1

Introduction

1.1 Introduction to Asymptotics

Before venturing into our examination of Mellin-Barnes integrals, we present an overview of some of the basic definitions and ideas found in asymptotic analysis. The treatment provided here is not intended to be comprehensive, and several high quality references exist which can provide a more complete treatment than is given here: in particular, we recommend the tracts by Olver (1974), Bleistein & Handelsman (1975) and Wong (1989) as particularly good treatments of asymptotic analysis, each with their own strengths.†

1.1.1 Order Relations

Let us begin our survey by defining the *Landau symbols* O and o and the notion of asymptotic equality.

Let f and g be two functions defined in a neighbourhood of x_0. We say that $f(x) = O(g(x))$ as $x \to x_0$ if there is a constant M for which

$$|f(x)| \le M\,|g(x)|$$

for x sufficiently close to x_0. The constant M depends only on how close to x_0 we wish the bound to hold. The notation $O(g)$ is read as 'big-oh of g', and the constant M, which is often not explicitly calculated, is termed the *implied constant.*

In a similar fashion, we define $f(x) = o(g(x))$ as $x \to x_0$ to mean that

$$|f(x)/g(x)| \to 0$$

† Olver provides a good balance between techniques used in both integrals and differential equations; Bleistein & Handelsman present a relatively unified treatment of integrals through the use of Mellin convolutions; and Wong develops the theory and application of (Schwartz) distributions in the setting of developing expansions of integrals.

as $x \to x_0$, subject to the proviso that $g(x)$ be nonzero in a neighbourhood of x_0. The expression $o(g)$ is read as 'little-oh of g', and from the preceding definition, it is immediate that $f = o(g)$ implies that $f = O(g)$ (merely take the implied constant to be any (arbitrarily small) positive number).

The last primitive asymptotic notion required is that of asymptotic equality. We write

$$f(x) \sim g(x)$$

as $x \to x_0$ to mean that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1,$$

provided, of course, that g is nonzero sufficiently close to x_0. The tilde here is read 'is asymptotically equal to'. An equivalent formulation of asymptotic equality is readily available: for $x \to x_0$,

$$f(x) \sim g(x) \qquad \text{iff} \qquad f(x) = g(x)\{1 + o(1)\}.$$

Example 1. The function $\log x$ satisfies the order relation $\log x = O(x - 1)$ as $x \to \infty$, since the ratio $(\log x)/(x - 1)$ is bounded for all large x. In fact, it is also true that $\log x = o(x - 1)$ for large x, and for $x \to 1$, $\log x \sim x - 1$.

Example 2. Stirling's formula is a well-known asymptotic equality. For large n, we have

$$n! \sim (2\pi)^{\frac{1}{2}} e^{-n} n^{n+\frac{1}{2}}.$$

This result follows from the asymptotic expansion of the gamma function, a result carefully developed in §2.1.

Example 3. The celebrated Prime Number Theorem is an asymptotic equality. If we denote by $\pi(x)$ the number of primes less than or equal to x, then for large positive x we have the well-known result

$$\pi(x) \sim \frac{x}{\log x}.$$

With the aid of Gauss' logarithmic integral,†

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$$

we also have the somewhat more accurate form

$$\pi(x) \sim \mathrm{li}(x) \qquad (x \to \infty).$$

† We note here that $\mathrm{li}(x)$ is also used to denote the same integral, but taken over the interval $(0, x)$, with $x > 1$. With this larger interval, the integral is a Cauchy principal value integral. The notation in this example appears to be in use by some number theorists, and is also sometimes written $\mathrm{Li}(x)$.

That both forms hold can be seen from a simple integration by parts:

$$\mathrm{li}(x) = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{(\log t)^2}.$$

An application of l'Hôpital's rule reveals that the resulting integral on the right-hand side is $o(x/\log x)$, from which the $x/\log x$ form of the Prime Number Theorem follows.

