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Introduction

Kähler manifolds and projective manifolds. The goal of this first volume is to explain the existence of special structures on the cohomology of Kähler manifolds, namely, the Hodge decomposition and the Lefschetz decomposition, and to discuss their basic properties and consequences. The second volume will be devoted to the systematic application of these results in different directions, relating Hodge theory, topology and the study of algebraic cycles on smooth projective complex manifolds.

Indeed, smooth projective complex manifolds are special cases of compact Kähler manifolds. A Kähler manifold is a complex manifold equipped with a Hermitian metric whose imaginary part, which is a 2-form of type $(1, 1)$ relative to the complex structure, is closed. This 2-form is called the Kähler form of the Kähler metric. As complex projective space (equipped, for example, with the Fubini–Study metric) is a Kähler manifold, the complex submanifolds of projective space equipped with the induced metric are also Kähler. We can indicate precisely which members of the set of Kähler manifolds are complex projective, thanks to Kodaira's theorem:

Theorem 0.1 *A compact complex manifold admits a holomorphic embedding into complex projective space if and only if it admits a Kähler metric whose Kähler form is of integral class.*

In this volume, we are essentially interested in the class of Kähler manifolds, without particularly emphasising projective manifolds. The reason is that our goal here is to establish the existence of the Hodge decomposition and the Lefschetz decomposition on the cohomology of such a manifold, and for this, there is no need to assume that the Kähler class is integral. However, the Lefschetz decomposition will be defined on the rational cohomology only in the projective case, and this is already an important reason to restrict ourselves, later, to the case of projective manifolds. Indeed, in this text, we will introduce the notions of polarised Hodge structure and the polarised period domain parametrising polarised Hodge structures. These polarised period domains have curvature properties which the non-polarised period domains do not possess. The Lefschetz decomposition, when it is defined on the rational or integral cohomology, splits the cohomology of a Kähler manifold into a direct sum of polarised Hodge structures.

In studying the applications of Hodge theory, another reason to restrict ourselves to projective manifolds is the fact that a Kähler manifold does not, in general, have complex submanifolds, whereas projective manifolds have many, so many that in fact it is currently conjectured, as a vast generalisation of the Hodge conjecture, that the Hodge structures on a projective manifold X are governed by, and determine in a sense to be explained later, the geometry of the algebraic subvarieties of X, and more precisely the Chow groups of X.

The Hodge decomposition. If X is a complex manifold, the tangent space to X at each point x is equipped with a complex structure J_x. The data consisting of this complex structure at each point is what is known as the underlying almost complex structure. The J_x provide a decomposition

$$T_{X,x} \otimes \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}, \tag{0.1}$$

where $T^{0,1}_{X,x}$ is the vector space of complexified tangent vectors $u \in T_{X,x}$ such that $J_x u = -iu$ and $T^{1,0}_{X,x}$ in the complex conjugate of $T^{0,1}_{X,x}$. From the point of view of the complex structure, i.e. of the local data of holomorphic coordinates, the vector fields of type (0, 1) are those which kill the holomorphic functions.

The decomposition (0.1) induces a similar decomposition on the bundles of complex differential forms

$$\Omega^k_{X,\mathbb{C}} := \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C} = \bigoplus\nolimits_{p+q=k} \Omega^{p,q}_X, \tag{0.2}$$

where

$$\Omega^{p,q}_X \cong \bigwedge\nolimits^p \Omega^{1,0}_X \otimes \bigwedge\nolimits^q \Omega^{0,1}_X$$

and

$$\Omega_{X,\mathbb{R}} \otimes \mathbb{C} = \Omega^{1,0}_X \oplus \Omega^{0,1}_X$$

is the dual decomposition of (0.1). The decomposition (0.2) has the property of Hodge symmetry

$$\overline{\Omega^{p,q}_X} = \Omega^{q,p}_X,$$

where complex conjugation acts naturally on $\Omega^k_{X,\mathbb{C}} = \Omega^k_{X,\mathbb{R}} \otimes \mathbb{C}$.

If we let $A^k_{\mathbb{C}}(X)$ denote the space of complex differential forms of degree k on X, i.e. the $\mathcal{C}^\infty$ sections of the vector bundle $\Omega^k_{X,\mathbb{C}}$, then we also have the exterior differential

$$d : A^k_{\mathbb{C}}(X) \to A^{k+1}_{\mathbb{C}}(X),$$

which satisfies $d \circ d = 0$. We then define the kth de Rham cohomology group of X by

$$H^k(X, \mathbb{C}) = \frac{\operatorname{Ker}\left(d : A^k_{\mathbb{C}}(X) \to A^{k+1}_{\mathbb{C}}(X)\right)}{\operatorname{Im}\left(d : A^{k-1}_{\mathbb{C}}(X) \to A^k_{\mathbb{C}}(X)\right)}.$$

The main theorem proved in this book is the following.

