An Analog Electronics Companion

Basic Circuit Design for Engineers and Scientists

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1.1 Trigonometry

The power of instruction is seldom of much efficacy except in those happy dispositions where it is almost superfluous.

Edward Gibbon

It may seem unexpected to find a section on trigonometry, but in electronics you cannot get away from sine waves. The standard definitions of sine, cosine and tangent in terms of the ratio of the sides of a right-angled triangle are shown in Fig. 1.1.1 and Eq. (1.1.1).

For angle θ and referring to the sides of the triangle as opposite (o), adjacent (a) and hypotenuse (h) we have:

$$\sin\theta = \frac{o}{h}, \qquad \cos\theta = \frac{a}{h}, \qquad \tan\theta = \frac{o}{a} \tag{1.1.1}$$

A common way to represent a sinusoidal wave is to rotate the phasor OA around the origin O at the appropriate rate ω (in radians per second) and take the projection of OA as a function of time as shown in Fig. 1.1.2.

The corresponding projection along the x-axis will produce a cosine wave. This allows us to see the values of the functions at particular points, e.g. at $\omega t = \pi/2$, π, $3\pi/2$ and 2π as well as the signs in the four quadrants (Q). These are summarized in Table 1.1.1.

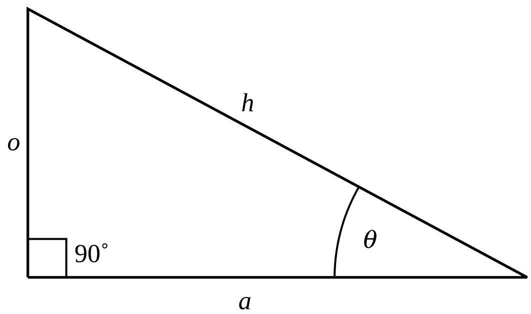

Fig. 1.1.1 Right-angled triangle.

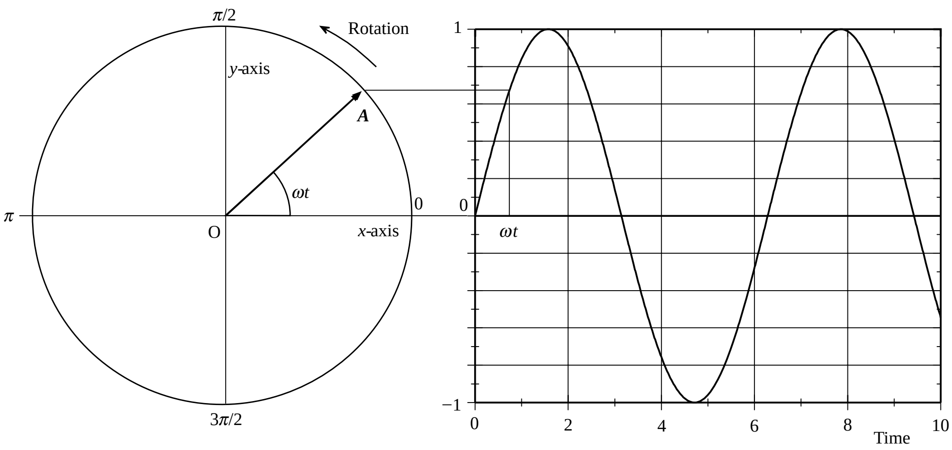

Fig. 1.1.2 Projection of a rotating vector.

Table 1.1.1 *Values and signs of trigonometrical functions in the four quadrants*

Angle	0	$\pi/2$	π	$3\pi/2$	2π	1Q	2Q	3Q	4Q
Sin	0	1	0	1	0	+	+	−	−
Cos	1	0	1	0	1	+	−	−	+
Tan	0	∞	0	∞	0	+	−	+	−

Some useful relationships for various trigonometrical expressions are:

(a) $\sin(-\theta) = -\sin\theta$

(b) $\cos(-\theta) = \cos\theta$

(c) $\tan(-\theta) = -\tan\theta$

(d) $\cos(\theta+\phi) = \cos\theta\cdot\cos\phi - \sin\theta\cdot\sin\phi$

(e) $\sin(\theta+\phi) = \sin\theta\cdot\cos\phi + \cos\theta\cdot\sin\phi$

(f) $\sin\theta + \sin\phi = 2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)$

(g) $\cos\theta + \cos\phi = 2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)$

(h) $\sin\theta\cdot\cos\phi = \frac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$

(i) $\cos\theta\cdot\cos\phi = \frac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$

(j) $\sin^2\theta + \cos^2\theta = 1$

(k) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ (1.1.2)

(l) $1 + \cos\theta = 2\cos^2(\theta/2)$

(m) $1 - \cos\theta = 2\sin^2(\theta/2)$

(n) $\sin 2\theta = 2\sin\theta\cos\theta$

(o) $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = \lim\limits_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$

(p) $\cos(\theta - \phi) = \cos \theta \cdot \cos \phi + \sin \theta \cdot \sin \phi$

(q) $\sin(\theta - \phi) = \sin \theta \cdot \cos \phi - \cos \theta \cdot \sin \phi$

(r) $\sin \theta - \sin \phi = 2\cos \tfrac{1}{2}(\theta + \phi) \sin \tfrac{1}{2}(\theta - \phi)$

(s) $\cos \theta - \cos \phi = 2\sin \tfrac{1}{2}(\phi + \theta) \sin \tfrac{1}{2}(\phi - \theta)$

(t) $\cos \theta \cdot \sin \phi = \tfrac{1}{2}[\sin(\theta + \phi) - \sin(\theta - \phi)]$

(u) $\sin \theta \cdot \sin \phi = \tfrac{1}{2}[\cos(\theta - \phi) - \cos(\theta + \phi)]$

We will have occasion to refer to some of these in other sections and it should be remembered that the complex exponential expressions (Section 1.5) are often easier to use.