A number of useful relationships exist for manipulating the Landau symbols. The following selections are all easily obtained from the above definitions, and are not established here:

$$\begin{array}{ll} \text{(a) } O(O(f)) = O(f) & \text{(e) } O(f) + O(f) = O(f) \\ \text{(b) } o(o(f)) = o(f) & \text{(f) } o(f) + o(f) = o(f) \\ \text{(c) } O(fg) = O(f)\cdot O(g) & \text{(g) } o(f) + O(f) = O(f) \\ \text{(d) } O(f)\cdot o(g) = o(fg) & \text{(h) } O(o(f)) = o(O(f)) = o(f). \end{array} \tag{1.1.1}$$

It is easy to deduce linearity of Landau symbols using these properties, and it is a simple matter to establish asymptotic equality as an equivalence relation. In the transition to calculus, however, some difficulties surface.

A moment's consideration reveals that differentiation is, in general, often badly behaved in the sense that if $f = O(g)$, then it does not necessarily follow that $f' = O(g')$, as the example $f(x) = x + \sin e^x$ aptly illustrates: for large, real x, we have $f = O(x)$, but the derivative of f is not bounded (i.e., not $O(1)$).

The situation for integration is a good deal better. It is possible to formulate many results concerning integrals of order estimates, but we content ourselves with just two.

Example 4. For functions f and g of a real variable x satisfying $f = O(g)$ as $x \to x_0$ on the real line, we have

$$\int_{x_0}^x f(t)\,dt = O\left(\int_{x_0}^x |g(t)|\,dt\right) \qquad (x \to x_0).$$

A proof can be fashioned along the following lines: for $f(t) = O(g(t))$, let M be the implied constant so that $|f(t)| \le M\,|g(t)|$ for t sufficiently close to x_0, say $|t - x_0| \le \eta$. (For $x_0 = \infty$, a suitable interval would be $t \ge N$ for some large positive N.) Then

$$-M\,|g(t)| \le f(t) \le M\,|g(t)| \qquad (|t - x_0| \le \eta),$$

whence the result follows upon integration.

Example 5. If f is an integrable function of a real variable x, and $f(x) \sim x^\nu$, $\mathrm{Re}(\nu) < -1$ as $x \to \infty$, then

$$\int_x^\infty f(t)\,dt \sim -\frac{x^{\nu+1}}{\nu+1} \qquad (x \to \infty).$$

A proof of this claim follows from $f(x) = x^\nu\{1 + \psi(x)\}$ where $\psi(x) = o(1)$ as $x \to \infty$, for then

$$\int_x^\infty f(t)\,dt = -\frac{x^{\nu+1}}{\nu+1} + \int_x^\infty t^\nu \psi(t)\,dt.$$

But $\psi(t) = o(1)$ implies that for $\epsilon > 0$ arbitrarily small, there is an $x_0 > 0$ for which $|\psi(t)| < \epsilon$ whenever $t > x_0$. Thus, the remaining integral may be bounded as

$$\left|\int_x^\infty t^\nu \psi(t)\,dt\right| < \epsilon \int_x^\infty |t^\nu|\,dt \qquad (x > x_0).$$

Accordingly, we find

$$\int_x^\infty f(t)\,dt = -\frac{x^{\nu+1}}{\nu+1} + o\left(\frac{x^{\nu+1}}{\nu+1}\right) = -\frac{x^{\nu+1}}{\nu+1}\{1 + o(1)\},$$

from which the asymptotic equality is immediate. □

It is in the complex plane that we find differentiation of order estimates becomes better behaved. This is due, in part, to the fact that the Cauchy integral theorem allows us to represent holomorphic functions as integrals which, as we have noted, are better behaved in the setting of Landau symbols. A standard result in this direction is the following:

Lemma 1.1. *Let f be holomorphic in a region containing the closed annular sector $\mathcal{S} = \{z : \alpha \le \arg(z - z_0) \le \beta,\ |z - z_0| \ge R \ge 0\}$, and suppose $f(z) = O(z^\nu)$ (resp. $f(z) = o(z^\nu)$) as $z \to \infty$ in the sector, for fixed real ν. Then $f^{(n)}(z) = O(z^{\nu-n})$ (resp. $f^{(n)} = o(z^{\nu-n})$) as $z \to \infty$ in any closed annular sector properly interior to $\mathcal{S}$ with common vertex z_0.*

The proof of this result follows from the Cauchy integral formula for $f^{(n)}$, and is available in Olver (1974, p. 9).

