Chapter 1
Fluctuations, renormalization and universality

The idea of a book on defects and geometry in condensed matter physics slowly nucleated as I prepared contributions to various reviews, schools and workshops during the period 1982 to 1996. Although I have other interests in theoretical physics, I kept coming back at regular intervals to the statistical mechanics of defects and to related problems in the physics of flexible lines and surfaces. A consistent picture of these phenomena began to emerge and it transpired that work published in various forums during the early 1990s built on research I had described, e.g., at a summer school in the 1980s. Because considerable time and effort went into these reviews and all areas described are still active fields of investigation, it seemed reasonable to combine eight of them with this new introductory chapter in book form. The result, I hope, is a reasonably coherent account of the fascinating interplay among defects, geometry and statistical mechanics which has played such a central role in condensed matter physics during the past quarter century [1].

All chapters emphasize research in which I had a direct role. I have not attempted exhaustive reviews of these subjects and I apologize in advance to those whose work I have overlooked or neglected. These chapters are aimed at graduate students in physics, physical chemistry and chemical engineering as well as at more advanced researchers. Whenever possible, I tried to make the material intelligible to experimenters as well as theorists and to mention the many ingenious experiments which motivate the theories.

Condensed matter theorists owe a tremendous debt to our friends in the experimental community. They challenge us to *predict* and not

merely *postdict* experimental phenomena. Our experimental colleagues will go to great lengths to design an experiment capable of testing an interesting new theoretical idea. Theorists must be careful what they say, because predictions in condensed matter physics can often be tested in a matter of months with relatively inexpensive table-top experiments. We owe our colleagues in the laboratory a great debt because they keep us honest and inspire us with their beautiful experiments.

I also owe a special personal debt to the numerous theoretical graduate students, postdoctoral fellows and scientific colleagues who have contributed to the ideas in this book. Without their enthusiasm, dedication and many crucial insights this work would not have been possible.

This book describes, among other things, the statistical mechanics of vortices, disclinations, dislocations, vacancies and interstitials. Excitation of these defects in crystals, superfluids, superconductors, liquid crystals and polymer arrays usually requires strong thermal fluctuations. Geometrical aspects of statistical mechanics, with or without defects, often become particularly interesting when these fluctuations entangle or crumple extended line-like or surface-like objects in three dimensions. Sometimes, as is the case for entangled vortices above the first-order flux lattice melting transition in type-II superconductors, the lines themselves are defects! Because modern ideas about the renormalization group and universality in the presence of fluctuations underpin most of the work on defects and the statistical mechanics of lines and surfaces in this book, the remainder of this chapter provides a brief introduction to this point of view with several illustrative examples. We conclude with a survey of subsequent chapters.

1.1 Fluctuations and universality in condensed matter physics

Condensed matter physics flourished in the second half of the twentieth century, due in part to the application of sophisticated tools for understanding thermal and quantum fluctuations to an astonishing variety of problems. The failure of uncontrolled "mean field" or decoupling approximations is particularly evident close to equilibrium critical points, where fluctuations occur over multiple length scales, from an atomic dimension of order ångström units to a (diverging) correlation length which can be micrometers or more. The renormalization group, which was exported from particle physics by Kenneth Wilson in the early 1970s, allows a systematic understanding of such nested length scales (see Appendix A for an elementary introduction to the renormalization group in the "hydrodynamic" context considered here). A particularly striking result is that most of the detailed physics of matter at

microscopic length scales is *irrelevant* for critical-point phenomena. One can make precise quantitative predictions about certain "universal" critical exponents or scaling functions without getting the microscopic physics right in detail. What matters is symmetry, conservation laws, the range of interactions and the dimensionality of space. The physics of the diverging fluctuations at a critical point on large length scales is largely decoupled from the physics on atomic scales of order ångströms. The idea of new universal laws of physics governing fluctuations at a critical point was nicely summarized by A. Z. Patashinskii and V. L. Pokrovskii [2], two pioneers of scaling ideas at critical points in the USSR, who wrote that

> When fluctuations, these shapeless amoebas, overlap in large numbers to form a continuous, undescribable soup, new and sharply defined laws . . . come into play, cutting through the chaos.