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1.2 Geometry

For geometry by itself is a rather heavy and clumsy machine. Remember its history, and how it went forward with great bounds when algebra came to its assistance. Later on, the assistant became the master.

Oliver Heaviside (1899): *Electromagnetic Theory*, April 10, Vol. II, p. 124

The relationship between equations and their geometric representation is outlined in Section 1.10. In various places we will need to make use of Cartesian (coordinate) geometry either to draw graphical responses or to determine various parameters from the graphs.

A commonly used representation of the first order, or single pole, response of an operational amplifier in terms of the zero frequency (z.f.) gain A_0 and the corner frequency $\omega_c = 1/T$ is given by:

$$A = \frac{A_0}{1 + sT} \tag{1.2.1}$$

which on the log–log scales usually used is as shown in Fig. 1.2.1.

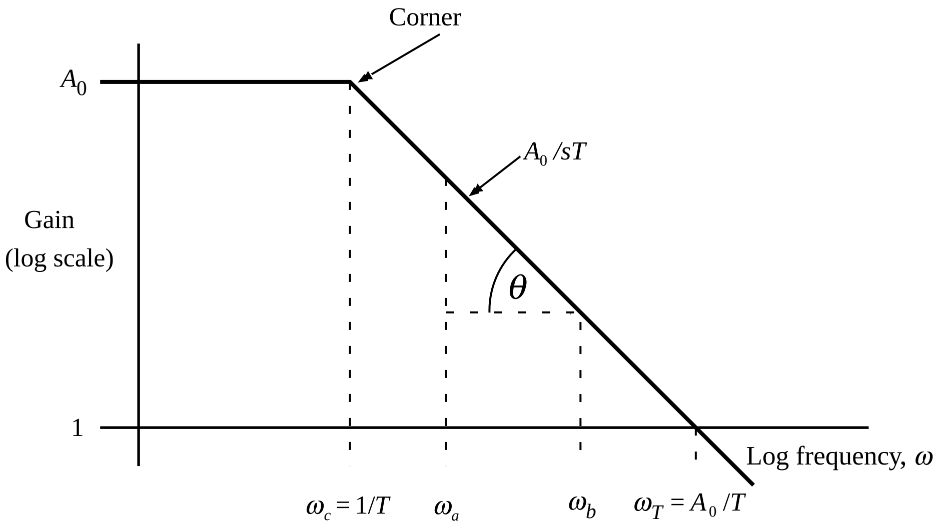

Fig. 1.2.1 Operational amplifier response.

At low frequency $sT \ll 1$ so the gain is just A_0. At high frequency $sT \gg 1$ so the gain is A_0/sT as shown. The graphs are slightly different from those with standard x- and y-axes, which are of course where $y=0$ and $x=0$, because here we have logarithmic scales. The traditional x- and y-axes would then be at $-\infty$, which is not useful. We can start by taking $G=1$ and $f=1$ for which the log values are zero ($\log 1=0$). You can of course draw your axes at any value you wish, but decade intervals are preferable. The slope of A_0/sT can be determined as follows. Take two frequencies ω_a and ω_b as shown. The slope, which is the tangent of the angle θ, will be given (allowing for the sense of the slope) by:

$$\begin{aligned}
\tan\theta &= \frac{\log\left(\dfrac{A_0}{\omega_a T}\right) - \log\left(\dfrac{A_0}{\omega_b T}\right)}{\log(\omega_b) - \log(\omega_a)} \\
&= \frac{\log\left(\dfrac{\omega_b}{\omega_a}\right)}{\log\left(\dfrac{\omega_b}{\omega_a}\right)} \qquad (1.2.2) \\
&= 1 \qquad \text{and} \qquad \theta = \tan^{-1}(1) = 45°
\end{aligned}$$

where we have used the fact that the difference of two logs is the log of the quotient (Section 1.4). Since the slope does not depend on ω then A_0/sT must be a straight line on the log–log scales.

A point of interest on the amplifier response is the unity-gain (or transition) frequency ω_T as this defines the region of useful performance and is particularly relevant to considerations of stability. We need to find the value of ω for which $G=1$. Note that though we use the more general complex frequency s we can simply substitute ω for s since we are dealing with simple sine waves and are not concerned with phase since this is a graph of amplitude. Thus we have:

$$1 = \frac{A_0}{\omega_T T} \qquad \text{or} \qquad \omega_T = \frac{A_0}{T} \qquad (1.2.3)$$

and remember that ω is an angular frequency in rad s^{-1} and $f=\omega/2\pi$ Hz. Say we now wish to draw the response for a differentiator which has $G=sR_fC_i$ (Section 5.6). The gain will be 1 when $\omega_D = 1/R_fC_i$ and the slope will be 45°. So fixing point ω_D and drawing a line at 45° will give the frequency response. To find where this line meets the open-loop response we have:

$$sR_fC_i = \frac{A_0}{sT} \qquad \text{or} \qquad s^2 = \frac{A_0}{R_fC_iT} \qquad \text{so} \qquad \omega_P = \left(\frac{A_0}{R_fC_iT}\right)^{\frac{1}{2}} \qquad (1.2.4)$$

which is shown in Fig. 1.2.2.