1.1.2 Asymptotic Expansions

Let a sequence of continuous functions $\{\phi_n\}$, $n = 0, 1, 2, \ldots$, be defined on some domain, and let x_0 be a (possibly infinite) limit point of this domain. The sequence $\{\phi_n\}$ is termed an *asymptotic scale* if it happens that $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to x_0$, for every n. If f is some continuous function on the common domain of the asymptotic scale, then by an (infinite) *asymptotic expansion* of f with respect to the asymptotic scale $\{\phi_n\}$ is meant the formal series $\sum_{n=0}^\infty a_n\phi_n(x)$, provided the coefficients a_n, independent of x, are chosen so that for any nonnegative integer N,

$$f(x) = \sum_{n=0}^{N} a_n\phi_n(x) + O(\phi_{N+1}(x)) \qquad (x \to x_0). \tag{1.1.2}$$

In this case we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \qquad (x \to x_0).$$

Such a formal series is uniquely determined in view of the fact that the coefficients a_n can be computed from

$$a_N = \lim_{x \to x_0} \frac{1}{\phi_N(x)} \left\{ f(x) - \sum_{n=0}^{N-1} a_n \phi_n(x) \right\} \qquad (N = 0, 1, 2, \ldots).$$

The formal series so obtained is also referred to as an asymptotic expansion of *Poincaré type*, or an asymptotic expansion in the sense of Poincaré or, more simply, a Poincaré expansion. Examples of asymptotic scales and asymptotic expansions built with them are easy to come by. The most commonplace is the asymptotic power series: an *asymptotic power series* is a formal series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^{\nu_n},$$

where the appropriate asymptotic scale is the sequence $\{(x - x_0)^{\nu_n}\}$, $n = 0, 1, 2, \ldots$, and the ν_n are constants for which $(x - x_0)^{\nu_{n+1}} = o\big((x - x_0)^{\nu_n}\big)$ as $x \to x_0$. Any convergent Taylor series expansion of an analytic function f serves as an example of an asymptotic power series, with x_0 a point in the domain of analyticity of f, $\nu_n = n$ for any nonnegative integer n, and the coefficients in the expansion are the familiar Taylor coefficients $a_n = f^{(n)}(x_0)/n!$.

Asymptotic expansions, however, need not be convergent, as the next two examples illustrate.

Example 1. WATSON'S LEMMA. A well-known result of Laplace transform theory is that the Laplace transform of a piecewise continuous function on the interval $[0, +\infty)$ is $o(1)$ as the transform variable grows without bound. By imposing more structure on the small parameter behaviour of the function being transformed, a good deal more can be said about the growth at infinity of the transform.

Lemma 1.2. *Let $g(t)$ be an integrable function of the variable $t > 0$ with asymptotic expansion*

$$g(t) \sim \sum_{n=0}^{\infty} a_n t^{(n+\lambda-\mu)/\mu} \quad (t \to 0+)$$

for some constants $\lambda > 0$, $\mu > 0$. Then, provided the integral converges for all sufficiently large x, the Laplace transform of g, $\mathcal{L}[g; x]$, has the asymptotic behaviour

$$\mathcal{L}[g; x] \equiv \int_0^{\infty} e^{-xt} g(t)\, dt \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\lambda}{\mu}\right) \frac{a_n}{x^{(n+\lambda)/\mu}} \quad (x \to \infty).$$

Proof. To see this, let us put, for positive integer N and $t > 0$,

$$g_N(t) = g(t) - \sum_{n=0}^{N-1} a_n t^{(n+\lambda-\mu)/\mu}$$

so that the Laplace transform has a finite expansion with remainder given by

$$\mathcal{L}[g;x] = \sum_{n=0}^{N-1} \Gamma\left(\frac{n+\lambda}{\mu}\right) \frac{a_n}{x^{(n+\lambda)/\mu}} + \int_0^\infty e^{-xt} g_N(t)\,dt. \tag{1.1.3}$$

Since $g_N(t) = O(t^{(N+\lambda-\mu)/\mu})$, there are constants K_N and t_N for which

$$|g_N(t)| \le K_N\, t^{(N+\lambda-\mu)/\mu} \quad (0 < t \le t_N).$$

Use of this in the remainder term in our finite expansion (1.1.3) allows us to write

$$\begin{aligned}\left|\int_0^{t_N} e^{-xt} g_N(t)\,dt\right| &\le K_N \int_0^{t_N} e^{-xt} t^{(N+\lambda-\mu)/\mu} dt\\ &< \Gamma\left(\frac{N+\lambda}{\mu}\right) \frac{K_N}{x^{(N+\lambda)/\mu}}.\end{aligned} \tag{1.1.4}$$