Theorem 0.2 *Let $H^{p,q}(X) \subset H^k(X, \mathbb{C})$ be the set of classes which are representable by a closed form α which is of type (p, q) at every point x in the decomposition (0.2). Then we have a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus\nolimits_{p+q=k} H^{p,q}(X). \tag{0.3}$$

Note that by definition, we have the Hodge symmetry

$$H^{p,q}(X) = \overline{H^{q,p}(X)},$$

where complex conjugation acts naturally on $H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C}$. Here $H^k(X, \mathbb{R})$ is defined by replacing the complex differential forms by real differential forms in the above definition.

This theorem immediately gives constraints on the cohomology of a Kähler manifold, which reveal the existence of compact complex manifolds which are not Kähler. For example, the decomposition (0.3) and the Hodge symmetry imply that the dimensions $\dim_{\mathbb{C}} H^k(X, \mathbb{C})$ (called the Betti numbers) are even for odd k, a property not satisfied by Hopf surfaces. These surfaces are the quotients of $\mathbb{C}^2 - \{0\}$ by the fixed-point-free action of a group isomorphic to $\mathbb{Z}$, where a generator g acts via

$$g(z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2),$$

where the λ_i are non-zero complex numbers of modulus strictly less than 1. These surfaces are compact, equipped with the quotient complex structures, and their π_1 is isomorphic to $\mathbb{Z}$ since $\mathbb{C}^2 - \{0\}$ is simply connected. Thus, their first Betti number is equal to 1, which implies that they are not Kähler.

The Lefschetz decomposition. The Lefschetz decomposition is another decomposition of the cohomology of a compact Kähler manifold X, this time of a topological nature. It depends only on the cohomology class of the Kähler form

$$[\omega] \in H^2(X, \mathbb{R}).$$

The exterior product on differential forms satisfies Leibniz' rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{d^0\alpha}\alpha \wedge d\beta,$$

so the exterior product with ω sends closed forms (i.e. forms killed by d) to closed forms and exact forms (i.e. forms in the image of d) to exact forms. Thus it induces an operator, called the Lefschetz operator,

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R}).$$

The following theorem is sometimes called the hard Lefschetz theorem.

Theorem 0.3 *For every $k \leq n = \dim X$, the map*

$$L^{n-k} : H^k(X, \mathbb{R}) \to H^{2n-k}(X, \mathbb{R}) \tag{0.4}$$

is an isomorphism.

(Note that the spaces on the right and on the left are of the same dimension by Poincaré duality, which is valid for all compact oriented manifolds.)

A very simple consequence of the above isomorphism is the following result, which is an additional topological constraint satisfied by Kähler manifolds.

Corollary 0.4 *The morphism*

$$L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$$

is injective for $k < n = \dim X$. Thus, the odd Betti numbers $b_{2k-1}(X)$ increase with k for $2k - 1 \leq n$, and similarly, the even Betti numbers $b_{2k}(X)$ increase for $2k \leq n$.

An algebraic consequence of Lefschetz' theorem is the Lefschetz decomposition, which as we noted earlier is particularly important in the case of projective manifolds. Let us define the primitive cohomology of a compact Kähler manifold X by

$$H^k(X, \mathbb{R})_{\mathrm{prim}} := \operatorname{Ker}(L^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n-k+2}(X, \mathbb{R}))$$

for $k \leq n$. (One can extend this definition to the cohomology of degree $> n$ by using the isomorphism (0.4).)

Theorem 0.5 *The natural map*

$$i : \bigoplus_{k-2r\geq 0} H^{k-2r}(X, \mathbb{R})_{\text{prim}} \to H^k(X, \mathbb{R})$$
$$(\alpha_r) \mapsto \sum_r L^r \alpha_r$$

is an isomorphism for $k \leq n$.

Once again, we can extend this decomposition to the cohomology of degree $> n$ by using the isomorphism (0.4).

