Cambridge University Press 0521801648 - Harmonic Superspace A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev Excerpt More information

1 Introductory overview

We start by overviewing the origins, motivations, basic ideas and results of the harmonic superspace (and space) approach. Our major aim here is to give the reader a preliminary impression of the subject before immersion into the main body of the book.

1.1 Brief motivations

It is hardly possible to overestimate the rôle of symmetries in the development of physics. The place they occupy is becoming more and more important every year. The very family of symmetries is getting richer all the time: Besides the old symmetries based on Lie algebras we are now exploiting new kinds of symmetries. These include supersymmetries which mix bosons with fermions and are based on superalgebras, symmetries associated with non-linear algebras of Zamolodchikov's type, symmetries connected to quantum groups, etc. To date, the supersymmetric models have been studied in most detail. They turn out to have quite remarkable features. They open a new era in the search for a unified theory of all interactions including gravity. They help to solve the hierarchy problem in the grand unification theories. For the first time in the history of quantum field theory, supersymmetry has led to the discovery of a class of ultraviolet-finite local four-dimensional field theories. In these finite theories the ultraviolet divergences in the boson and fermion loops 'miraculously' cancel against each other. Supersymmetries underlie the superstring theories, which provide the first consistent scheme for quantization of gravity. The research programs of the leading accelerator laboratories include searches for supersymmetric partners of the known particles (predicted by supersymmetry but not yet discovered).

In view of this impressive development, it is imperative to be able to formulate the supersymmetric theories in a systematic, consistent and clear way. There already exist several reviews [F2, F3, F4, F7, N2, O4, S13, V1]

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and textbooks [B17, G26, W7, W12] devoted to the simplest kind of supersymmetry, N = 1 (i.e., containing one spinor generator in its superalgebra). It was this supersymmetry that was first discovered in the pioneering articles [G38, G39, V4, V5, V6, W8]. The superfield approach appropriate to this case was developed in the 1970s. However, extended supersymmetries (i.e., those containing more than one spinor generator) turned out much more difficult. Each new step in understanding them requires new notions and approaches. Even in the simplest extended N = 2 supersymmetry, until 1984 no way to formulate all such theories off shell, in a manifestly supersymmetric form and in terms of unconstrained superfields, was known. Such formulations are preferable not only because of their intrinsic beauty, but also since they provide an efficient technique, in particular, in quantum calculations or in the proof of finiteness. The invention of a new, harmonic superspace [G4, G13] made it possible to develop off-shell unconstrained formulations of all the N = 2 supersymmetric theories (matter, Yang–Mills and supergravity) and of N = 3 Yang–Mills theory.

N = 2 harmonic superspace is standard superspace augmented by the two-dimensional sphere $S^2 \sim SU(2)/U(1)$. In such an enlarged superspace it is possible to introduce a new kind of analyticity, Grassmann harmonic [G4, G13]. This proved to be the key to the adequate off-shell unconstrained formulations, just like chirality [F13], the simplest kind of Grassmann analyticity [G8], is a keystone in N = 1 supersymmetry. This new analyticity amounts to the existence of an analytic subspace of harmonic superspace whose odd dimension is half of that of the full superspace. All N = 2 theories mentioned above are naturally described by *Grassmann analytic* superfields, i.e., the unconstrained superfields in this subspace. A similar kind of analyticity underlies the N = 3 gauge theory [G5, G6].

A most unusual and novel feature of the analytic superfields is the unavoidable presence of infinite sets of auxiliary and/or gauge degrees of freedom in their component expansions. They naturally emerge from the harmonic expansions on the two-sphere S^2 with respect to a new sort of bosonic coordinates, the harmonic variables, which describe S^2 in a parametrization-independent way. These infinite sets, instead of being a handicap, proved to be very helpful indeed. It is due to their presence in the analytic superfield describing the N = 2 scalar multiplet (hypermultiplet) [F1, S12] off shell that one can circumvent the so-called 'no-go' theorem [H18, S21] claiming that such a formulation is not possible. In fact, the no-go theorems always implicitly assume the existence of a *finite* set of auxiliary fields.