It turns out that not just critical points but entire phases of matter are described by a "universal," coarse-grained, long-wavelength theory. This point was recognized by Wilson [3], who argued that Landau's hydrodynamic treatment of magnets *far* from critical points (carried out in the 1930s) was itself representative of a particularly simple renormalization group fixed point. One can make similar statements about the hydrodynamic laws derived for fluids in the nineteenth century. Upon systematically integrating out the high-frequency, short-wavelength modes associated with atoms and molecules, one should be able to arrive at, say, the Navier–Stokes equations. One does not have to be at a critical point to have universal physical laws insensitive to the microscopic details. We now have many concrete calculations well away from critical points that support this point of view. Ignorance about microscopic details is typically packaged into a few phenomenological parameters characterizing the "fixed point," such as the density and viscosity of an incompressible fluid like water in the case of the Navier–Stokes equations.

The modern theory of critical phenomena has interesting implications for our understanding of what constitutes "fundamental" physics. For many important problems, a fundamental understanding of the physics involved does *not* necessarily lie in the science of the smallest available time or length scale. The extreme insensitivity of the hydrodyamics of fluids to the precise physics at high frequencies and short distances is highlighted when we remember that the Navier–Stokes equations were derived in the early nineteenth century, at a time when even the discrete atomistic nature of matter was in doubt. The same equations would have resulted had matter been continuous at all length

scales. The existence of atoms and molecules is irrelevant to the profound (some might even say "fundamental") problems of understanding, say, turbulence in the Navier–Stokes equations at high Reynolds numbers [4]. We would face almost identical problems in constructing a theory of turbulence if quantum mechanics did not exist, or if the discreteness of matter first became noticeable at length scales of order fermis instead of ångström units.

Many aspects of condensed matter physics, by which I mean the study of matter at everyday length and energy scales, do, of course, depend crucially on quantum mechanics and the particulate nature of matter. We cannot begin to understand phonons in solids, the specific heat of metals, localization in semiconductors, the quantum Hall effect and high-temperature superconductivity without knowing about the quantum mechanics of protons, neutrons, electrons and, occasionally, muons and positrons. There comes a point, however, when a more traditional reductionist approach burrows down to such short length scales and high energies that its conclusions become largely irrelevant to the physics of the world around us. This is why most condensed matter physicists are not aiming to discover the "fundamental" laws at the smallest length scales. The reductionist school of high-energy physics continues to be a noble intellectual enterprise, but is now virtually decoupled from physics at ångström-unit scales, just as atomic physics is decoupled from the Navier–Stokes equations. New particles discovered in high-energy physics are unlikely to help us understand problems like turbulence or how itinerant magnetism arises from the Hubbard model; neither will they unravel other hard problems like the complexities of reptation dynamics in entangled polymer melts [5].

Although the precise nature of physics at very short length scales need not have a profound impact on deep unresolved questions at much larger scales, knowledge of the correct short-distance theory is of course far from useless in condensed matter physics. A first-principles calculation of the viscosity and density of water, for example, would require a molecular or atomic starting point. Deriving hydrodynamic parameters such as the viscosity from an atomistic framework is the task of kinetic theory, in which significant progress has been made during the last century, at least for weakly interacting gases; and we are impressed when *ab initio* band-structure experts are able to correctly predict the lattice constant and crystal structure of silicon via numerical solutions of Schrödinger's equation. Nevertheless, there will always be important problems that a strict *ab initio* approach based on a more fundamental theory are unlikely to resolve.

The problems discussed in this book are all represented by coarse-grained long-wavelength "hydrodynamic" models, with the detailed

physics packaged into a small number of phenomenological coupling constants. To illustrate the approach, we discuss two interesting but unusual situations in which ratios of long-wavelength hydrodynamic parameters are themselves *universal constants*, just like the universal exponents at a critical point. The first is the universal value of the Prandtl number of a two-dimensional incompressible fluid (it equals $(\sqrt{17}-1)/2$) and the second is the universal (negative!) value of Poisson's ratio associated with polymerized membranes or "tethered surfaces" (it is about $-\frac{1}{3}$). We then review the role of topological defects in destabilizing the hydrodynamic *surface* of fixed points associated with two-dimensional crystals. Here, the physics associated with strong thermal disorder leads to models dominated by fluctuations in the *phase* of the translational and orientational order parameters, instead of the usual amplitude fluctuations associated with mean field or Landau theories. Amplitude fluctuations now reside only in the cores of defects such as dislocations and disclinations. We discuss the new fixed point that takes over in membranes characterized by a bending rigidity, where defects such as dislocations can buckle easily out of the plane. This introduction concludes with an overview of the contents of the rest of the book.