By hypothesis, $\mathcal{L}[g;x]$ exists for all sufficiently large x, so the Laplace transform of g_N must also exist for all sufficiently large x, by virtue of (1.1.3). Let X be such that $\mathcal{L}[g_N;x]$ exists for all $x \ge X$, and put

$$G_N(t) = \int_{t_N}^t e^{-Xv} g_N(v)\,dv.$$

The function G_N so defined is a bounded continuous function on $[t_N, \infty)$, whence the bound

$$L_N = \sup_{[t_N,\infty)} |G_N(t)|$$

exists. Then for $x > X$, we have

$$\begin{aligned}\int_{t_N}^\infty e^{-xt} g_N(t)\,dt &= \int_{t_N}^\infty e^{-(x-X)t} e^{-Xt} g_N(t)\,dt\\ &= (x-X)\int_{t_N}^\infty e^{-(x-X)t} G_N(t)\,dt\end{aligned}$$

after one integration by parts. After applying the uniform bound L_N to the integral that remains, we arrive at

$$\left|\int_{t_N}^\infty e^{-xt} g_N(t)\,dt\right| \le (x-X)L_N \int_{t_N}^\infty e^{-(x-X)t}\,dt = L_N e^{-(x-X)t_N} \tag{1.1.5}$$

for $x > X$.

Together, (1.1.4) and (1.1.5) yield

$$\left|\int_0^\infty e^{-xt} g_N(t)\,dt\right| < \Gamma\left(\frac{N+\lambda}{\mu}\right)\frac{K_N}{x^{(N+\lambda)/\mu}} + L_N e^{-(x-X)t_N}$$

which, since $L_N e^{-(x-X)t_N}$ is $o(x^{-\nu})$ for any positive ν, establishes the asymptotic expansion for $\mathcal{L}[g; x]$. □

As a simple illustration of the use of Watson's lemma, consider the Laplace transform of $(1+t)^{\frac{1}{2}}$. From the binomial theorem, we have the convergent expansion as $t \to 0$

$$(1+t)^{\frac{1}{2}} = 1 + \tfrac{1}{2}t - \tfrac{1}{8}t^2 + \sum_{n=3}^{\infty}(-)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}t^n.$$

Since $(1+t)^{\frac{1}{2}}$ is of algebraic growth, its Laplace transform clearly exists for $x > 0$, and Watson's lemma produces the asymptotic expansion

$$\mathcal{L}[(1+t)^{\frac{1}{2}}; x] \sim \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{4x^3} + \sum_{n=3}^{\infty}(-)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n x^{n+1}}$$

as $x \to \infty$. The resulting asymptotic series is divergent, since the ratio of the $(n+1)$th to nth terms in absolute value is $(2n-1)/(2x)$ which, for fixed x, tends to ∞ with n. The reason for this divergence is a simple consequence of our applying the binomial expansion for $(1+t)^{\frac{1}{2}}$ (valid in $0 \le t \le 1$) in the Laplace integral beyond its interval of convergence.

Example 2. The confluent hypergeometric function† $U(1; 1; z)$ (which equals the exponential integral $e^z E_1(z)$) has the integral representation

$$U(1; 1; z) = \int_0^\infty \frac{e^{-t}\,dt}{t+z} \tag{1.1.6}$$

for z not a negative number or zero. In fact, it is relatively easy to show that this integral representation converges uniformly in the closed annular sector $\mathcal{S}_{\epsilon,\delta} = \{z : |z| \ge \epsilon, |\arg z| \le \pi - \delta\}$ for every positive ϵ and every positive $\delta < \pi$. Such a demonstration can proceed along the following lines.