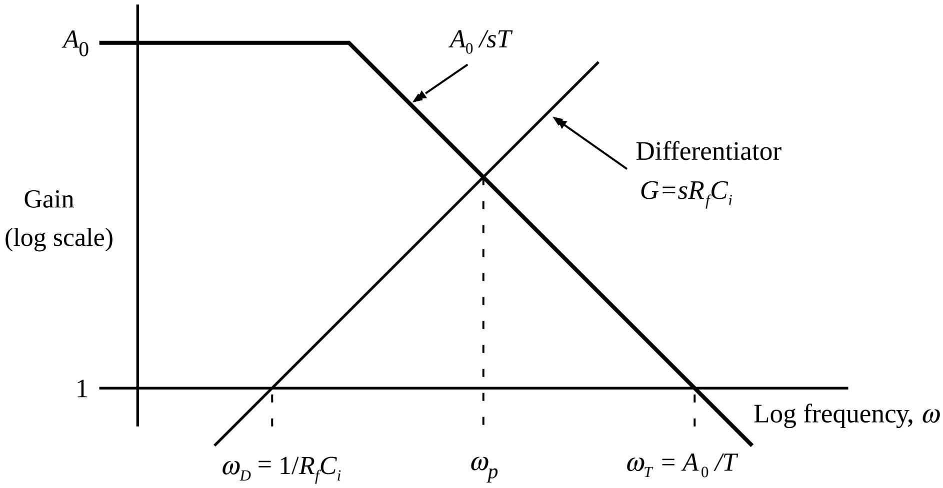

Fig. 1.2.2 Geometry of differentiator frequency response.

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1.3 Series expansions

Prof. Klein distinguishes three main classes of mathematicians – the intuitionists, the formalists or algorithmists, and the logicians. Now it is intuition that is most useful in physical mathematics, for that means taking a broad view of a question, apart from the narrowness of special mathematics. For what a physicist wants is a good view of the physics itself in its mathematical relations, and it is quite a secondary matter to have logical demonstrations. The mutual consistency of results is more satisfying, and exceptional peculiarities are ignored. It is more useful than exact mathematics.

But when intuition breaks down, something more rudimentary must take its place. This is groping, and it is experimental work, with of course some induction and deduction going along with it. Now, having started on a physical foundation in the treatment of irrational operators, which was successful, in seeking for explanation of some results, I got beyond the physics altogether, and was left without any guidance save that of untrustworthy intuition in the region of pure quantity. But success may come by the study of failures. So I made a detailed study and close examination of some of the obscurities before alluded to, beginning with numerical groping. The result was to clear up most of the obscurities, correct the errors involved, and by their revision to obtain correct formulae and extend results considerably.

Oliver Heaviside (1899): *Electromagnetic Theory*, April 10, Vol. II, p. 460

Expansion of functions in terms of infinite (usually) series is often a convenient means of obtaining an approximation that is good enough for our purposes. In some cases it also allows us to obtain a relationship between apparently unconnected functions, and one in particular has been of immense importance in our and many other fields. We will list here some of the more useful expansions without derivation (Boas 1966):

$$\begin{aligned}\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\end{aligned} \tag{1.3.1}$$

where

$$n! \equiv n(n-1)(n-2)(n-3)\cdots 1$$

and $n!$ is known as n factorial. Note that, odd though it may seem, $0! = 1$. The variable θ must be in radians.

The binomial series is given by:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots \tag{1.3.2}$$

valid for n positive or negative and $|x| < 1$
which is most frequently used for $x \ll 1$ to give a convenient approximation:

$$(1+x)^n \cong 1 + nx \tag{1.3.3}$$

The geometric series in x has a sum S_n to n terms given by:

$$a + ax^2 + ax^3 + ax^4 + \cdots + ax^n + \cdots +, \quad \text{with sum } S_n = \frac{a(1-x^n)}{1-x} \tag{1.3.4}$$

and for $|x| < 1$ the sum for an infinite number of terms is:

$$S = \frac{a}{1-x} \tag{1.3.5}$$

In some circumstances, when we wish to find the value of some function as the variable goes to a limit, e.g. zero or infinity, we find that we land up with an indeterminate value such as $0/0$, ∞/∞ or $0 \times \infty$. In such circumstances, if there is a proper limit, it may be determined by examining how the function approaches the limit rather than what it appears to do if we just substitute the limiting value of the variable. As an example consider the function (Boas 1966, p. 27):

$$\lim_{x\to 0} \frac{1-e^x}{x}, \text{ which becomes } \frac{0}{0} \text{ for } x = 0 \tag{1.3.6}$$

If we expand the exponential using Eq. (1.3.1), then remembering that x is going to become very small:

$$\lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{1 - \left(1 + x + \dfrac{x^2}{2!} + \cdots\right)}{x} = \lim_{x\to 0}\left(-1 - \frac{x}{2!} - \cdots\right) = -1 \tag{1.3.7}$$

Expansion in terms of a series, as in this case, is generally most useful for cases where $x \to 0$, since in the limit the series is reduced to the constant term. There is another approach, known as l'Hôpital's rule (or l'Hospital), which makes use of a Taylor series expansion in terms of derivatives. If the derivative of $f(x)$ is $f'(x)$, then:

$$\lim_{x\to 0} \frac{f(x)}{\phi(x)} = \lim_{x\to 0} \frac{f'(x)}{\phi'(x)} \tag{1.3.8}$$

but the ease of use depends on the complexity of the differentials. The Taylor series referred to allows expansion of a function $f(x)$ around a point $x = a$:

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) \tag{1.3.9}$$

where $f(a)$, $f'(a)$, etc. is the value of the quantity for $x = a$. If the expansion is around the origin, $x = 0$, the series is sometimes referred to as Maclaurin's series.