Harmonic forms and cohomology. Let us now express the main principle of Hodge theory, which has immense applications. The study of the cohomology of Kähler manifolds and the proof of the theorems 0.2 and 0.3, which are the main content of this book, are among the most important applications, but the principle applies in various other situations. The vanishing theorems for the cohomology of line bundles equipped with a Chern connection with positive curvature, whose proofs will only be sketched here, provide another example of possible applications. The applications to the topology of compact Riemannian manifolds under certain curvature hypotheses are also very important, but they lie outside of the scope of this book.

Following Weil (1957), we restrict ourselves here to giving an explanation of the main idea, which is the notion of a harmonic form, and the application of the theory of elliptic operators which makes it possible to represent the cohomology classes by harmonic forms, but we will omit the proof of the fundamental theorem on elliptic operators, which uses estimations and notions from analysis (Sobolev spaces), which are in different directions from the aims of this book. The delicate point consists in passing from spaces of L^2 differential forms, in which the Hodge decomposition is algebraically obvious, to spaces of $\mathcal{C}^\infty$ differential forms. One of the problems we encounter is the fact that the operators considered here are differential operators, and thus do not define continuous operators on the spaces of L^2 forms. We refer to Demailly (1996) for a presentation of this analytic aspect of Hodge theory.

The idea that we want to explain here is the following: using the metric on X, we can define the L^2 metric on the spaces of differential forms

$$(\alpha, \beta)_{L^2} = \int_X \langle \alpha, \beta \rangle_x \mathrm{Vol},$$

where α, β are differential forms of degree k and the scalar product $\langle \alpha, \beta \rangle_x$ at a point $x \in X$ is induced by the evaluation of the forms at the point x and by the metric at the point x.

The operator $d : A^k(X) \to A^{k+1}(X)$ is a differential operator, and we can construct its formal adjoint $d^* : A^k(X) \to A^{k-1}(X)$, which is also a differential operator, and satisfies the identity

$$(\alpha, d\beta)_{L^2} = (d^*\alpha, \beta)_{L^2}$$

for $\alpha \in A^k(X)$, $\beta \in A^{k-1}(X)$. This adjunction relation only makes d^* into a formal adjoint, since these operators are not defined on the Hilbert space $L^2(\Omega^*_X)$ of L^2 differential forms, which is the completion of $A^*(X)$ for the L^2 metric.

The idea of Hodge theory consists in using the adjoint d^* to write the decompositions

$$A^k(X) = \operatorname{Im} d \oplus \operatorname{Im} d^{\perp} = \operatorname{Im} d \oplus \operatorname{Ker} d^*,$$
$$A^k(X) = \operatorname{Ker} d \oplus \operatorname{Ker} d^{\perp} = \operatorname{Ker} d \oplus \operatorname{Im} d^*,$$

and finally, using the inclusion $\operatorname{Im} d \subset \operatorname{Ker} d$,

$$A^k(X) = \operatorname{Im} d \oplus \operatorname{Im} d^* \oplus \operatorname{Ker} d \cap \operatorname{Ker} d^*.$$

Of course, these identities, which would be valid on finite-dimensional spaces or Hilbert spaces since the operator d has closed image there, require the analysis mentioned above in order to justify them here.

Apart from this issue, if we accept these identities, we see that the space

$$\mathcal{H}^k := \operatorname{Ker} d \cap \operatorname{Ker} d^* \subset A^k(X)$$

of harmonic forms projects bijectively onto $H^k(X, \mathbb{R})$ (or $H^k(X, \mathbb{C})$ if we study the cohomology with complex coefficients), since it is a supplementary subspace of $\operatorname{Im} d$ inside $\operatorname{Ker} d$.

Another characterisation of harmonic forms uses the Laplacian

$$\Delta_d = dd^* + d^*d.$$

Indeed, it is very easy to see that we have

$$\mathcal{H}^k = \operatorname{Ker} \Delta_d.$$

The operator Δ_d is an elliptic operator. This property of a differential operator can be read directly from its symbol, which is essentially its homogeneous term of largest order (which is 2 for the Laplacian). The decompositions written above are special cases of the decomposition associated to an elliptic operator.

Kähler identities. The Hodge decomposition (0.3) is obtained by combining the Hodge theory sketched above and the study of the properties of the Laplacian of a Kähler manifold. We have already mentioned various operators acting on the spaces of differential forms of a Kähler manifold, namely d, L and their formal adjoints d^*, Λ for the L^2 metric. Moreover, the complex structure makes it possible to decompose d as

$$d = \partial + \overline{\partial},$$

where the Dolbeault operator $\overline{\partial}$ sends $\alpha \in A^{p,q}(X)$ to the component of bidegree $(p, q+1)$ of $d\alpha$. Here $A^{p,q}(X)$ is the space of differential forms of bidegree (p, q) at every point of x; it is also the space of sections of the bundle $\Omega_X^{p,q}$ which appears in the decomposition (0.2) given by the complex structure. The differential operators ∂ and $\overline{\partial}$ are differential operators of order 1, and have formal adjoint operators ∂^* and $\overline{\partial}^*$.