The Grassmann analytic superfields with their infinite towers of components can be handled in much the same way as ordinary superfields, using a set of simple rules and tools. In [G14, G15] we worked out the quantization scheme for the N = 2 matter and gauge theories in harmonic superspace. The crucial importance of formulating quantum perturbation theory in supersymmetric

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models in terms of *unconstrained* off-shell superfields has repeatedly been pointed out in the literature (see, e.g., [H16]). Such formulations allow one to understand the origin of many remarkable properties of quantum supersymmetric theories which seem miraculous in the context of the component or constrained superfield formulations. Above all, this concerns the cancellation of ultraviolet divergences. Harmonic superspace is the only known approach which provides unconstrained off-shell formulations of both the matter and gauge N = 2 multiplets and as such it is indispensable for quantum calculations in the theories involving these multiplets. A particular representative of this class of theories is N = 4 super-Yang–Mills theory which, from the N = 2perspective, is just the minimal coupling of the hypermultiplet in the adjoint representation of the gauge group to the N = 2 super-Yang–Mills multiplet.

It is worthwhile to emphasize that the harmonic superspace approach is very close to the twistor one which is an effective tool for solving the self-dual Yang–Mills and Einstein equations. In fact, harmonic superspace could be regarded as an isotwistor superspace. However, even when applied to the purely bosonic self-duality problems, the harmonic space approach has some advantages, one of them being as follows. We use harmonics (the fundamental isospin 1/2 spherical functions) as abstract global coordinates spanning the whole two-sphere. This is in contrast with, e.g., polar or stereographic coordinates which require two charts on the sphere. So, if one succeeds in solving a self-duality equation in terms of harmonics, there will be no need to attack the famous Riemann–Hilbert problem which is central in conventional twistor approaches. We also wish to stress that the harmonic (super)space formalism heavily uses the Cartan coset technique, transparent and familiar to many physicists.

A surge of interest in the harmonic superspace methods and, above all, in the methods for off-shell quantum calculations was mainly motivated by two remarkable developments in our understanding of supersymmetric field theories during the 1990s.

The first one stems from the seminal paper by Seiberg and Witten [S5] where it was suggested that N = 2 gauge theories are exactly solvable at the full quantum level under some reasonable hypotheses like *S* duality intimately related to extended supersymmetry [W17]. The study of the structure of the quantum low-energy effective actions of N = 2 gauge theories, in both the perturbative and non-perturbative sectors, is of great importance in this respect. The quantum harmonic superspace methods were successfully applied for this purpose, in particular for computing the holomorphic and non-holomorphic contributions to the effective action (see [B14, B15, B16, I8] and references therein).

The second source of interest is the famous Maldacena AdS/CFT conjecture [G42, M1, W16]. This is the idea that the quantum N = 4 super Yang–Mills theory in the limit of large number of colors and strong coupling is dual to the type IIB superstring on $AdS_5 \times S^5$ and contains the corresponding supergravity

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as a sub-sector of its Hilbert space. This conjecture greatly stimulated thorough analysis of the structure of this exceptional gauge theory from different points of view using different calculational means. The harmonic superspace methods, as was shown in several recent papers [E1, E3, E4, E5, H14], can drastically simplify the calculations and allow one to make far reaching predictions in N = 4 super-Yang–Mills theory.

All this justifies the need for a comprehensive introduction to the harmonic superspace approach. We hope that the present book will meet, at least partly, this quest. Here we do not discuss the latest developments but prefer to concentrate on the basics of the harmonic superspace method. Some developments are briefly addressed in the Conclusions. When reading this book one may find it helpful to consult the reviews and books mentioned above. We also point out that there are a few papers devoted to the mathematical aspects of harmonic superspace and, in particular, to a more rigorous definition of it, e.g., [H3, H10, H12, R6, S4]. We do not address these special issues in our rather elementary exposition.