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1.4 Logarithms

God created the whole numbers: all the rest is man's work.

Leopold Kronecker (1823–1891)

Logarithms, originally developed to help in complex calculations, are now largely superseded for hand calculation by the ubiquitous calculator or mathematical software on your PC. However, we still need to know about them in a number of circumstances and most particularly in electronics they help us cope with numbers spanning a wide range. In a Bode plot of gain, for example, a linear scale will show only a small portion of the range with any resolution and low gains will be inaccessible. Log scales are very common therefore and the decibel 'unit' is used in many areas from aircraft noise to attenuation in optical fibres.

We list now some of the basic relations for logarithms (Abramowitz and Stegun 1970). Logarithms relate to a base value, say β, in the following way. For generality we will write **lgm** for reference to an arbitrary base. If:

$$y = \beta^x \qquad \text{then} \qquad \mathrm{lgm}_\beta(y) = x \tag{1.4.1}$$

and on this basis we can write:

$$\begin{aligned}
&\mathrm{lgm}(ab) = \mathrm{lgm}(a) + \mathrm{lgm}(b) && \mathrm{lgm}\left(\frac{a}{b}\right) = \mathrm{lgm}(a) - \mathrm{lgm}(b) \\
&\mathrm{lgm}(a^n) = n\mathrm{lgm}(a) && \mathrm{lgm}\left(a^{\frac{1}{n}}\right) = \frac{1}{n}\mathrm{lgm}(a) \\
&\mathrm{lgm}_\beta(a) = \frac{\mathrm{lgm}_\alpha(a)}{\mathrm{lgm}_\alpha(\beta)}\,, && \text{for conversion between bases } \beta \text{ and } \alpha \\
&\mathrm{lgm}(1) = 0 && \mathrm{lgm}(0) = -\infty
\end{aligned} \tag{1.4.2}$$

The two common bases are $\beta = 10$, for which is written **log**, and $\beta = e = 2.71828$ . . . , which is written **ln** and which are called natural logarithms. We can then list the additional relations for **ln**:

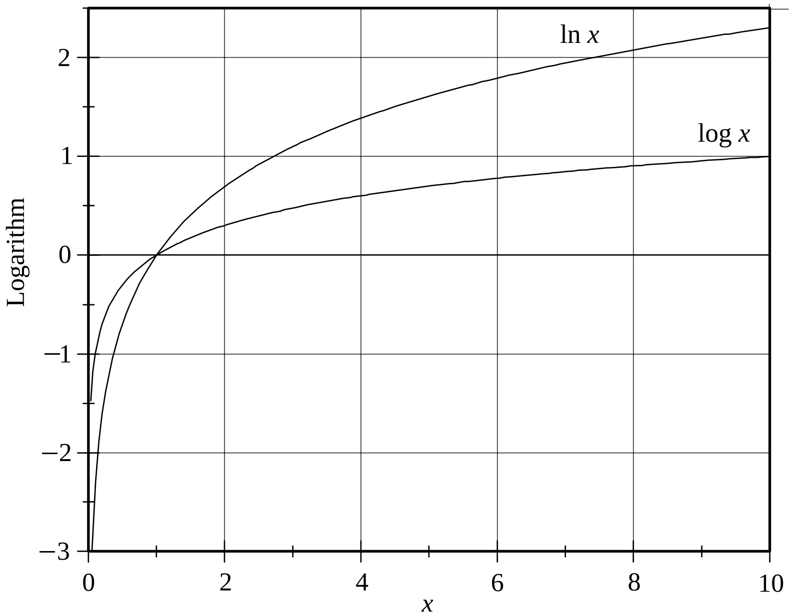

Fig. 1.4.1 Graph of the logarithmic functions.

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$$\ln(n+1)=\ln(n)+2\left[\frac{1}{(2n+1)}+\frac{1}{3}\frac{1}{(2n+1)^3}+\frac{1}{5}\frac{1}{(2n+1)^5}+\cdots\right]\quad \text{for } n>0 \quad (1.4.3)$$

$$\frac{d}{dx}[\ln(x)]=\frac{1}{x} \qquad \int\frac{dx}{x}=\ln(x)$$

The value of $\mathbf{ln}(0)=-\infty$ is the reason *PROBE* in PSpice will refuse to display on a log scale if the data include zero. The use of logarithmic scales is discussed below. The form of the logarithm is shown in Fig. 1.4.1.

