose to show that for a point z in the interior of D such that $z_1 \neq 0$, Cauchy's formula

$$f(z) = \left(\frac{1}{2i\pi}\right)^n \int_{\partial D} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}, \tag{1.13}$$

where

$$\partial D := \{(\zeta_1, \ldots, \zeta_n) \mid |\zeta_i| = r_i, \forall i\},$$

holds. Note that the right-hand term in (1.13) is well-defined, since the integration locus is contained in the locus of definition of f.

Let $\epsilon_1 \in \mathbb{R}$, $0 < \epsilon_1 < |z_1|$ be such that the closed disk of radius ϵ_1 and centre z_1 is contained in the disk $\{\zeta \mid |\zeta| < r_1\}$. Then the polydisk

$$D_{\epsilon_1} := \{(\zeta_1, \ldots, \zeta_n) \mid |\zeta_1 - z_1| \leq \epsilon_1,\ |z_i| \leq r_i, \quad i \geq 2\}$$

is contained in $D - \{\zeta_1 = 0\}$, so that Cauchy's formula gives

$$f(z) = \left(\frac{1}{2i\pi}\right)^n \int_{\partial D_{\epsilon_1}} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}, \tag{1.14}$$

where

$$\partial D_{\epsilon_1} := \{(\zeta_1, \ldots, \zeta_n) \mid |\zeta_1 - z_1| = \epsilon_1,\ |\zeta_i| = r_i,\ i \geq 2\}.$$

Consider, also, the product of circles

$$\partial D'_\epsilon := \{(\zeta_1, \ldots, \zeta_n) |\ |\zeta_1| = \epsilon,\ |\zeta_i| = r_i, i \geq 2\}.$$

Then when ϵ is sufficiently small, $\partial D - \partial D_{\epsilon_1} - \partial D'_\epsilon$ is the boundary of the manifold

$$M = \{(\zeta_1, \ldots, \zeta_n) |\ |\zeta_1 - z_1| \geq \epsilon_1,\ |\zeta_1| \geq \epsilon,\ |\zeta_i| = r_i,\ i \geq 2\},$$

which is contained in D and intersects neither the hypersurface $\{\zeta_1 = 0\}$ nor the hypersurfaces $\{\zeta_i = z_i\}$. Here, the signs given to the components of the boundary are positive when the orientation as part of the boundary of M coincides with the natural orientation, negative otherwise. Stokes' formula and (1.14) then give

$$f(z) = \left(\frac{1}{2i\pi}\right)^n \left[\int_{\partial D} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n} - \int_{\partial D'_\epsilon} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}\right].$$

The proof of formula (1.13) can then be finished using the following lemma.

Lemma 1.24 *When f is bounded, and for z such that $z_1 \neq 0, |z_i| < r_i$, we have*

$$\lim_{\epsilon \to 0} \int_{\partial D'_\epsilon} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n} = 0. \tag{1.15}$$

Proof Let us parametrise the product of circles $\partial D'_\epsilon$ by $[0, 1]^n$, $(t_1, \ldots, t_n) \mapsto (\epsilon e^{2i\pi t_1}, r_2 e^{2i\pi t_2}, \ldots, r_n e^{2i\pi t_n})$. The integral (1.15) is thus equal to

$$(2i\pi)^n \int_0^1 \cdots \int_0^1 \epsilon r_2 \cdots r_n \prod_j e^{2i\pi t_j} \frac{f(\epsilon e^{2i\pi t_1}, r_2 e^{2i\pi t_2}, \ldots, r_n e^{2i\pi t_n})}{(\epsilon e^{2i\pi t_1} - z_1) \cdots (r_n e^{2i\pi t_n} - z_n)} dt_1 \cdots dt_n.$$

As f is bounded, under the hypotheses on z, the integrand in this formula tends uniformly to 0 with ϵ, and thus the integral in the formula tends to 0 with ϵ.

□

As Cauchy's formula (1.13) is proved, Riemann's extension theorem follows immediately, since it is clear that the function defined by the right-hand term in (1.13) extends holomorphically to D. □

To conclude this section, we will prove the following version of Hartogs' extension theorem.

Theorem 1.25 *Let U be an open set of $\mathbb{C}^n$ and f a holomorphic function on $U - \{z \mid z_1 = z_2 = 0\}$. Then f extends to a holomorphic function on U.*

Remark 1.26 *This implies the more general theorem mentioned above, using theorem 11.11, which proves that an analytic subset of codimension 2 can be stratified into smooth analytic submanifolds of codimension at least 2; this theorem will be proved in section 11.1.1.*