The Kähler identities establish commutation relations between these operators. For example, we have the identity

$$[\Lambda, \partial] = i\overline{\partial}^*,$$

and the other identities follow from this one via passage to the complex conjugate or to the adjoint.

From these identities, and from the fact that L commutes with d while ∂ and $\overline{\partial}$ anticommute, we deduce the following result.

Theorem 0.6 *The Laplacians Δ_d, Δ_∂ and $\Delta_{\overline{\partial}}$ associated to the operators d, ∂ and $\overline{\partial}$ respectively satisfy the equalities*

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}. \tag{0.5}$$

We deduce that the harmonic forms for d are also harmonic for ∂ and $\overline{\partial}$, and in particular are also ∂- and $\overline{\partial}$-closed. Finally, as the operators ∂ and $\overline{\partial}$ are bihomogeneous (of bidegree $(1, 0)$ and $(0, 1)$ respectively) for the bigraduation of the spaces of differential forms given by the decomposition (0.2), it follows easily that each of the Laplacians Δ_∂ and $\Delta_{\overline{\partial}}$ is bihomogeneous of bidegree $(0, 0)$, i.e. preserves the forms of type (p, q) for every bidegree (p, q). The same then holds for Δ_d by the equality (0.5). The Hodge decomposition is then obtained simply by the decomposition of the harmonic forms as sums of forms of type (p, q):

Corollary 0.7 *Let X be a compact Kähler manifold. If ω is a harmonic form (for the Laplacian associated to the operator d and to the metric), its components of type (p, q) are harmonic. Thus, we have a decomposition*

$$\mathcal{H}^k(X) = \bigoplus \mathcal{H}^{p,q}, \tag{0.6}$$

where $\mathcal{H}^{p,q}$ is the space of harmonic forms of type (p, q) at every point of X.

The Hodge decomposition (0.3) is obtained by combining the theorem of representation of cohomology classes by harmonic forms with the decomposition (0.6).

The Lefschetz decomposition is also an easy consequence of the decomposition (0.6). Indeed, we first show that theorem 0.3 holds for the operator L acting on differential forms. Furthermore, the Kähler identities show that L commutes with the Laplacian, so that the operators L^r send harmonic forms to harmonic forms, and once the theorem is proved on the level of forms, it remains valid on the level of harmonic forms, and thus also on cohomology classes.

De Rham cohomology and Betti cohomology. The Hodge decomposition (0.3) gives an extremely interesting structure when it is combined with the integral structure on the cohomology

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

For this equality, which follows from the change of coefficients theorem, one must adopt a different definition of cohomology, which does not make use of differential forms.

For one possible definition, we can introduce the singular cohomology

$$H^k_{\text{sing}}(X, \mathbb{Z}).$$

We start from the complex

$$C_*(X), \quad \partial : C_k(X) \to C_{k-1}(X)$$

of singular chains, where $C_k(X)$ is the free abelian group generated by the continuous maps from the simplex Δ_k of dimension k to X. The map ∂ is given by the restriction to the boundary

$$\partial\phi = \sum_i (-1)^i \phi_{|\partial\Delta_{k,i}},$$

where $\Delta_{k,i}$ is the ith face of Δ_k. The complex $(C^*_{\text{sing}}(X), d)$ of singular cochains is then defined as the dual complex of $(C_*(X), \partial)$. Its cohomology is the singular cohomology $H^*_{\text{sing}}(X, \mathbb{Z})$. We have the following theorem, due to de Rham.

Theorem 0.8 *For $K = \mathbb{R}$ or $K = \mathbb{C}$, we have*

$$H^k(X, K) = H^k_{\text{sing}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} K.$$

If we consider the complex of differentiable chains, we can prove this theorem by using the natural map from $A^k(X)$ to $C^k_{\text{sing}}(X)$ given by integration:

$$\alpha \mapsto \left(\phi \mapsto \int_{\Delta_k} \phi^*\alpha\right).$$

Sheaves and cohomology. A much more conceptual proof of de Rham's theorem can be given by using the language of sheaf theory and sheaf cohomology, which we present here, and whose usefulness will appear frequently throughout this book: it will be used, for example, in the Hodge decomposition, to describe the spaces $H^{p,q}$ as the Dolbeault cohomology groups $H^q(X, \Omega_X^p)$, which are defined for every complex manifold X as the qth cohomology group of X with values in the sheaf Ω_X^p of holomorphic differential forms of degree p. (Note, however, that this identification is valid only in the Kähler case. In general, without the Kähler hypothesis, we cannot identify $H^q(X, \Omega_X^p)$ with the space of cohomology classes of degree $p + q$ which are representable by a closed form of type (p, q) at every point.)