It might appear that there are no logarithms for negative numbers, but this is only true if we are restricted to real numbers (Section 1.7). For complex numbers negative arguments are allowed though we will not make use of this possibility. In fact a number, positive or negative will now have an infinite set of logarithms. For example (n is an integer):

$$\ln(-1)=\ln(re^{j\theta})=\ln(r)+\ln(e^{j\theta})=\ln(1)+j(\pi+2n\pi)=j\pi,\ -j\pi,\ 3j\pi,\ \ldots \quad (1.4.4)$$

As mentioned above the very wide range of many of the quantities met in electronics, together with the convenience of simple addition, rather than multiplication, for sequential gains when expressed in a logarithmic scale, prompted the widespread adoption of logarithmic measures. Early measures of attenuation were made in terms of a length of standard cable (Everitt and Anner 1956). To obtain a more generally usable and convenient measure a unit was chosen that closely matched the older version and later it was named the decibel in honour of Alexander Graham Bell. Though the actual unit is the bel, closer agreement with the earlier measures was achieved at one-tenth of this, i.e. the decibel and this has over time become the most commonly used unit. The proper definition is given as a power ratio and could represent both attenuation or gain. If the two powers are P_1 and P_2 then:

$$\text{Power gain or ratio } G_p = 10 \log\left(\frac{P_1}{P_2}\right) \text{dB} \tag{1.4.5}$$

If P_1 is greater than P_2 then we have gain expressed as $+$dB whereas if the powers are in the opposite sense then we have attenuation expressed as $-$dB. If the resistances R at which the powers are measured are the same, then since $P = V^2/R$ we may also write:

$$G_p = 10 \log\left(\frac{V_1^2}{V_2^2}\right) = 20 \log\left(\frac{V_1}{V_2}\right) \text{dB} \tag{1.4.6}$$

The convenience of logarithmic scales has led to widespread use, sometimes bending the rules, which has caused argument in the literature (Simons 1973; Page 1973), but the improper usages have by now become so established that political correctness has been discarded. When we talk of voltage gains using Eq.(1.4.6) to determine the dB value, the usual difference of impedance levels is ignored. So long as we are all agreed and understand the usage there should be no confusion, but it is as well to be aware of the approximation. It should always be made clear whether power or voltage (or current) is being referred to as the measure will be different. Many derivative units have subsequently been defined such as dBm which refers to a power gain where the reference level (e.g. P_2 above) is 1 mW, so that 0 dBm = 1 mW.

An associated reason for using logarithmic scales is that (some of) our senses, e.g. hearing or vision, are logarithmic in sensitivity. This was expressed in the Weber–Fechner law, which holds that 'the minimum change in stimulus necessary to produce a perceptible change in response is proportional to the stimulus already existing' (Everitt and Anner 1956, p. 244). Such responses are illustrated by the audibility sensitivity curves averaged over many subjects in the early 1930s (e.g. Terman 1951); one wonders if measurements at the present time on subjects exposed to the extreme volumes of modern 'music' would reveal the same results.

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1.5 Exponentials

There must be an ideal world, a sort of mathematician's paradise where everything happens as it does in textbooks.

Bertrand Russell

The exponential function occurs frequently in electronics. It represents phenomena where the rate of change of a variable is proportional to the value of the variable. In a more abstract form we will meet it in Section 1.7 where we will find a most useful relation between it and the trigonometrical functions. Let us consider a more practical circumstance, the charging of a capacitor (Fig. 1.5.1).

We assume that the capacitor is uncharged (this is not essential) and at time $t=0$ the switch is closed. At this instant V_C is zero so the current $i=V_{in}/R=i_0$. We require to find the variation of V_C with time. At any time when the current is i and the charge on C is Q, V_C will be given by:

$$V_C=\frac{Q}{C} \quad \text{so that} \quad dV_C=\frac{dQ}{C}=\frac{i\,dt}{C}, \quad \text{where} \quad i=\frac{V_{in}-V_C}{R}$$
$$\text{so } \frac{dV_C}{dt}=\frac{i}{C}=\frac{V_{in}-V_C}{RC} \tag{1.5.1}$$

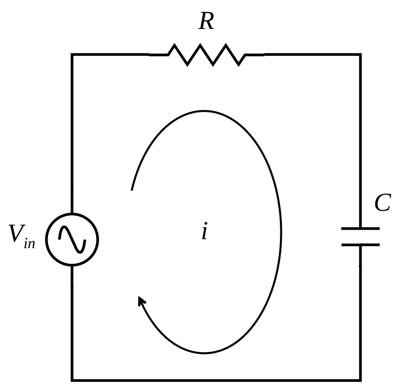

Fig. 1.5.1 Current flow in a capacitor.

This fits with the statement above about the exponential function except that the rate of change dV_c/dt is proportional to $V_{in} - V_C$, so that as V_C increases towards its final value V_{in} the rate of change will decrease. There is a formal method of solving this differential equation for V_C but we will take the easier path by guessing (knowing) the answer and showing that it agrees with Eq. (1.5.1). We try:

$$V_C = V_{in}(1 - e^{-t/RC}) \tag{1.5.2}$$

To see if this is in agreement with (1.5.1) we differentiate to give:

$$\begin{aligned} \frac{dV_C}{dt} &= 0 + \frac{V_{in}}{RC} e^{-t/RC} \\ &= \frac{V_{in}}{RC}\left(\frac{V_{in} - V_C}{V_{in}}\right) = \frac{V_{in} - V_C}{RC} \end{aligned} \tag{1.5.3}$$

where we have substituted for $e^{-t/RC}$ from Eq. 1.5.2, and see that (1.5.2) is a solution of Eq. (1.5.1). We also have:

$$i = \frac{V_{in} - V_C}{R} = \frac{V_{in}}{R} e^{-t/RC} = i_0\, e^{-t/RC}, \qquad \text{where} \qquad i_0 = \frac{V_{in}}{R} \tag{1.5.4}$$