1.2 The universal Prandtl number in two-dimensional hydrodynamics

Understanding chaotic fluid flows, particularly those at high Reynolds numbers, remains one of the most challenging problems in theoretical physics and fluid mechanics, despite the concerted efforts of many experts during the past half century [4]. Under many circumstances, it is expected that the fluid velocity field $\mathbf{v}(\mathbf{r}, t)$ and the concentration $\psi(\mathbf{r}, t)$ of tracer particles are described by the equations

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v}, \tag{1.1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{1.2}$$

$$\partial_t \psi + (\mathbf{v}\cdot\nabla)\psi = D\nabla^2\psi, \tag{1.3}$$

where ν is the kinematic viscosity, ρ is the density of the fluid, D is the diffusivity of the tracer particles and the condition of incompressibility (which is valid in the limit of velocities much less than the speed of sound) is enforced by Eq. (1.2). This condition can be used to eliminate the term involving the pressure field, $p(\mathbf{r}, t)$, in Eq. (1.1). The dynamics becomes insensitive to the density. Equations (1.1)–(1.3) then represent a "universal" long-wavelength, low-frequency description of a large number of atomic and molecular fluids, parameterized only by the

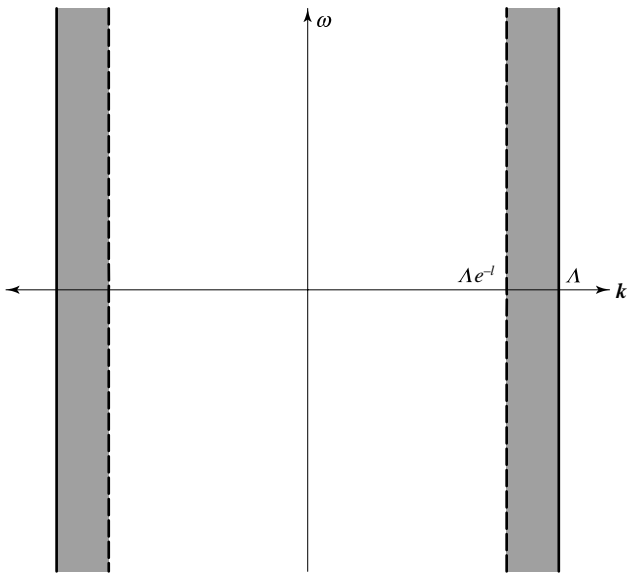

Fig. 1.1. The degrees of freedom for a randomly stirred Navier–Stokes fluid, indexed by wavevector and frequency, with wavevectors above a cutoff Λ excluded. A renormalization group transformation focusing on the long-wavelength hydrodynamic behavior can be constructed by iterative elimination of Fourier modes in a shell of wavevectors in the range $\Lambda\exp(-\ell)<k<\Lambda$ indicated by the shaded region.

viscosity ν and the diffusivity D. If a colored dye of tracer particles is injected into water, one has $\nu \approx 10^{-2}$ cm^2 s^{-1} for the viscosity and $D \approx 10^{-6}$ cm^2 s^{-1} for the molecular diffusivity and thus the dimensionless "Prandtl number" is

$$Pr = \nu/D \approx 10^4. \tag{1.4}$$

However, in general, one would expect that the Prandtl number could assume any positive value in a three-dimensional fluid. This is not the case in two dimensions!