Put $\theta = \arg z$ for $z \in \mathcal{S}_{\epsilon,\delta}$ and observe that for any nonnegative t, $|t+z|^2 = t^2 + |z|^2 + 2|z|t\cos\theta \ge t^2 + |z|^2 - 2|z|t\cos\delta \ge |z|^2\sin^2\delta$. Thus, the integrand of (1.1.6) admits the simple bound

$$e^{-t}|t+z|^{-1} \le e^{-t}|z|^{-1}\operatorname{cosec}\delta$$

whence we have, upon integrating the bound,

$$|U(1; 1; z)| \le |z|^{-1}\operatorname{cosec}\delta$$

† An alternative notation for this function is $\Psi(1; 1; z)$.

for $z \in \mathcal{S}_{\epsilon,\delta}$. The uniform convergence of the integral follows, from which we see that $U(1; 1; z)$ is holomorphic in the z plane cut along the negative real axis.

Through repeated integration by parts, differentiating in each case the factor $(t + z)^{-k}$ appearing at each step, we arrive at

$$U(1; 1; z) = \sum_{k=1}^{n} (-)^{k-1}(k-1)!z^{-k} + R_n(z), \tag{1.1.7}$$

where the remainder term $R_n(z)$ is

$$R_n(z) = (-)^n n! \int_0^\infty \frac{e^{-t}}{(t+z)^{n+1}}\, dt. \tag{1.1.8}$$

Evidently, each term produced in the series in (1.1.7) is a term from the asymptotic scale $\{z^{-j}\}$, $j = 1, 2, \ldots$, so that if we can show that for any n, $R_n(z) = O(z^{-n-1})$, we will have established the asymptotic expansion

$$U(1; 1; z) \sim \sum_{k=1}^{\infty} (-)^{k-1}(k-1)!z^{-k}, \tag{1.1.9}$$

for $z \to \infty$ in the sector $|\arg z| \leq \pi - \delta < \pi$.

To this end, we observe that the bound used in establishing the uniform convergence of the integral (1.1.6), namely $1/|t + z| \leq 1/|z| \sin \delta$, can be brought to bear on (1.1.8) to yield

$$|R_n(z)| \leq \frac{n!}{(|z| \sin \delta)^{n+1}}.$$

The expansion (1.1.9) is therefore an asymptotic expansion in the sense of Poincaré. It is, however, quite clearly a divergent series, as ratios of consecutive terms in the asymptotic series diverge to ∞ as $(n!/|z|^{n+1})/((n-1)!/|z|^n) = n/|z|$, as $n \to \infty$, irrespective of the value of z. Nevertheless, the divergent character of this asymptotic series does not detract from its computational utility. □

In Tables† 1.1 and 1.2, we have gathered together computed and approximate values of $U(1; 1; z)$, with approximate values derived from the finite series approximation

$$S_n(z) = \sum_{k=1}^{n} (-)^{k-1}(k-1)!z^{-k},$$

obtained by truncating the asymptotic expansion (1.1.9) after n terms. It is apparent from the tables that the calibre of even modest approximations to $U(1; 1; z)$ becomes quite good once $|z|$ is of the order of 100, and is good to two or more significant digits for values of $|z|$ as small as 10. This naturally leads one to

† In Tables 1.1 and 1.2 we have adopted the convention of writing $x(y)$ in lieu of the more cumbersome $x \times 10^y$.

Table 1.1. *Computed and approximate values of* $U(1; 1; z)$ *for real values of* z

z	$U(1; 1; z)$	$S_5(z)$	$S_{10}(z)$
10	0.915633(−1)	0.916400(−1)	0.915456(−1)
50	0.196151(−1)	0.196151(−1)	0.196151(−1)
100	0.990194(−2)	0.990194(−2)	0.990194(−2)

Table 1.2. *Computed and approximate values of* $U(1; 1; z)$ *for imaginary values of* z

z	$U(1; 1; z)$
$10i$	$0.948854(-2) - 0.981910(-1)i$
$50i$	$0.399048(-3) - 0.199841(-1)i$
$100i$	$0.999401(-4) - 0.999800(-2)i$
z	$S_5(z)$
$10i$	$0.940000(-2) - 0.982400(-1)i$
$50i$	$0.399040(-3) - 0.199841(-1)i$
$100i$	$0.999400(-4) - 0.999800(-2)i$
z	$S_{10}(z)$
$10i$	$0.950589(-2) - 0.982083(-1)i$
$50i$	$0.399048(-3) - 0.199841(-1)i$
$100i$	$0.999401(-4) - 0.999800(-2)i$

wonder how the best approximation can be obtained, in view of the utility of these finite approximations and the divergence of the full asymptotic expansion: how can we select n so that the approximation furnished by $S_n(z)$ is the best possible?

