Part 1

Mathematical techniques

Philosophy is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these one is wandering about in a dark labyrinth.

Galileo Galilei (1564–1642)

As indicated in the preface, this book is substantially about design and hence prediction. The tools that allow us to extrapolate to create a new design are an understanding of the physical characteristics and limitations of components, mathematical techniques that allow us to determine the values of components and responses to input signals, and of course as much experience as one can get. The latter of course includes making as much use as possible of the experience of others either by personal contact or by consulting the literature.

This part covers much of the basic mathematics that is generally found useful in analysing electronic circuits. There is a fairly widely propagated view that you can get by without much mathematical knowledge but I evidently do not subscribe to this. Many do indeed do very well without recourse to mathematics but they could do so much better with some knowledge, and this book is, in part, an attempt to persuade them to make the effort. We do not present a course on these techniques as that would expand the book far beyond an acceptable size, but rather provide an indication and reminder of what we think is important and useful. Much of the reluctance in this direction is possibly caused by the unattractiveness of heavy numerical computation but this is nowadays generally unnecessary since we have the assistance of many mathematical computational packages and, in our case, the enormous power and convenience of electronic simulation software. With the spread of the ubiquitous PC it is now uncommon for an electronicist to be without access to one.

When carrying out algebraic analysis it is all too easy to make mistakes and great care must be taken when writing out equations. It is often of assistance to check your units to see that they are consistent as this can often be of great use in catching errors. You also need to be prepared to make approximations as the equations for even quite simple circuits become more complex than can be analysed. SPICE can be of considerable assistance in that you may use it to determine at least approximate values for parameters that then allow you to determine the relative magnitudes of terms and hence which may be neglected without serious error. You can then check your final result against SPICE which is able to carry out the analysis without significant approximation. The benefit of the algebraic analysis is that it makes the function of each component evident and provides parameterized design formulae.

Though some of the topics may at first sight seem unexpected, I hope that as you progress through later sections you will come to appreciate their relevance. Some are treated in terms of simply a reminder and some are delved into in a little more detail. As far as possible references to further sources of information are provided.

1.1 Trigonometry

The power of instruction is seldom of much efficacy except in those happy dispositions where it is almost superfluous.

Edward Gibbon

It may seem unexpected to find a section on trigonometry, but in electronics you cannot get away from sine waves. The standard definitions of sine, cosine and tangent in terms of the ratio of the sides of a right-angled triangle are shown in Fig. 1.1.1 and Eq. (1.1.1).

For angle θ and referring to the sides of the triangle as opposite (o), adjacent (a) and hypotenuse (h) we have:

$$\sin\theta = \frac{o}{h}, \qquad \cos\theta = \frac{a}{h}, \qquad \tan\theta = \frac{o}{a} \tag{1.1.1}$$

A common way to represent a sinusoidal wave is to rotate the phasor OA around the origin O at the appropriate rate ω (in radians per second) and take the projection of OA as a function of time as shown in Fig. 1.1.2.

The corresponding projection along the x-axis will produce a cosine wave. This allows us to see the values of the functions at particular points, e.g. at $\omega t = \pi/2$, π, $3\pi/2$ and 2π as well as the signs in the four quadrants (Q). These are summarized in Table 1.1.1.

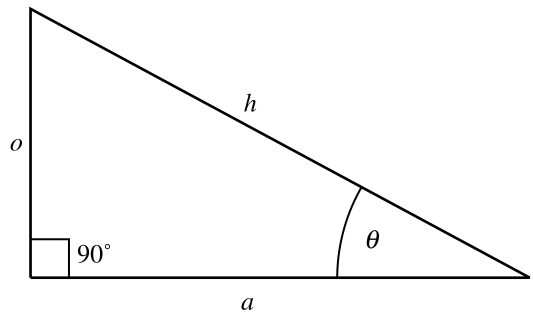

Fig. 1.1.1 Right-angled triangle.