By including various additive forcing terms on the right-hand side of Eqs. (1.1) and (1.3), one can simulate the effects of thermal fluctuations or even random stirring which can provoke chaos and high-Reynolds-number turbulence. The influence of the nonlinear terms ($(\mathbf{v}\cdot\nabla)\mathbf{v}$ and $(\mathbf{v}\cdot\nabla)\psi$) on the physics at long wavelengths can be assessed by the iterative coarse-graining procedure embodied in the renormalization group. The idea behind this perturbative renormalization procedure is reviewed in Fig. 1.1 [6, 7]. The velocity field of a d-dimensional fluid is first decomposed into Fourier modes according to

$$\mathbf{v}(\mathbf{r}, t) = \int_{k<\Lambda} \frac{d^d k}{(2\pi)^d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathbf{v}(\mathbf{k},\omega) e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}. \tag{1.5}$$

The spatial modes are cut off above a spatial wavevector $k = \Lambda$ of order the inverse interparticle separation and the frequencies are unconstrained. The equations of motion for modes in the range $\Lambda/e^{\ell} < k < \Lambda$ are formally solved (via a diagrammatic perturbation theory) and these solutions then substituted into the equations of motion for the remaining

modes in the range $0<k<\Lambda/e^{\ell}$. After averaging over forcing contributions in the range $\Lambda/e^{\ell}<k<\Lambda$ and rescaling frequencies and wavevectors, one finds a coupled set of equations for the remaining modes which have the same form as the initial set. Thus, this procedure can be iterated indefinitely. By tracking the effect of repeated transformations on the coupling constants and correlation functions, this implementation of the renormalization group often allows a systematic resummation of Feynman diagrams in regimes for which conventional perturbation theory fails. This renormalization procedure also predicts situations in which it, too, fails, i.e., when the perturbative coupling constants iterate outside the range accessible to perturbation theory. Related implementations of this type of renormalization group underpin much of this book.

Although some progress is possible when the fluid is driven far out of equilibrium with appropriate forcing functions [7], we concentrate here on equilibrium forcing representing thermal fluctuations that obey the fluctuation–dissipation theorem. In three dimensions, one finds that the nonlinearities are "irrelevant variables," in that short-distance fluctuations lead only to finite renormalizations of the viscosity and diffusivity. However, in two dimensions the situation is more interesting. Here, thermal fluctuations couple into long-range hydrodynamic backflow, which causes the renormalized wavevector-dependent kinematic viscosity $\nu_{\mathrm{R}}(k)$ to diverge at small wavevectors k [7],

$$\nu_{\mathrm{R}}(k)=\nu\left[1+\frac{k_{\mathrm{B}}T}{\rho\nu^{2}}\ln\left(\frac{\Lambda}{k}\right)\right]^{1/2}, \tag{1.6}$$

where k_{B} is Boltzmann's constant and T is the temperature. Here, $\nu(k)$ controls the rate of relaxation of a velocity fluctuation on wavelength $2\pi/k$, which decays like $\exp[-\nu(k)k^{2}t]$ as $t\to\infty$. Thus, the apparent viscosity will depend (logarithmically) on the length scale $\sim k^{-1}$ of the measurement! The renormalized diffusivity $D_{\mathrm{R}}(k)$ is also affected by the velocity fluctuations and it too diverges as $k\to 0$,

$$D_{\mathrm{R}}(k)\sim\ln^{1/2}(\Lambda/k). \tag{1.7}$$

Although both ν_{R} and D_{R} diverge, it can be shown that the Prandtl number Pr remains well defined and approaches a *universal* constant characteristic of all incompressible fluids in the limit $k\to 0$ [8],

$$\begin{aligned} Pr &\equiv \lim_{k\to 0}\frac{\nu_{\mathrm{R}}(k)}{D_{\mathrm{R}}(k)} \\ &= \tfrac{1}{2}\left(\sqrt{17}-1\right). \end{aligned} \tag{1.8}$$

As is evident from Eq. (1.6), the dimensionless coupling constant which controls the strength of nonlinear effects in two-dimensional hydrodynamics is

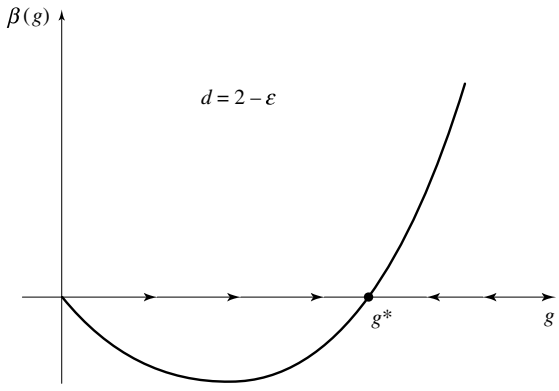

Fig. 1.2. The function $\beta(g)$ which determines the flow of the dimensionless coupling constant $k_B T/(\rho\nu^2)$ describing driven Navier–Stokes fluids in $2-\varepsilon$ dimensions, with $0<\varepsilon \ll 1$. The fixed point at $g=0$ describes conventional linearized hydrodynamics. The stable nontrivial fixed point at $g=g^*=O(\varepsilon)$ indicates a breakdown of conventional hydrodynamics. The two fixed points merge for $d=2$ and the fixed point at the origin dominates the physics for all $d\geq 2$.