1.2 Brief summary

The present book has been conceived as a pedagogical review of all the extended supersymmetric N = 2 theories and of N = 3 Yang–Mills theory in the framework of harmonic superspace. The details of these theories are discussed, as well as some applications. A special emphasis is put on their geometrical origin and on the relationship with hyper-Kähler and quaternionic complex manifolds which appear as the target manifolds of N = 2 supersymmetric sigma models in a flat background and in the presence of supergravity, respectively. The Cartan coset techniques are used systematically with emphasis on their power and simplicity. The self-duality Yang–Mills and Einstein equations are treated in this language with stress on their deep affinity with N = 2 supersymmetric theories and on comparing the harmonic space approach with the twistor one.

A detailed outline of the content of this book is given at the end of Chapter 1. In order to help the reader, we preface the main body of the book with an overview of the basic ideas, notions and origins. We begin with a discussion of spaces and superspaces for the realization of symmetries and supersymmetries, emphasizing the importance of making the right choice: *The same symmetry can be realized in different ways, one of them being much more appropriate for a given problem than the others.*

1.3 Spaces and superspaces

Manifestly invariant formulations of field theories make use of some space (or

1.3 Spaces and superspaces 5

superspace) where a given symmetry (or supersymmetry) is realized geometrically by coordinate transformations. Two examples are well known:

(i) In Minkowski space $\mathbb{M}^4 = (x^a)$ the Poincaré group transformations have the form

$$x^{\prime a} = \Lambda^a_b x^b + c^a \,. \tag{1.1}$$

In classical and quantum field theories the action principle and the equations of motion are manifestly invariant under (1.1), the form of the corresponding field transformations being completely fixed by (1.1) and the tensor properties of the field, e.g.,

$$f'(x') = f(x)$$
 (1.2)

for a scalar field f(x). It is important that this transformation law does not depend on the model under consideration.

(ii) One usually attempts to formulate manifestly invariant *N*-extended supersymmetric theories in the standard superspace [S1]

$$\mathbb{R}^{4|4N} = (x^a, \theta_i^{\alpha}, \bar{\theta}^{\dot{\alpha}i}), \qquad i = 1, 2, \dots, N$$
(1.3)

involving the spinor *anticommuting* coordinates θ_i^{α} , $\bar{\theta}^{\dot{\alpha}i}$ in addition to x^a . Their transformation rules under the Poincaré group are evident, while the transformations under supersymmetry (supertranslations with anticommuting parameters ϵ_i^{α} , $\bar{\epsilon}^{\dot{\alpha}i}$) are given by

$$\delta x^{a} = i(\epsilon^{i}\sigma^{a}\bar{\theta}_{i} - \theta^{i}\sigma^{a}\bar{\epsilon}_{i}), \qquad \delta \theta^{\alpha}_{i} = \epsilon^{\alpha}_{i}, \qquad \delta \bar{\theta}^{\dot{\alpha}i} = \bar{\epsilon}^{\dot{\alpha}i}.$$
(1.4)

Superfields $\Phi(x, \theta, \overline{\theta})$ are defined as functions on this superspace and their transformation law is completely determined by (1.2). For example, for a scalar superfield

$$\Phi'(x',\theta',\theta') = \Phi(x,\theta,\theta).$$
(1.5)

Of course, this law is model-independent. Expanding a superfield $\Phi(x, \theta, \overline{\theta})$ in powers of the spinor (anticommuting, hence nilpotent) variables $\theta, \overline{\theta}$ yields a finite set of usual fields $f(x), \psi^{\alpha}(x), \ldots$, called components of the superfield.