The initial slope of V_C is given by the value of dV_C/dt at $t = 0$. From Eq. (1.5.3):

$$\left(\frac{dV_C}{dt}\right)_{t=0} = \frac{V_{in}}{RC} \tag{1.5.5}$$

The quantity $RC = \tau$ is called the time constant. The initial slope tangent will reach V_{in} at time τ. The exponent $-t/RC$ must be dimensionless so that the units of RC must be time. This can be checked:

$$\begin{aligned} &C = \frac{Q}{V} = \frac{\text{Coulomb}}{\text{Volt}} = \frac{\text{Amp sec}}{\text{Volt}} \qquad \text{and} \qquad R = \frac{\text{Volt}}{\text{Amp}} \\ &\text{so} \qquad RC = \frac{\text{Volt}}{\text{Amp}}\,\frac{\text{Amp sec}}{\text{Volt}} = \text{sec} \end{aligned} \tag{1.5.6}$$

It is often a useful check when doing some complex algebra to examine the consistency of the units of all the terms to see if they are compatible. Any inconsistency can alert you to errors in your algebra. Units are discussed in Section 2.12.

The voltage V_R across R is just the difference between V_{in} and V_C. Thus we have from Eq. (1.5.2):

$$V_R = V_{in} - V_C = V_{in}\, e^{-t/RC} \tag{1.5.7}$$

The form of the various functions are shown in Fig. 1.5.2.

Theoretically V_C, for example, never reaches V_{in}. The time to reach within a given percentage of V_{in} can be calculated and must be allowed for when making more

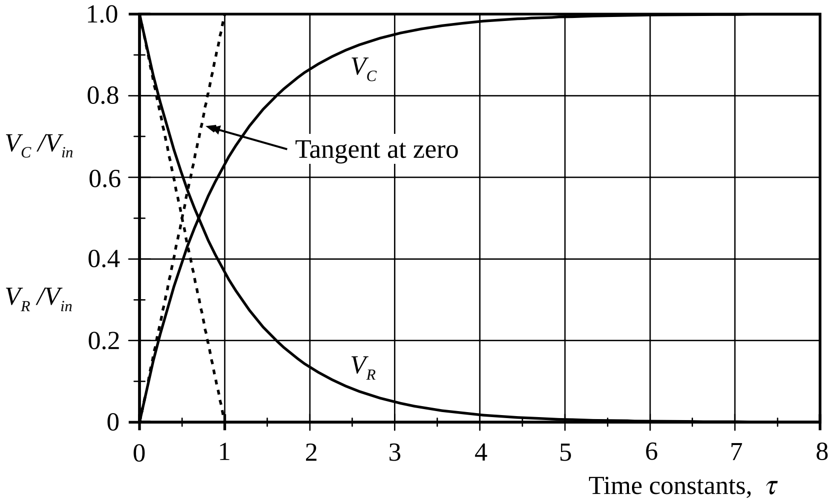

Fig. 1.5.2 Exponential responses and initial slopes.

Table 1.5.1 *Approach of an exponential function to the final value*

Time τ	0	1	2	3	4	5	6	7
V_C/V_{in}	0	0.632	0.865	0.950	0.982	0.993	0.998	0.999
V_R/V_{in}	1	0.368	0.135	0.050	0.018	0.007	0.002	0.001

accurate measurements. Table 1.5.1 shows the difference as a function of multiples of the time constant τ. For example at time $t=\tau=RC$ the value of V_C is $\cong$63% of V_{in} and you must wait for seven time constants to be within 0.1% of V_{in}.

If we consider a similar circuit to Fig. 1.5.1 with an inductor replacing the capacitor then a similar analysis leads to the result:

$$i_L = i_0(1-e^{-tR/L}), \qquad \text{with} \qquad i_0 = V_{in}/R \tag{1.5.8}$$

and in this case the time constant is $\tau = L/R$.

References and additional sources 1.5

Courant R. (1937): *Differential and Integral Calculus*, 2nd Edn, Vols 1 and 2, London: Blackie and Son. See Chapter III, Section 6.

Lambourne R., Tinker M. (Eds) (2000): *Basic Mathematics for the Physical Sciences*, New York: John Wiley. ISBN 0-471-85207-4.

Walton A. K. (1987): *Network Analysis and Practice*, Cambridge: Cambridge University Press. ISBN 0-521-31903-X. See Appendix 2.

1.6 Vectors

> Also, he should remember that unfamiliarity with notation and processes may give an appearance of difficulty that is entirely fictitious, even to an intrinsically easy matter; so that it is necessary to thoroughly master the notation and ideas involved. The best plan is to sit down and work; all that books can do is to show the way.
>
> Oliver Heaviside (1891): *Electromagnetic Theory*, Nov. 13, Vol. I, p. 139

In discussing electromagnetic topics it is necessary to make use of vectors since many of the quantities involved have both magnitude and direction. The algebra of vectors is a little messy but if we can understand the vector relationships things become much neater and easier to write. There are a number of very useful theorems which allow us to transform relations to suit our purposes: we will state and describe how they work. Vectors are written in bold italic type, e.g. $\boldsymbol{A}$.

The addition and subtraction of vectors follows the simple parallelogram geometry as discussed in Section 1.7 and the possible circumstance of three dimensions simply requires two successive operations. Subtraction also follows the same technique. Vectors may be resolved along any suitable set of coordinates, such as Cartesian or polar, but for our purposes we can restrict our choice to Cartesian.

Multiplication presents us with two different possibilities. Any vector $\boldsymbol{A}$, say, may be resolved into components along a chosen set of rectangular coordinates x, y and z, with components A_x, A_y and A_z (Fig. 1.6.1).