$$g=\frac{k_B T}{\rho\nu^2}. \tag{1.9}$$

The effective coupling constant $g(\ell)$ in d dimensions after the short-distance cutoff has been reduced from Λ to Λ/e^{ℓ} is the solution of the renormalization-group differential equation,

$$\begin{aligned}\frac{dg(\ell)}{d\ell} &= \tfrac{1}{2}(2-d)g - A_d g^2 \\ &\equiv -\beta(g),\end{aligned} \tag{1.10}$$

where A_d is a positive constant. As illustrated in Fig. 1.2, conventional linearized hydrodynamics is described by a "Navier–Stokes" fixed point $g=g^*=0$. For $d>2$, $g(\ell)$ decays rapidly to zero and linearized hydrodynamics is stable. In two dimensions, the slow decay to zero of $g(\ell)$ for large ℓ ($g(\ell)\sim 1/\ell$) leads to logarithmic corrections to linearized hydrodynamics discussed above. There is an interesting nontrivial stable fixed point with $g^*>0$ for $d<2$. Unfortunately, incompressible fluid flow is not very meaningful in the physically relevant limit $d=1$. However, nontrivial hydrodynamic fixed points do appear in two and three dimensions for forcing functions that do not obey the fluctuation–dissipation theorem [7]. In the next section, we discuss the nontrivial fixed point which describes the anomalous long-wavelength elastic properties and bending rigidity of a two-dimensional solid membrane, free to fluctate in three dimensions and subject to thermal fluctuations.

1.3 The universal Poisson ratio in fluctuating polymerized membranes

Like fluid mechanics, the continuum elastic description of solids subject to arbitrary deformations has a long and rich history. It reached its

current form early in the twentieth century. The theory of the bending of thin plates resembles a simplified form of general relativity and both subjects use the language of differential geometry. A thin elastic plate can be parameterized by two internal indices, x_1 and x_2, which we can take as the particle positions in the flat undeformed state. Its deformations in the xy-plane are described by phonon coordinates $u_1(x_1,x_2)$ and $u_2(x_1,x_2)$. Out-of-plane deformations are parametrized via a perpendicular displacement $f(x_1,x_2)$. The actual position in three dimensions of a particle labeled by (x_1,x_2) in the deformed membrane is thus given by

$$\mathbf{r}(x_1,x_2)=\begin{pmatrix} x_1+u_1(x_1,x_2) \\ x_2+u_2(x_1,x_2) \\ f(x_1,x_2). \end{pmatrix}. \tag{1.11}$$

The energy of a thin plate for small deformations is characterized by a bending rigidity κ and in-plane elastic constants μ and λ [9],

$$F=\tfrac{1}{2}\kappa\int d^2x(\nabla^2 f)^2+\tfrac{1}{2}\int d^2x(2\mu u_{ij}^2+\lambda u_{kk}^2), \tag{1.12}$$

where $u_{ij}(x_1,x_2)$ is the nonlinear strain matrix. (For more details, see Chapters 5 and 6.) Like general relativity, the theory is intrinsically nonlinear, which shows up at this level in the relation between the strain matrix and the displacements $\mathbf{u}(x_1,x_2)$ and $f(x_1,x_2)$ [9],

$$u_{ij}(x_1,x_2)=\tfrac{1}{2}(\partial_i u_j+\partial_j u_i)+\tfrac{1}{2}(\partial_i f)(\partial_j f). \tag{1.13}$$