As an alternative to $\mathbb{R}^{4|4N}$, *N*-extended supersymmetry can also be realized in the so-called chiral superspace $\mathbb{C}^{4|2N}$ which is *complex* and contains only *half* of the spinor coordinates [F13]:

$$\delta x_L^a = -2i\theta_L^i \sigma^a \bar{\epsilon}_i \,, \qquad \delta \theta_{Li}^\alpha = \epsilon_i^\alpha \,. \tag{1.6}$$

In fact, the real superspace $\mathbb{R}^{4|4N}$ can be viewed as a real hypersurface in the complex superspace $\mathbb{C}^{4|2N}$:

$$x_L^a = x^a + i\theta^i \sigma^a \bar{\theta}_i , \qquad \theta_{Li}^\alpha = \theta_i^\alpha , \qquad \overline{\theta_{Li}^\alpha} = \bar{\theta}^{\dot{\alpha}\,i} . \tag{1.7}$$

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1.4 Chirality as a kind of Grassmann analyticity

The superfields $\Phi(x_L, \theta_L) = \Phi(x + i\theta\sigma\bar{\theta}, \theta)$ defined in $\mathbb{C}^{4|2N}$ can be treated as Grassmann analytic superfields. Indeed, they obey the constraint

$$\bar{D}_{\dot{\alpha}i}\Phi = \left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} - i(\theta_i\sigma^a)_{\dot{\alpha}}\frac{\partial}{\partial x^a}\right)\Phi = 0, \qquad (1.8)$$

where $\bar{D}^i_{\dot{\alpha}}$ is the covariant (i.e., commuting with the supersymmetry transformations) spinor derivative. In the basis $(x_L, \theta_L, \bar{\theta})$ this derivative simplifies to $\bar{D}^i_{\dot{\alpha}} = -\partial/\partial \bar{\theta}^{\dot{\alpha}i}$. Then the constraint (1.8) takes the form of a Cauchy–Riemann condition,

$$\frac{\partial \Phi}{\partial \bar{\theta}_i^{\dot{\alpha}}} = 0, \qquad (1.9)$$

which means that Φ is a function of θ_L but is independent of $\overline{\theta}$ (cf. the standard theory of analytic functions where the Cauchy–Riemann condition $\partial f(z)/\partial \overline{z} = 0$ means that the function depends on the variable z and is independent of its conjugate \overline{z}). The notion of Grassmann analyticity [G8] in this simplest form is most useful in N = 1 supersymmetry. In this book the reader will see that there exist non-trivial generalizations of this concept which underlie the N = 2 and N = 3 supersymmetric theories.

It should by emphasized that finding the adequate superspace for a given theory is, as a rule, a non-trivial problem. The above superspaces $\mathbb{R}^{4|4N}$ and $\mathbb{C}^{4|2N}$ prove to be appropriate for off-shell formulations only in the simplest case of N = 1 supersymmetry. These 'standard' superspaces cease to be so useful in the extended (N > 1) supersymmetric theories. Finding and using the adequate superspaces for N = 2, 3 is the main subject of this book.

Now, before approaching the main problem, we recall in a few words some key points in N = 1 supersymmetry.

1.5 N = 1 chiral superfields

As already said, N = 1 supersymmetric theories can be formulated in the superspaces $\mathbb{R}^{4|4}$ or $\mathbb{C}^{4|2}$. Consider, for example, the simplest N = 1 supermultiplet, the matter one. On shell it contains a spin 1/2 field ψ_{α} and a complex scalar field A(x). In $\mathbb{R}^{4|4}$ it can be described by a scalar superfield $\Phi(x, \theta, \overline{\theta})$. However, the latter involves too many fields in its θ expansion: Four real scalars, two Majorana spinors and a vector of various dimensions. To eliminate the extra fields, it is necessary to impose a *constraint* on the superfield, which turns out to be just the chirality (Grassmann analyticity) condition

$$\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{1.10}$$

As explained above, this constraint means that Φ is an analytic superfield. In the N = 1 case the expansion of such a superfield (written down in the chiral basis)

1.6 Auxiliary fields

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is very short [F13]:

$$\Phi(x,\theta,\bar{\theta}) = \Phi(x_L,\theta_L) = \phi(x_L) + \theta_L^{\alpha} \psi_{\alpha}(x_L) + \theta_L^{\alpha} \theta_{L\alpha} F(x_L).$$
(1.11)

The fields ϕ , ψ_{α} , F form the off-shell N = 1 matter supermultiplet.