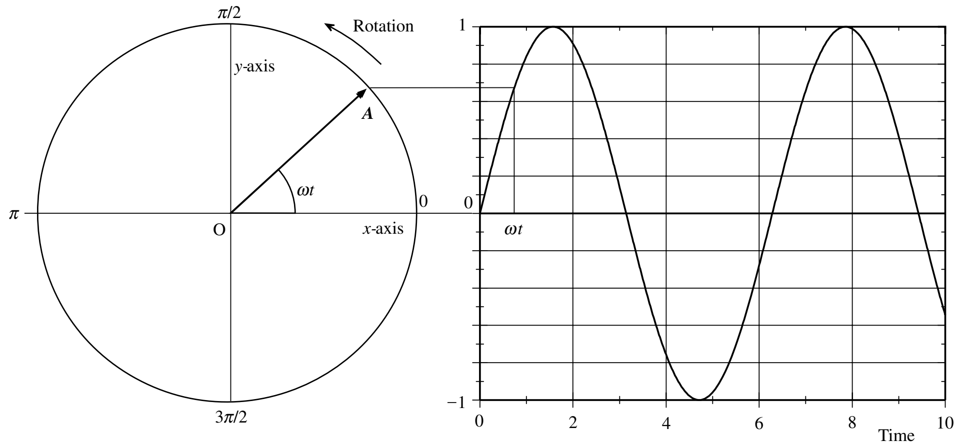

Fig. 1.1.2 Projection of a rotating vector.

Table 1.1.1 *Values and signs of trigonometrical functions in the four quadrants*

Angle	0	$\pi/2$	π	$3\pi/2$	2π	1Q	2Q	3Q	4Q
Sin	0	1	0	1	0	+	+	−	−
Cos	1	0	1	0	1	+	−	−	+
Tan	0	∞	0	∞	0	+	−	+	−

Some useful relationships for various trigonometrical expressions are:

(a) $\sin(-\theta) = -\sin\theta$

(b) $\cos(-\theta) = \cos\theta$

(c) $\tan(-\theta) = -\tan\theta$

(d) $\cos(\theta+\phi) = \cos\theta\cdot\cos\phi - \sin\theta\cdot\sin\phi$

(e) $\sin(\theta+\phi) = \sin\theta\cdot\cos\phi + \cos\theta\cdot\sin\phi$

(f) $\sin\theta + \sin\phi = 2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)$

(g) $\cos\theta + \cos\phi = 2\cos\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi)$

(h) $\sin\theta\cdot\cos\phi = \frac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$

(i) $\cos\theta\cdot\cos\phi = \frac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$

(j) $\sin^2\theta + \cos^2\theta = 1$

(k) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ (1.1.2)

(l) $1 + \cos\theta = 2\cos^2(\theta/2)$

(m) $1 - \cos\theta = 2\sin^2(\theta/2)$

(n) $\sin 2\theta = 2\sin\theta\cos\theta$

(o) $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=\lim_{\theta\to 0}\dfrac{\tan\theta}{\theta}=1$

(p) $\cos(\theta-\phi)=\cos\theta\cdot\cos\phi+\sin\theta\cdot\sin\phi$

(q) $\sin(\theta-\phi)=\sin\theta\cdot\cos\phi-\cos\theta\cdot\sin\phi$

(r) $\sin\theta-\sin\phi=2\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi)$

(s) $\cos\theta-\cos\phi=2\sin\tfrac{1}{2}(\phi+\theta)\sin\tfrac{1}{2}(\phi-\theta)$

(t) $\cos\theta\cdot\sin\phi=\tfrac{1}{2}[\sin(\theta+\phi)-\sin(\theta-\phi)]$

(u) $\sin\theta\cdot\sin\phi=\tfrac{1}{2}[\cos(\theta-\phi)-\cos(\theta+\phi)]$

We will have occasion to refer to some of these in other sections and it should be remembered that the complex exponential expressions (Section 1.5) are often easier to use.