To determine the deformation of plates subjected to edge forces with various boundary conditions, one must minimize Eq. (1.12). Upon eliminating $u_{ij}(x_1,x_2)$ in favor of the Airy stress function $\chi(x_1,x_2)$, via $\sigma_{ij}(x_1,x_2)\equiv 2\mu u_{ij}(x_1,x_2)+\lambda u_{kk}(x_1,x_2)\delta_{ij}\equiv\varepsilon_{im}\varepsilon_{jn}\partial_m\partial_n\chi(x_1,x_2)$, where $\varepsilon_{ij}=\delta_{i1}\delta_{j2}-\delta_{j1}\delta_{i2}$ is the antisymmetric unit matrix in two dimensions, one obtains the von Karman equations,

$$\kappa\nabla^4 f=\frac{\partial^2\chi}{\partial y^2}\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2\chi}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-2\frac{\partial^2\chi}{\partial x\,\partial y}\frac{\partial^2 f}{\partial x\,\partial y}, \tag{1.14}$$

$$\frac{2\mu+\lambda}{4\mu(\mu+\lambda)}\nabla^4\chi=-\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}+\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2. \tag{1.15}$$

The negative of the Gaussian curvature $\det(\partial^2 f/\partial x_i\partial x_j)$ of the deformed surface appears as a source on the right-hand side of (1.15). According to Landau and Lifshitz [9], these nonlinear equations are "very complicated, and cannot be solved exactly, even in very simple cases." However, at least the elastic constants κ, μ and λ are indeed *constant*s for long-wavelength deformations in the von Karman theory! When thermal fluctuations are taken into account, *these parameters depend on the*

wavelength, even more strongly than do the diverging viscosity and diffusivity in Eqs. (1.6) and (1.7).

Equation (1.12) also describes biological sheet polymers that are one molecule thick, such as the spectrin skeleton on the inside walls of red blood cells [10, 11]. When these polymers are extracted from red blood cells and put into water solution, the nodes of this "biological fishnet" are subject to violent thermal fluctuations in the form of Brownian motion [12]. The equilibrium statistical mechanics of this structure (similar to the fluctuations of semiflexible linear polymer chains) can be modelled by assuming that configurations occur with probability proportional to $\exp[-F/(k_B T)]$. Because the out-of-plane displacement $f(x_1,x_2)$ occurs in the generalized strain matrix (1.13), there is a nontrivial coupling between in-plane and out-of-plane displacements.

This nonlinearity can be studied by first expanding u_1, u_2 and f in Fourier modes in the range $0<k<\Lambda$, where Λ is a suitable microscopic cutoff. Upon integrating Fourier modes in the interval $\Lambda/e^{\ell}<k<\Lambda$ out of the partition function (defined as a functional integral over u_1, u_2 and f)

$$Z=\int \mathcal{D}\mathbf{u}(x_1,x_2)\int \mathcal{D}f(x_1,x_2)\exp(-F/k_B T), \tag{1.16}$$

and rescaling degrees of freedom and wavevectors, one can implement a renormalization group similar to that for the Navier–Stokes equations discussed in the previous section. Iterating this transformation determines effective dimensionless coupling constants $\bar{\lambda}(\ell)=k_B T\lambda/(\Lambda^2\kappa^2)$ and $\bar{\mu}(\ell)=k_B T\mu/(\Lambda^2\kappa^2)$ that describe the statistical mechanics for wavevectors $k<\Lambda/e^{\ell}$. Figure 1.3 shows schematically the solutions of the renormalization-group differential equations for $\bar{\lambda}(\ell)$ and $\bar{\mu}(\ell)$ for fluctuating membranes embedded in three dimensions [13]. The "von Karman" fixed point at the origin describes the physics at zero temperature, i.e., situations to which Eqs. (1.14) and (1.15) apply with *constant* elastic parameters κ, μ and λ. However, this fixed point is unstable at any finite temperature relative to a fixed point where conventional elasticity theory breaks down at long wavelengths. Indeed, at this fixed point the renormalized wavevector-dependent bending rigidity $\kappa_R(k)$ *diverges* strongly as k goes to zero [14], while the renormalized wavevector-dependent elastic parameters $\mu_R(k)$ and $\lambda_R(k)$ *vanish* in this limit [13]. Among the best current estimates of these singularities are those of Radzihovsky and Le Doussal [15],

$$\kappa_R(k)\sim 1/k^{0.82},\qquad \mu_R(k)\sim k^{0.36},\qquad \lambda_R(k)\sim k^{0.36}, \tag{1.17}$$

where these power-law singularities are governed by universal critical exponents.