The chiral (N = 1 analytic) superspace $\mathbb{C}^{4|2}$ is the cornerstone of all the N = 1 theories: They are either formulated in terms of chiral superfields (matter and its self-couplings) or are based on gauge principles which respect chirality (Yang–Mills and supergravity and their couplings to matter). The reader will see that for N = 2 and N = 3 the suitably modified concept of Grassmann analyticity will also be crucial.

1.6 Auxiliary fields

Besides the physical fields $\phi(x_L)$, $\psi_{\alpha}(x_L)$, the superfield $\Phi(x_L, \theta_L)$ also contains an *auxiliary* complex scalar field $F(x_L)$ of non-physical dimension 2. As a consequence, this field can only appear in an action without derivatives and thus can be eliminated by its equation of motion. In the presence of auxiliary fields the supersymmetry transformations are *model-independent* and so have the same form off and on shell. They form a *closed* supersymmetry algebra. For example, in the case of the chiral scalar superfield above one obtains from (1.5), (1.6) and (1.11)

$$\begin{split} \delta\phi(x) &= -\epsilon^{\alpha}\psi_{\alpha}(x),\\ \delta\psi_{\alpha}(x) &= -2i\sigma^{a}_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\,\partial_{a}\phi(x) - 2\epsilon_{\alpha}F(x),\\ \delta F(x) &= -i\bar{\epsilon}^{\dot{\alpha}}\sigma^{a}_{\alpha\dot{\alpha}}\,\partial_{a}\psi^{\alpha}(x). \end{split}$$
(1.12)

The commutator of two such supertranslations yields an ordinary translation with a parameter composed in accordance with the supersymmetry algebra. (See Chapter 2 for more details on the realization of supersymmetry in terms of fields.)

Of course, one can find a realization of supersymmetry on the physical fields only, with the auxiliary fields eliminated by the equations of motion of a given model. In fact, the first known realizations of supersymmetry were of just such a kind, and it was to some extent an 'art' to simultaneously find the invariant action and the supersymmetry transformations leaving it invariant. In contrast with the transformations in the presence of auxiliary fields, now one has:

- (i) Supersymmetry transformations depending on the choice of the specific field model. They are in general non-linear and the structure of this non-linearity varies from one action to another.
- (ii) The algebra of these transformations closes only modulo the equations of motion, i.e., on shell. Such algebras are referred to as *open* or *soft*.

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These complications cause difficulties when trying to exploit the consequences of supersymmetry, in particular, in studying the ultraviolet behavior. Working in a manifestly invariant manner, in terms of the appropriate superfields, has undeniable advantages for such purposes. Note that some people prefer to avoid the use of superfields and instead work directly with the off-shell supermultiplets of fields including the auxiliary ones (e.g., $\phi(x)$, $\psi_{\alpha}(x)$, F(x) in our N = 1 example). Then one needs a set of rules for handling such multiplets, known as *tensor calculus*. The superfield approach automatically reproduces all such rules in a nice geometrical way. This concerns the composition rule for supermultiplets (it amounts to multiplication of superfields), the building of invariant actions, etc.

The reader should realize that the notion of auxiliary fields is not peculiar to supersymmetry, it also appears in the usual non-supersymmetric theories. For instance, the Coulomb field is auxiliary in quantum electrodynamics.

The auxiliary fields play an extremely important rôle in the theories with extended supersymmetry, their number there may even become infinite. The reader will learn from the present book that this is due to a new feature of the harmonic superspace: It involves auxiliary bosonic coordinates. This superspace of a new kind is the only one that provides us with a systematic tool for off-shell realizations of all the N = 2 extended supersymmetries and the N = 3 Yang–Mills theory.