The strategy we detail here, called *optimal truncation*, is easily stated: for a fixed z, the successive terms in the asymptotic expansion will reach a minimum in absolute value, after which the terms must necessarily increase without bound given the divergent character of the full expansion; see Fig. 1.1. It is readily shown that the terms in $S_n(z)$ attain their smallest absolute value when $k \sim |z|$ (except when $|z|$ is an integer, in which case there are two equally small terms corresponding to $k = |z| - 1$ and $k = |z|$). If the full series is truncated just before this minimum modulus term is reached, then the finite series that results is the optimally truncated series, and will yield the best approximation to the original function, in the present case, $U(1; 1; z)$.

To see that this is so, observe for $U(1; 1; z)$ that for $z > 0$ the remainder in the approximation after n terms of the asymptotic series,

$$R_n(z) = U(1; 1; z) - S_n(z) = (-)^n n! \int_0^\infty \frac{e^{-t}\,dt}{(t+z)^{n+1}},$$

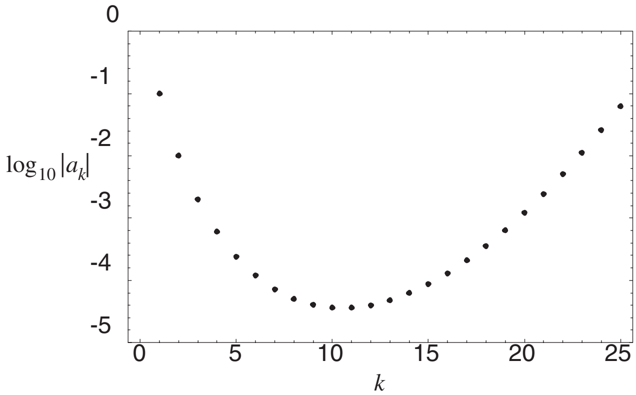

Fig. 1.1. Magnitude of the terms $a_k = (-)^{k-1}\Gamma(k)z^{-k}$ in the expansion $S_n(z)$ against ordinal number k when $z = 10$.

is of the sign opposite to that in the last term in $S_n(z)$ and further, is of the same sign as the first term left in the full asymptotic series after excising $S_n(z)$. In absolute value, we also have

$$|R_n(z)| = \frac{n!}{z^{n+1}} \int_0^\infty \frac{e^{-t}dt}{(1+t/z)^{n+1}} < \frac{n!}{z^{n+1}},$$

so the remainder term is numerically smaller in absolute value than the modulus of the first neglected term. Since the series $S_n(z)$ is an alternating series, it follows that $S_n(z)$ is alternately bigger than $U(1;1;z)$ and less than $U(1;1;z)$ as n increases. The sum $S_n(z)$ will therefore be closest in value to $U(1;1;z)$ precisely when we truncate the full expansion just before the numerically smallest term (in absolute value) in the full expansion. From the preceding inequality, it is easy to note that the remainder term will then be bounded by this minimal term.

To see the order of the remainder term at optimal truncation, we substitute $n \sim z$ ($\gg 1$) in the above bound for $R_n(z)$, and employ Stirling's formula to approximate the factorial, to find

$$|R_n(z)| < \frac{n!}{z^{n+1}} \simeq (2\pi)^{\frac{1}{2}} \frac{e^{-n}n^{n+\frac{1}{2}}}{z^{n+1}} \simeq \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} e^{-z}.$$

This shows that at optimal truncation the remainder term for $U(1;1;z)$ is of order $z^{-\frac{1}{2}}e^{-z}$ as $z \to +\infty$ and consequently that evaluation of the function by this scheme will result in an error that is *exponentially small* in z; these results can be extended to deal with complex values of z – see Olver (1974, p. 523) for a more detailed treatment. We remark that this principle is found to apply to a wide range of asymptotic series yielding in each case an error term at optimal truncation that is typically exponentially small in the asymptotic variable.

We observe that not all asymptotic series present the regular behaviour of the coefficients depicted in Fig. 1.1. In certain compound expansions, with coefficients