The magnitude of the vector is then given by:

$$\text{Magnitude} = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}} \tag{1.6.1}$$

We can define the *scalar product*, shown by $\boldsymbol{A} \bullet \boldsymbol{A}$ (sometimes also called the dot product) by:

$$\boldsymbol{A} \bullet \boldsymbol{A} = A_x^2 + A_y^2 + A_z^2 \tag{1.6.2}$$

which is a scalar, i.e. it has no direction, only magnitude. It is just the square of the length of the vector $\boldsymbol{A}$ and, though the components will change, $\boldsymbol{A} \bullet \boldsymbol{A}$ is independent of the axes chosen. The scalar product of two different vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined in a similar way as:

$$\boldsymbol{A} \bullet \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.6.3}$$

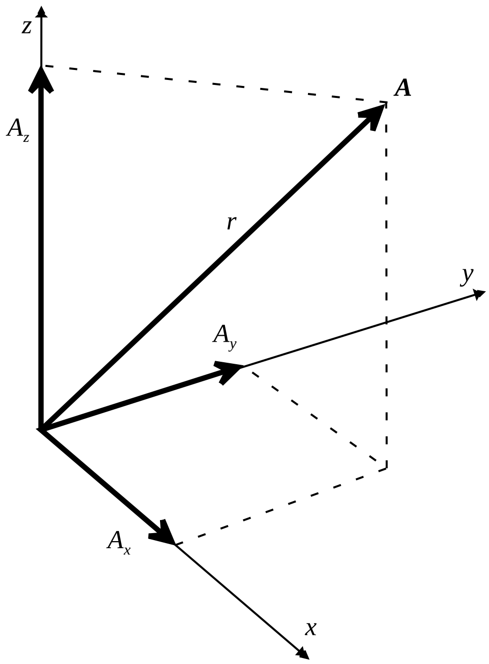

Fig. 1.6.1 Vector components.

and an alternative form can be expressed in terms of the lengths, A and B, of the two vectors and the angle θ between them (Fig. 1.6.2):

$$\boldsymbol{A} \bullet \boldsymbol{B} = AB\cos\theta \tag{1.6.4}$$

and it is evident that the order of the vectors is immaterial, i.e. $\boldsymbol{A} \bullet \boldsymbol{B} = \boldsymbol{B} \bullet \boldsymbol{A}$.

It is often convenient to make use of *unit vectors* along each of the three axes x, y and z. These are usually represented by the vector symbols $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, respectively (it is perhaps unfortunate that i is also used in complex numbers, and that we use j instead in electronics, but the context should make the meaning clear; there are just not enough symbols to go round for everything to have its own). The scalar products of these vectors can be readily deduced from Eq. (1.6.4) to give:

$$\begin{array}{lll} \boldsymbol{i} \bullet \boldsymbol{i} = 1 & \boldsymbol{j} \bullet \boldsymbol{j} = 1 & \boldsymbol{k} \bullet \boldsymbol{k} = 1 \\ \boldsymbol{i} \bullet \boldsymbol{j} = 0 & \boldsymbol{j} \bullet \boldsymbol{k} = 0 & \boldsymbol{k} \bullet \boldsymbol{i} = 0 \end{array} \tag{1.6.5}$$

and we can use the unit vectors to express any vector in the form:

$$\boldsymbol{A} = \boldsymbol{i}A_x + \boldsymbol{j}A_y + \boldsymbol{k}A_z \tag{1.6.6}$$

Some quantities multiply in a quite different way. You may be aware that the force acting on a charge moving in a magnetic field is proportional to the product of velocity and field but acts in a direction normal to both velocity and field. This

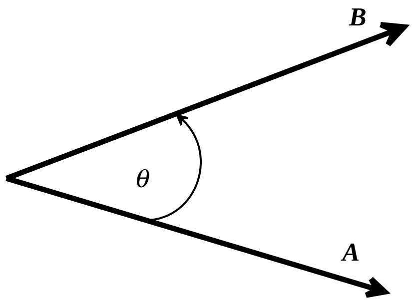

Fig. 1.6.2 Vector scalar or dot product.

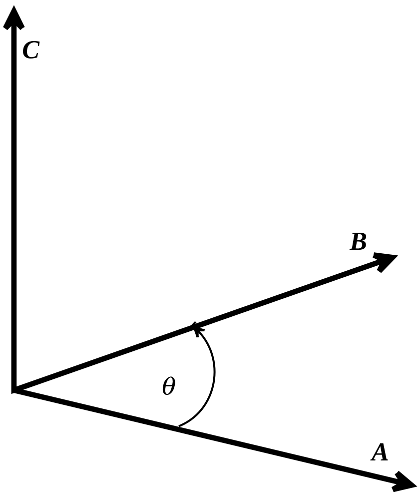

Fig. 1.6.3 Vector cross product.

requires the definition of another form of multiplication known as the *vector or cross product* shown by $\boldsymbol{A} \times \boldsymbol{B}$ which is itself a vector (some books use the symbol $\wedge$ instead of $\times$). The magnitude is most directly defined by:

$$\boldsymbol{A} \times \boldsymbol{B} = AB \sin \theta \tag{1.6.7}$$

and the direction of the vector is normal to the plane containing $\boldsymbol{A}$ and $\boldsymbol{B}$ and in the sense of the advancement of a right-handed screw rotated from $\boldsymbol{A}$ to $\boldsymbol{B}$, and shown as $\boldsymbol{C}$ in Fig. 1.6.3. It is evident that reversing the order of $\boldsymbol{A}$ and $\boldsymbol{B}$ gives the same magnitude but the opposite direction, i.e. $\boldsymbol{B} \times \boldsymbol{A} = -\boldsymbol{C}$.