References and additional sources 1.1

Fink D., Christansen D. (1989): *Electronics Engineer's Handbook*, 3rd Edn, New York: McGraw-Hill. ISBN 0-07-020982-0

Korn G. A., Korn T. M. (1989): Mathematics, formula, definitions, and theorems used in electronics engineering. In Fink D., Christansen D. *Electronics Engineer's Handbook*, 3rd Edn, Section 2, New York: McGraw-Hill.

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Langford-Smith F. (1954): *Radio Designer's Handbook*, London: Illife and Sons.

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1.2 Geometry

For geometry by itself is a rather heavy and clumsy machine. Remember its history, and how it went forward with great bounds when algebra came to its assistance. Later on, the assistant became the master.

Oliver Heaviside (1899): *Electromagnetic Theory*, April 10, Vol. II, p. 124

The relationship between equations and their geometric representation is outlined in Section 1.10. In various places we will need to make use of Cartesian (coordinate) geometry either to draw graphical responses or to determine various parameters from the graphs.

A commonly used representation of the first order, or single pole, response of an operational amplifier in terms of the zero frequency (z.f.) gain A_0 and the corner frequency $\omega_c = 1/T$ is given by:

$$A = \frac{A_0}{1 + sT} \tag{1.2.1}$$

which on the log–log scales usually used is as shown in Fig. 1.2.1.

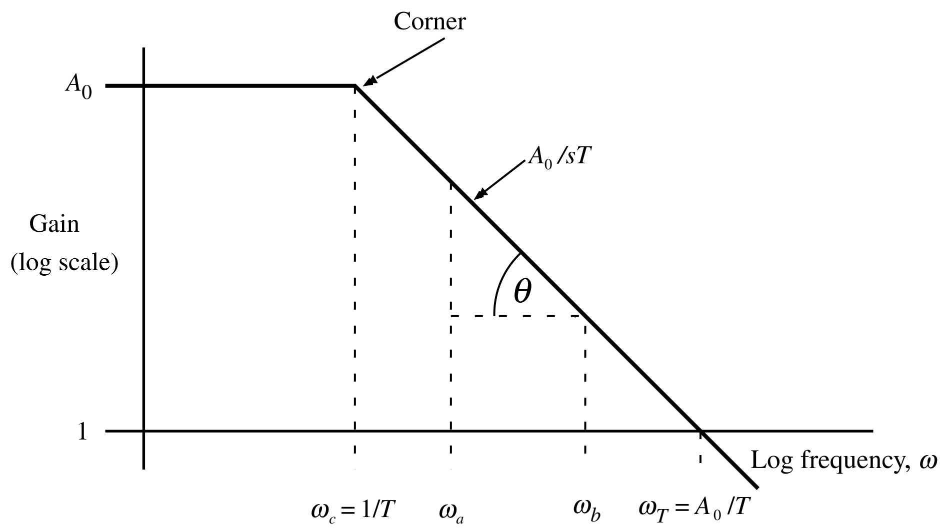

Fig. 1.2.1 Operational amplifier response.

At low frequency $sT \ll 1$ so the gain is just A_0. At high frequency $sT \gg 1$ so the gain is A_0/sT as shown. The graphs are slightly different from those with standard x- and y-axes, which are of course where $y=0$ and $x=0$, because here we have logarithmic scales. The traditional x- and y-axes would then be at $-\infty$, which is not useful. We can start by taking $G=1$ and $f=1$ for which the log values are zero (log $1=0$). You can of course draw your axes at any value you wish, but decade intervals are preferable. The slope of A_0/sT can be determined as follows. Take two frequencies ω_a and ω_b as shown. The slope, which is the tangent of the angle θ, will be given (allowing for the sense of the slope) by:

$$\begin{aligned}\tan\theta &= \frac{\log\left(\dfrac{A_0}{\omega_a T}\right) - \log\left(\dfrac{A_0}{\omega_b T}\right)}{\log(\omega_b) - \log(\omega_a)} \\ &= \frac{\log\left(\dfrac{\omega_b}{\omega_a}\right)}{\log\left(\dfrac{\omega_b}{\omega_a}\right)} \\ &= 1 \qquad \text{and} \qquad \theta = \tan^{-1}(1) = 45°\end{aligned} \tag{1.2.2}$$

where we have used the fact that the difference of two logs is the log of the quotient (Section 1.4). Since the slope does not depend on ω then A_0/sT must be a straight line on the log–log scales.