Proof Let D be a closed polydisk contained in U:

$$D = \{(z_1, \dots, z_n) | \ |z_i| \leq r_i\}.$$

Let $z \in D - \{\zeta \mid \zeta_1 = \zeta_2 = 0\}$. As in the preceding proof, we will show that Cauchy's formula

$$f(z) = \left(\frac{1}{2i\pi}\right)^n \int_{\partial D} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n} \tag{1.16}$$

is satisfied, and this will enable us to conclude, as above, that the function $f(z)$, given in the form of an integral as in (1.16), extends holomorphically to D. Let ϵ_1, ϵ_2 be two positive real numbers, sufficiently small for the polydisk

$$D_\epsilon = \{\zeta | \ |\zeta_i - z_i| \leq \epsilon_i, \ i = 1, \ 2, |\zeta_i| \leq r_i, \ i > 2\}$$

to be contained in $D - \{\zeta \mid \zeta_1 = \zeta_2 = 0\}$. Then we have Cauchy's formula

$$f(z) = \left(\frac{1}{2i\pi}\right)^n \int_{\partial D_\epsilon} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}, \tag{1.17}$$

where

$$\partial D_\epsilon = \{\zeta | \ |\zeta_i - z_i| = \epsilon_i, \ i = 1, \ 2, |\zeta_i| = r_i, \ i > 2\}.$$

It thus suffices to show that $\partial D - \partial D_\epsilon$ is a boundary in

$$D - (\{\zeta \mid \zeta_1 = \zeta_2 = 0\} \cup \bigcup_i \{\zeta \mid \zeta_i = z_i\}),$$

in order to apply Stokes' formula and conclude that (1.16) holds.

Let $\alpha_1(t), \alpha_2(t), \ t \in [0, 1]$, be two positive-valued differentiable functions such that $\alpha_i(1) = \epsilon_i, \ \alpha_i(0) = r_i$. For every $t \in [0, 1]$, let

$$\partial D_t = \{\zeta | \ |\zeta_i - tz_i| = \alpha_i(t), \ i = 1, \ 2, |\zeta_i| = r_i, \ i > 2\}.$$

Lemma 1.27 *For a suitable choice of functions α_1, α_2, ∂D_t is contained in*

$$D - (\{\zeta \mid \zeta_1 = \zeta_2 = 0\} \cup \bigcup_i \{\zeta \mid \zeta_i = z_i\})$$

for every $t \in [0, 1]$.

Proof Firstly, ∂D_t still lies in D if $\alpha_i(t) + t\ |z_i| \leq r_i$, $i = 1, 2$. Moreover, ∂D_t still lies in $D - \bigcup_i \{\zeta \mid \zeta_i = z_i\}$ if $\alpha_i(t) \neq (1-t)\ |z_i|$, $i = 1, 2$. Now, note that $(1-t)\ |z_i| < r_i - t\ |z_i|$, since $|z_i| < r_i$. Furthermore, the conditions $\alpha_i(t) \leq r_i - t\ |z_i|$ and $\alpha_i(t) > (1-t)\ |z_i|$ are both satisfied for $t = 1$ and $t = 0$. It thus suffices to take functions $\alpha_i(t)$ satisfying

$$(1-t)\ |z_i| < \alpha_i(t) \leq r_i - t\ |z_i|, \quad \alpha_i(0) = r_i, \quad \alpha_i(1) = \epsilon_i.$$

It remains to see that D_t does not meet $\{\zeta \mid \zeta_1 = \zeta_2 = 0\}$ for any $t \in [0, 1]$, for a suitable choice of the pair (α_1, α_2). But D_t meets $\{\zeta \mid \zeta_1 = \zeta_2 = 0\}$ if we have $\alpha_i(t) = t\ |z_i|$ for $i = 1$ and 2. For fixed t, this imposes two conditions on the pair (α_1, α_2), and t varies in a segment, so it is clear that this last condition is not satisfied by a pair of sufficiently general functions. □

Lemma 1.27 gives a differentiable homotopy in $D - (\{\zeta \mid \zeta_1 = \zeta_2 = 0\} \cup \{\zeta \mid \zeta_i = z_i\})$ from ∂D to ∂D_ϵ, so we can conclude by Stokes' formula that (1.16) holds. Thus theorem 1.25 is proved. □

1.3 The equation $\frac{\partial g}{\partial \bar{z}} = f$

The following theorem will play an essential role in the proof of the local exactness of the operator $\bar{\partial}$.

Theorem 1.28 *Let f be a $\mathcal{C}^k$ function (for $k \geq 1$) on an open set of $\mathbb{C}$. Then, locally on this open set, there exists a $\mathcal{C}^k$ function g (for $k \geq 1$), such that*

$$\frac{\partial g}{\partial \bar{z}} = f. \tag{1.18}$$

Remark 1.29 *Such a function g is defined up to the addition of a holomorphic function.*

Proof As the statement is local, we may assume that f has compact support, and thus is defined and $\mathcal{C}^k$ on $\mathbb{C}$. Now set

$$g = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

□

Remark 1.30 *This is a singular integral. By definition, it is equal to the limit, as ϵ tends to 0, of the integrals*

$$\frac{1}{2i\pi} \int_{\mathbb{C} - D_\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta},$$

where D_ϵ is a disk of radius ϵ centred at z. It is easy to see that this limit exists (the function $\frac{1}{\zeta - z}$ is L^1).