The consequences for unit vectors are:

$$\begin{aligned} &\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0 \\ &\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \qquad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \qquad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j} \end{aligned} \tag{1.6.8}$$

If we write the vectors in the form of Eq. (1.6.6) and carry out the multiplications using (1.6.8), then we have:

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B} &= (\boldsymbol{i}A_x+\boldsymbol{j}A_y+\boldsymbol{k}A_z)\times(\boldsymbol{i}B_x+\boldsymbol{j}B_y+\boldsymbol{k}B_z)\\ &= \boldsymbol{i}(A_yB_z-A_zB_y)+\boldsymbol{j}(A_zB_x-A_xB_z)+\boldsymbol{k}(A_xB_y-A_yB_x)\\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\end{aligned} \tag{1.6.9}$$

where the last line expresses the relation in the form of a determinant (Section 1.10) and is much easier to recall.

We will be making use of vector algebra for dealing with electromagnetic field quantities, which vary both with respect to position and with time. We therefore need to examine how we can differentiate and integrate vectors. Taking a vector in the form of Eq. (1.6.6), then if say the vector is a function of t we have:

$$\frac{d\boldsymbol{A}}{dt}=\boldsymbol{i}\frac{dA_x}{dt}+\boldsymbol{j}\frac{dA_y}{dt}+\boldsymbol{k}\frac{dA_z}{dt} \tag{1.6.10}$$

which is a vector whose components are the derivatives of the components of $\boldsymbol{A}$. If we have a field representing a simple scalar quantity $\phi(x, y, z)$, (temperature is a good example), then we can ask what is the steepest slope or gradient at any point and this will evidently depend on the slope in the direction of each of the axes. The result is found to be the gradient of ϕ:

$$\text{grad }\phi=\boldsymbol{i}\frac{\partial\phi}{\partial x}+\boldsymbol{j}\frac{\partial\phi}{\partial y}+\boldsymbol{k}\frac{\partial\phi}{\partial z} \tag{1.6.11}$$

and is evidently also a vector. The use of ∂ signifies that when differentiating only quantities depending on what you are differentiating with respect to are relevant, e.g. in the first term those depending on x are relevant while those depending on y or z are considered constants. The form found here arises frequently and it is convenient to define a symbol to represent this in the form of an operator, for example just like say d/dt:

$$\boldsymbol{\nabla}=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z} \tag{1.6.12}$$

The vector operator $\boldsymbol{\nabla}$ is called *del* and only has meaning when operating on something. Thus we have that $\boldsymbol{\nabla}\phi\equiv\text{grad }\phi$ as in Eq. (1.6.11). $\boldsymbol{\nabla}$ can also operate on a vector. If we have a field described by a vector function $\boldsymbol{V}(x, y, z)$ where the components V_x, V_y and V_z of $\boldsymbol{V}$ are functions of x, y and z:

$$\boldsymbol{V}(x, y, z)=\boldsymbol{i}V_x(x, y, z)+\boldsymbol{j}V_y(x, y, z)+\boldsymbol{k}V_z(x, y, z) \tag{1.6.13}$$

then we can define the *divergence* of $\boldsymbol{V}$ by:

$$\operatorname{div} \boldsymbol{V} \equiv \boldsymbol{\nabla} \cdot \boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{1.6.14}$$

which is a scalar. We can also define the *curl* of $\boldsymbol{V}$ by:

$$\operatorname{curl} \boldsymbol{V} \equiv \boldsymbol{\nabla} \times \boldsymbol{V} = \boldsymbol{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \tag{1.6.15}$$

and this is a vector. Since the gradient is a vector function we can define a further useful relation by taking the divergence of it:

$$\begin{aligned} \operatorname{div} \operatorname{grad} \phi \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \phi &= \frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \phi}{\partial z} \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \equiv \nabla^2 \phi \end{aligned} \tag{1.6.16}$$

and this is a very important expression. The operator ∇^2 is called the Laplacian and is a scalar operator. For example an equation of the form:

$$\nabla^2 \phi = \frac{1}{a^2}\frac{\partial^2 \phi}{\partial t^2} \tag{1.6.17}$$

is a wave equation as we will come across in examining the consequences of Maxwell's equations. If we reverse the order of div and grad and apply this to a vector, then:

$$\begin{aligned} &\operatorname{grad} \operatorname{div} \boldsymbol{V} \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) \\ &= \boldsymbol{i}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z}\right) + \boldsymbol{j}\left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z}\right) + \boldsymbol{k}\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2}\right) \end{aligned} \tag{1.6.18}$$

and now it becomes more evident that the symbolic forms can save a lot of writing. Since ∇^2 is a scalar, the operation on a vector is simply a vector with components:

$$\nabla^2 \boldsymbol{V} \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{V} = (\nabla^2 V_x,\ \nabla^2 V_y,\ \nabla^2 V_z) \tag{1.6.19}$$

One further relation will be needed which is defined by:

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{V}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\boldsymbol{V} \\ &= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) - \nabla^2 \boldsymbol{V} \end{aligned}$$

or $\quad \operatorname{curl} \operatorname{curl} \boldsymbol{V} = \operatorname{grad} \operatorname{div} \boldsymbol{V} - \mathrm{del}^2\, \boldsymbol{V} \qquad (1.6.20)$

The divergence of a cross product will be required in Section 2.2:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) \tag{1.6.21}$$