A point of interest on the amplifier response is the unity-gain (or transition) frequency ω_T as this defines the region of useful performance and is particularly relevant to considerations of stability. We need to find the value of ω for which $G=1$. Note that though we use the more general complex frequency s we can simply substitute ω for s since we are dealing with simple sine waves and are not concerned with phase since this is a graph of amplitude. Thus we have:

$$1 = \frac{A_0}{\omega_T T} \qquad \text{or} \qquad \omega_T = \frac{A_0}{T} \tag{1.2.3}$$

and remember that ω is an angular frequency in rad s^{-1} and $f=\omega/2\pi$ Hz. Say we now wish to draw the response for a differentiator which has $G=sR_fC_i$ (Section 5.6). The gain will be 1 when $\omega_D = 1/R_fC_i$ and the slope will be 45°. So fixing point ω_D and drawing a line at 45° will give the frequency response. To find where this line meets the open-loop response we have:

$$sR_fC_i = \frac{A_0}{sT} \qquad \text{or} \qquad s^2 = \frac{A_0}{R_fC_iT} \qquad \text{so} \qquad \omega_P = \left(\frac{A_0}{R_fC_iT}\right)^{\frac{1}{2}} \tag{1.2.4}$$

which is shown in Fig. 1.2.2.

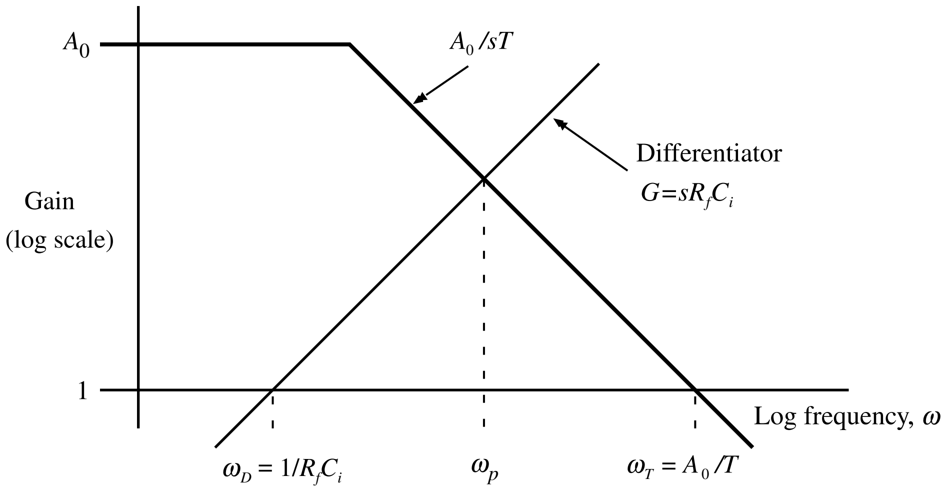

Fig. 1.2.2 Geometry of differentiator frequency response.

References and additional sources 1.2

Hambley A. R. (1994): *Electronics. A Top-Down Approach to Computer-Aided Circuit Design*, New York: Macmillan. ISBN 0-02-349335-6. See Appendix B.

Lambourne R., Tinker M. (Eds) (2000): *Basic Mathematics for the Physical Sciences*, New York: John Wiley. ISBN 0-471-85207-4.

Tobey G. E., Graeme J. G., Huelsman L. P. (1971): *Operational Amplifiers. Design and Applications*, New York: McGraw-Hill. Library of Congress Cat. No. 74-163297.