Making the change of variable $\zeta' = \zeta - z$, we also have

$$g(z) = \lim_{\epsilon \to 0} g_\epsilon(z), \quad g_\epsilon(z) = \frac{1}{2i\pi} \int_{\mathbb{C}_\epsilon} \frac{f(\zeta' + z)}{\zeta'} d\zeta' \wedge d\overline{\zeta'},$$

where

$$\mathbb{C}_\epsilon = \mathbb{C} - D'_\epsilon,$$

and D'_ϵ is a disk of radius ϵ centred at 0. The convergence of the g_ϵ when ϵ tends to 0 is uniform in z. Moreover, we can differentiate under the integral sign the (non-singular) integral defining g_ϵ

$$\frac{\partial g_\epsilon}{\partial \overline{z}} = \frac{1}{2i\pi} \int_{\mathbb{C}_\epsilon} \frac{\partial f(\zeta' + z)}{\partial \overline{z}} \frac{d\zeta' \wedge d\overline{\zeta'}}{\zeta'}.$$

As $\frac{\partial f(\zeta'+z)}{\partial \overline{z}}$ is $\mathcal{C}^{k-1}$, with $k - 1 \geq 0$, the functions $\frac{\partial g_\epsilon}{\partial \overline{z}}$ converge uniformly, and we conclude that g is at least $\mathcal{C}^1$ and satisfies

$$\frac{\partial g}{\partial \overline{z}} = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{\partial f(\zeta' + z)}{\partial \overline{z}} \frac{d\zeta' \wedge d\overline{\zeta'}}{\zeta'}.$$

By induction on k, the same argument actually shows that g is $\mathcal{C}^k$. Thus, it remains to show the equality $\frac{\partial g}{\partial \overline{z}} = f$. Again making the change of variable $\zeta = \zeta' + z$, we have

$$\frac{\partial g}{\partial \overline{z}}(z) = \lim_{\epsilon \to 0} \frac{1}{2i\pi} \int_{\mathbb{C} - D_\epsilon} \frac{\partial f}{\partial \overline{\zeta}}(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z}. \tag{1.19}$$

Now, we have the equality on $\mathbb{C} - D_\epsilon$

$$\frac{\partial f}{\partial \overline{\zeta}}(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z} = -d\left(f \frac{d\zeta}{\zeta - z} \right);$$

indeed, for a differentiable function $\phi(\zeta)$ we know that $d\phi = \frac{\partial \phi}{\partial \zeta} d\zeta + \frac{\partial \phi}{\partial \overline{\zeta}} d\overline{\zeta}$, and thus

$$d(\phi d\zeta) = -\frac{\partial \phi}{\partial \overline{\zeta}} d\zeta \wedge d\overline{\zeta}.$$

Stokes' formula thus gives

$$\frac{1}{2i\pi} \int_{\mathbb{C} - D_\epsilon} \frac{\partial f}{\partial \overline{\zeta}}(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta - z} = \frac{1}{2i\pi} \int_{\partial D_\epsilon} f(\zeta) \frac{d\zeta}{\zeta - z}. \tag{1.20}$$

Using lemma 1.14 and the equalities (1.19), (1.20) we have thus proved the equality (1.18).

Exercises

1. Let $\phi : U \to V$ be a holomorphic map from an open subset of $\mathbb{C}^n$ to an open subset of $\mathbb{C}^n$. Show that the set
$$R = \{x \in U \mid d\phi_x \text{ is not an isomorphism}\}$$
is defined in U by exactly one holomorphic equation.
This set is called the ramification divisor of ϕ, when it is different from U.
2. Let f be a holomorphic function defined over an open subset U of $\mathbb{C}^n$. We assume that f does not vanish outside the set
$$\{z = (z_1, \ldots, z_n) \in U \mid z_1 = z_2 = 0\}.$$
Show that f does not vanish at any point of U.
3. Let f be a meromorphic function defined on an open subset U of $\mathbb{C}$. This means that f is locally the quotient of two holomorphic functions.
(a) Show that for any compact subset $K \subset U$, the number of zeros or poles of f in K is finite.
(b) Let $x \in U$. Show that there exists an integer $k_x \in \mathbb{Z}$ such that f can be written as $(z - x)^{k_x}\phi$ in a neighbourhood of x, with ϕ holomorphic and invertible (that is non-zero).
The divisor of f is defined as the locally finite sum
$$\sum_{x \in U} k_x x.$$
(c) Let $x \in U$ and $D \subset U$ be a disk centred in x, such that x is the only pole or zero of f in D. Show that
$$k_x = \int_{\partial D} \frac{1}{2i\pi} \frac{df}{f}.$$