The notion of a sheaf $\mathcal{F}$ (of groups, for example) over a topological space X is a set-theoretic notion. It is given by the following data: the group $\mathcal{F}(U)$ of "sections of $\mathcal{F}$ over U" for every open subset U of X, and restriction maps

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

for every inclusion $V \subset U$. These restrictions are compatible in an obvious way when we take three open sets $W \subset V \subset U$. We also require that a section from $\mathcal{F}$ to U is determined exactly by its restrictions to the open subsets of an open cover of U, which of course must coincide on the intersections of two of these open sets. The first typical example of a sheaf is the sheaf of local sections of a vector bundle over X. Another example is given by the constant sheaves of stalk G, where G is a fixed abelian group; to an open set U, we associate the locally constant maps defined on U with values in G.

The sheaves of abelian groups over X form an abelian category which has "sufficiently many injective objects" (see chapter 4). Thus, the theory of derived functors applies to this category. The main functors which interest us here are the functors of global sections Γ of the category of sheaves of abelian groups on X to the category of abelian groups, or the direct image functor from the category of sheaves of abelian groups on X to the category of sheaves of abelian groups on Y, for a continuous map $\phi : X \to Y$.

These functors are left-exact. Given a left-exact functor $F : \mathcal{A} \to \mathcal{B}$ of an abelian category $\mathcal{A}$ having sufficiently many injective objects to an abelian category $\mathcal{B}$, we define $R^i F(M)$, $M \in \mathcal{O}b(\mathcal{A})$ as the ith cohomology group of the complex $(F(M\cdot), d)$, where $(M\cdot, d)$ is an injective resolution of M. In fact, more generally, we can take resolutions by F-acyclic objects. The important point is that given two such resolutions, we have a canonical isomorphism between the objects $R^i F(M)$ calculated via the two resolutions.

Returning to the case of the functor of global sections Γ, we show using Poincaré's theorem that the sheaves of differential forms form a Γ-acyclic resolution of the constant sheaf $\mathbb{C}_X$ (often written $\mathbb{C}$) of stalk $\mathbb{C}$, so that the space $H^k(X, \mathbb{C})$ defined above must be understood as the kth cohomology group of X with values in $\mathbb{C}_X$, i.e. $R^k\Gamma(\mathbb{C}_X)$. Similarly, we can interpret the singular cohomology as the cohomology of the complex of global sections of a Γ-acyclic resolution of the constant sheaf of stalk $\mathbb{Z}$. Thus, we have $H^k_{\mathrm{sing}}(X, \mathbb{Z}) = H^k(X, \mathbb{Z})$ canonically.

De Rham's theorem thus reduces to proving a change of coefficients theorem for the cohomology of the sheaves

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C},$$

which is not difficult.

These different interpretations of the cohomology, corresponding to different resolutions, are all equally important, since they carry different types of information. For example, the Hodge decomposition of the cohomology of a Kähler manifold requires the de Rham version of the cohomology, while that of the integral structure requires another version, singular or Čech for example.

The Frölicher spectral sequence. With the exception of the statement concerning the Hodge symmetry, the theorem of Hodge decomposition can be reformulated as a theorem of degeneracy of a spectral sequence. The justification of this reformulation, particularly in the case of projective manifolds, is that it consists in a completely algebraic translation, where in fact we may even use Serre's "GAGA" principle of comparison of algebraic geometry and analytic geometry to replace the sheaves of holomorphic differential forms and their cohomology relative to the usual topology by sheaves of algebraic differential forms and their cohomology relative to the Zariski topology. Thus, we can almost give meaning to Hodge's theorem 0.2 for smooth projective manifolds defined over an arbitrary field. Under certain "lifting" hypotheses, Deligne and Illusie prove this statement for manifolds in non-zero characteristic (Illusie 1996).

The differentiable de Rham complex of a differentiable manifold, i.e. the complex of sheaves of differential forms equipped with the exterior differential,