Van Valkenburg M. E. (1982): *Analog Filter Design*, New York: Holt, Rinehart and Winston. ISBN 0-03-059246-1, or 4-8338-0091-3 International Edn. See Chapter 3.

1.3 Series expansions

Prof. Klein distinguishes three main classes of mathematicians – the intuitionists, the formalists or algorithmists, and the logicians. Now it is intuition that is most useful in physical mathematics, for that means taking a broad view of a question, apart from the narrowness of special mathematics. For what a physicist wants is a good view of the physics itself in its mathematical relations, and it is quite a secondary matter to have logical demonstrations. The mutual consistency of results is more satisfying, and exceptional peculiarities are ignored. It is more useful than exact mathematics.

But when intuition breaks down, something more rudimentary must take its place. This is groping, and it is experimental work, with of course some induction and deduction going along with it. Now, having started on a physical foundation in the treatment of irrational operators, which was successful, in seeking for explanation of some results, I got beyond the physics altogether, and was left without any guidance save that of untrustworthy intuition in the region of pure quantity. But success may come by the study of failures. So I made a detailed study and close examination of some of the obscurities before alluded to, beginning with numerical groping. The result was to clear up most of the obscurities, correct the errors involved, and by their revision to obtain correct formulae and extend results considerably.

Oliver Heaviside (1899): *Electromagnetic Theory*, April 10, Vol. II, p. 460

Expansion of functions in terms of infinite (usually) series is often a convenient means of obtaining an approximation that is good enough for our purposes. In some cases it also allows us to obtain a relationship between apparently unconnected functions, and one in particular has been of immense importance in our and many other fields. We will list here some of the more useful expansions without derivation (Boas 1966):

$$\begin{aligned}\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\end{aligned} \tag{1.3.1}$$

where

$$n! \equiv n(n-1)(n-2)(n-3)\cdots 1$$

and $n!$ is known as n factorial. Note that, odd though it may seem, $0! = 1$. The variable θ must be in radians.

The binomial series is given by:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots \qquad (1.3.2)$$

valid for n positive or negative and $|x| < 1$
which is most frequently used for $x \ll 1$ to give a convenient approximation:

$$(1+x)^n \cong 1 + nx \qquad (1.3.3)$$

The geometric series in x has a sum S_n to n terms given by:

$$a + ax^2 + ax^3 + ax^4 + \cdots + ax^n + \cdots +, \quad \text{with sum } S_n = \frac{a(1-x^n)}{1-x} \qquad (1.3.4)$$

and for $|x| < 1$ the sum for an infinite number of terms is:

$$S = \frac{a}{1-x} \qquad (1.3.5)$$

In some circumstances, when we wish to find the value of some function as the variable goes to a limit, e.g. zero or infinity, we find that we land up with an indeterminate value such as $0/0$, ∞/∞ or $0 \times \infty$. In such circumstances, if there is a proper limit, it may be determined by examining how the function approaches the limit rather than what it appears to do if we just substitute the limiting value of the variable. As an example consider the function (Boas 1966, p. 27):

$$\lim_{x\to 0} \frac{1-e^x}{x}, \text{ which becomes } \frac{0}{0} \text{ for } x = 0 \qquad (1.3.6)$$

If we expand the exponential using Eq. (1.3.1), then remembering that x is going to become very small:

$$\lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{1 - \left(1 + x + \frac{x^2}{2!} + \cdots\right)}{x} = \lim_{x\to 0} \left(-1 - \frac{x}{2!} - \cdots\right) = -1 \qquad (1.3.7)$$

Expansion in terms of a series, as in this case, is generally most useful for cases where $x \to 0$, since in the limit the series is reduced to the constant term. There is another approach, known as l'Hôpital's rule (or l'Hospital), which makes use of a Taylor series expansion in terms of derivatives. If the derivative of $f(x)$ is $f'(x)$, then:

$$\lim_{x\to 0} \frac{f(x)}{\phi(x)} = \lim_{x\to 0} \frac{f'(x)}{\phi'(x)} \qquad (1.3.8)$$