1.7 Why standard superspace is not adequate for N = 2 supersymmetry

'Not adequate' means that in the framework of the standard superspaces $\mathbb{R}^{4|8}$ and $\mathbb{C}^{4|4}$ it is impossible to find off-shell actions for an unconstrained description of all the N = 2 supersymmetric theories. We illustrate this on the example of the Fayet–Sohnius matter hypermultiplet [F1, S12]. On shell this supermultiplet contains four scalar fields forming an SU(2) doublet $f^i(x)$ and two isosinglet spinor fields $\psi^{\alpha}(x)$, $\bar{\kappa}^{\dot{\alpha}}(x)$. To incorporate them as components of a standard superfield one has to use [S12] an isodoublet superfield $q^i(x, \theta, \bar{\theta})$ defined in $\mathbb{R}^{4|8}$. Due to the large number of spinor variables this superfield contains a lot of redundant field components in addition to the physical ones listed above. The extra fields are eliminated by imposing the constraint [S12]

$$D^{(i)}_{\alpha}q^{(j)} = \bar{D}^{(i)}_{\dot{\alpha}}q^{(j)} = 0, \qquad (1.13)$$

where (ij) means symmetrization and D^i_{α} , $\bar{D}^i_{\dot{\alpha}}$ are the supercovariant spinor derivatives obeying the algebra

$$\{D^{i}_{\alpha}, \bar{D}_{\dot{\alpha}j}\} = -2i\delta^{i}_{j}\sigma^{a}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{a}}$$
(1.14)

(for their precise definition see Chapter 3). These constraints eliminate the extra field components of q^i , leaving only the above physical fields (and their

1.8 Search for conceivable superspaces (spaces)

derivatives in the higher terms of the θ expansion):

$$q^{i}(x,\theta,\bar{\theta}) = f^{i}(x) + \theta^{i\alpha}\psi_{\alpha}(x) + \bar{\theta}^{i}_{\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}}(x) + \text{ derivative terms.}$$
(1.15)

However, at the same time the above constraints put all the physical fields on the free mass shell:

$$\Box f^{i}(x) = (\sigma^{a})^{\alpha \dot{\alpha}} \frac{\partial}{\partial x^{a}} \psi_{\alpha}(x) = \sigma^{a}_{\alpha \dot{\alpha}} \frac{\partial}{\partial x^{a}} \bar{\kappa}^{\dot{\alpha}}(x) = 0.$$
(1.16)

The reason for this is that the constraints (1.13) are not integrable off shell: The supercovariant spinor derivatives do not anticommute. Equations (1.16) follow from the constraints (1.13) and the algebra (1.14), taking into account the definitions

$$f^{i}(x) = q^{i}|_{\theta = \tilde{\theta} = 0}, \qquad \psi_{\alpha}(x) = \frac{1}{2}D^{i}_{\alpha}q_{i}|_{\theta = \tilde{\theta} = 0}, \qquad \kappa^{\dot{\alpha}(x)} = \frac{1}{2}\bar{D}^{\dot{\alpha}}_{i}q^{i}|_{\theta = \tilde{\theta} = 0}.$$
(1.17)

In order to extend this theory off shell and to introduce interactions it has been proposed to relax, in one way or another, the constraints (1.13) [H15, Y1]. However, according to the general no-go theorem [H18, S21] (see Chapter 2), this is impossible in the framework of the standard N = 2 superspaces $\mathbb{R}^{4|8}$ or $\mathbb{C}^{4|4}$ using a *finite number* of auxiliary fields (or, equivalently, a *finite number* of standard N = 2 superfields). A natural way out was to look for other superspaces.

1.8 Search for conceivable superspaces (spaces)

Above we saw that it is helpful to consider different superspaces even in the simplest case N = 1. For any (super)symmetry there exists a number of admissible (super)spaces. The inadequacy of the standard superspaces $\mathbb{R}^{4|8}$ and $\mathbb{C}^{4|4}$ for off-shell realizations of N = 2 supersymmetry suggested to start searching through the list of other available superspaces. This list is provided by the standard coset construction due to E. Cartan [C4]* that allows one to classify the different (super)spaces of some (super)group G and to handle them effectively. One has to examine the conceivable quotients (we prefer the term 'coset') G/H of the group G over some of its subgroups H. For instance, Minkowski space is the coset $\mathbb{M}^4 = \mathcal{P}/\mathcal{L} = (x^a)$ of the Poincaré group \mathcal{P} over its Lorentz subgroup \mathcal{L} . As we shall see later, the Poincaré group for the Euclidean space \mathbb{R}^4 can also be realized in another way, using the coset space $\mathcal{P}/SU(2) \times U(1)$, with $SU(2) \times U(1)$ being a subgroup of the rotation group $SO(4) = SU(2) \times SU(2)$. This space is closely related to the so-called *twistor space* (more precisely, the traditional twistor space is related by a similar procedure to the Poincaré group of the complexified Minkowski space \mathbb{M}^4).

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^{*} Subsequently rediscovered by physicists [C11, O3, V3].

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Analogously, the standard real superspaces $\mathbb{R}^{4|4N}$ are the coset spaces

$$\mathbb{R}^{4|4N} = \frac{Su\mathcal{P}_N}{\mathcal{L}} = (x^a, \theta_i^{\alpha}, \bar{\theta}^{\dot{\alpha}i}), \qquad (1.18)$$

where $Su\mathcal{P}_N$ is the *N*-extended super-Poincaré group involving the generators of the Poincaré group and the spinor supersymmetry generators Q_{α}^i , $\bar{Q}_{\alpha i}$. In the same way, the chiral superspaces are the following coset spaces

$$\mathbb{C}^{4|2N} = \frac{Su\mathcal{P}_N}{\{\mathcal{L}, \bar{Q}_{\dot{\alpha}i}\}} = (x^a, \theta_i^\alpha).$$
(1.19)

Note the important difference between (1.18) and (1.19). In the latter the stability *supergroup* contains half of the spinor generators in addition to the Lorentz group ones. In Chapter 3 the coset techniques [C4, C11, O3, V3] are presented in detail. These techniques provide simple rules on how to find explicit transformation laws, how to construct invariants making use of covariant derivatives (obtained from the appropriate Cartan forms), etc.

1.9 N = 2 harmonic superspace

Certainly, $\mathbb{R}^{4|4N}$ and $\mathbb{C}^{4|2N}$ do not exhaust the list of possible superspaces for realizations of *N*-extended supersymmetry. Let us briefly outline some general features of N = 2 harmonic superspace, our main topic of interest in this book.

The N = 2 superalgebra

$$\left\{\mathcal{Q}^{i}_{\alpha}, \bar{\mathcal{Q}}_{\dot{\alpha}j}\right\} = 2\delta^{i}_{j}(\sigma^{a})_{\alpha\dot{\alpha}}P_{a}, \qquad i, j = 1, 2$$
(1.20)

possesses an SU(2) group of automorphisms, Q_{α}^{i} , $\bar{Q}_{\alpha,j}$ being SU(2) doublets (indices *i*, *j*) and P_{a} being a singlet. In the standard case of eqs. (1.18) and (1.19) (with N = 2) this SU(2) can be viewed as present both in the numerator and the denominator, thus effectively dropping out. To obtain the harmonic superspace, one has to keep only the U(1) subgroup of SU(2) in the denominator instead of the whole SU(2):

$$\mathbb{H}^{4+2|8} = \frac{Su\mathcal{P}_2}{\mathcal{L}} \times \frac{SU(2)}{U(1)}.$$
(1.21)

In other words, one has to enlarge the N = 2 supersymmetry group by its automorphisms group SU(2) realized in the coset space SU(2)/U(1). The latter is a two-dimensional space known to have the topology of the two-sphere S^2 . So, harmonic superspace is a tensor product of $\mathbb{R}^{4|8}$ and a two-sphere S^2 .

1.10 Dealing with the sphere S^2

Before discussing the harmonic superspace as a whole it is instructive to study its much more familiar part SU(2)/U(1). Of course, one could choose polar