1

Ordinary Differential Equations

1.1 Introduction

The formulation of many problems in physical and social sciences involves differential equations that express the relationship among the derivatives of one or more unknown functions with respect to the independent variables (Refs. [1–7]). In an ordinary differential equation (ODE), all derivatives are with respect to a single independent variable. The order of a differential equation is the order of the *highest* derivative that appears in the equation. For example, Newton's law describing the angular position of an oscillating pendulum consisting of a massless string of length ℓ and a point mass m under the influence of gravity (see Fig. 1.1) takes the form of a second-order differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0,$$

where $\theta = \theta(t)$ and t denotes the time variable. Because of the term $\sin\theta$, the above is a nonlinear differential equation whose solution has been extensively studied. Unfortunately, to date, there is no comprehensive theory to solve general nonlinear differential equations analytically. This stands in contrast to the well-developed theory of linear differential equations. In the above example, if the angle of oscillation is small, then $\sin\theta \approx \theta$, and the solution can be approximated from the following linear differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0.$$

In most cases, a linear differential equation can be obtained by linearization of a nonlinear one. We are fortunate that many physical systems can be adequately described by linear differential equations.

Perhaps one of the most famous ODE is that of a linear spring–mass–damper system (see Fig. 1.2) whose dynamics is governed by

$$m\frac{d^2w}{dt^2} + \zeta\frac{dw}{dt} + kw = f,$$

where w denotes the position of the mass from its equilibrium position, $f(t)$ is the forcing function, and m, ζ, and k denote the mass, damping, and stiffness coefficients, respectively. In practice, many vibration problems are treated as linear even if they

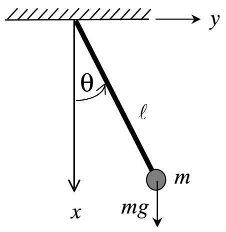

Figure 1.1. An oscillating pendulum.

involve a large number of degrees of freedom. This is because they typically deal with small-amplitude motion about an equilibrium position. The control of such linear systems can be handled conveniently by linear control, which represents a major portion of control theory in general.

In this chapter, we will show how to solve homogeneous ODEs with constant coefficients. The characteristic equation is derived and its solutions are discussed including all possible cases such as distinct roots, repeated roots, etc. In the section of the nonhomogeneous ODEs with constant coefficients, the solution of the nonhomogeneous equations and its properties are discussed. The last section briefly introduces coupled differential equations and the definition of a matrix differential equation.

1.2 Homogeneous ODE with Constant Coefficients

An nth-order linear homogeneous ODE has the form

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = 0, \tag{1.1}$$

where $a_0, a_1, \ldots, a_n$ are real constant coefficients. Note that the right-hand side is zero in this case. In a physical system, a homogeneous ODE may correspond to the situation in which the dynamic system does not have any input or force applied to it and its response is due to some nonzero initial conditions.

EXAMPLE 1.1
The following equation is a homogeneous ODE:

$$\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} + 2y = 0.$$

This may represent a spring–mass–dashpot system with mass $m = 1$, damping coefficient $\zeta = 3$, and spring constant $k = 2$.

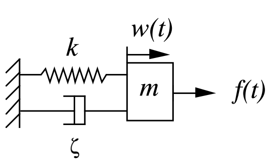

Figure 1.2. A spring–mass–damper system.

1.2.1 General Solution

Functions of $y(t)$ that satisfy Eq. (1.1) are called the homogeneous solutions of the ODE. It is anticipated that the solutions are of the form $y = e^{\gamma t}$ with appropriate values of γ. To find these values, we substitute $y = e^{\gamma t}$ into the differential equations and simplify the resultant expression to obtain

$$(a_0\gamma^n + a_1\gamma^{n-1} + \cdots + a_{n-1}\gamma + a_n)e^{\gamma t} = 0. \tag{1.2}$$

The equation

$$a_0\gamma^n + a_1\gamma^{n-1} + \cdots + a_{n-1}\gamma + a_n = 0. \tag{1.3}$$

is called the characteristic equation of the ODE. A polynomial of degree n has n roots, say, $\gamma_1, \gamma_2, \ldots, \gamma_n$. Therefore, the characteristic equation can be written in the form

$$a_0(\gamma - \gamma_1)(\gamma - \gamma_2)\cdots(\gamma - \gamma_n) = 0. \tag{1.4}$$

Solving the roots of characteristic equation (1.3) yields the values of $\gamma_1, \gamma_2, \ldots, \gamma_n$. Each value of γ represents one solution to the ODE. The general solution is a linear combination of these solutions:

$$y = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t} + \cdots + c_n e^{\gamma_1 n}. \tag{1.5}$$

The simplest situation is that in which all the characteristic roots $\gamma_1, \gamma_2, \ldots, \gamma_n$ are real and distinct. Minor complexities will occur if there are some complex roots or repeated roots. The following will summarize various possible cases.

CASE 1: REAL AND NONREPEATED ROOTS

If all the roots of characteristic equation (1.3) are distinct and real, the general solution is

$$y = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t} + \cdots + c_n e^{\gamma_1 n}.$$

The coefficients $c_1, c_2, \ldots, c_n$ are determined by initial conditions, which will be addressed later in this chapter.

EXAMPLE 1.2

The characteristic equation for the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

is

$$\gamma^2 + 3\gamma + 2 = 0,$$

which has two roots, $\gamma_1 = -1$ and $\gamma_2 = -2$. The general (homogeneous) solution is

$$y = c_1 e^{-1t} + c_2 e^{-2t}.$$

CASE 2: COMPLEX ROOTS

If the characteristic equation has complex roots, they must occur in complex-conjugate pairs, $\sigma \pm i\omega$, as the coefficients $a_0, a_1, \ldots, a_n$ are real numbers. Provided that the roots are not repeated, the corresponding solutions that make up the general solution will have the form $e^{(\sigma+i\omega)t}$, $e^{(\sigma-i\omega)t}$. We can carry the mathematics one step further by considering the linear combination

$$\begin{aligned} Ae^{(\sigma+i\omega)t} + Be^{(\sigma-i\omega)t} &= Ae^{\sigma t}e^{i\omega t} + Be^{\sigma t}e^{-i\omega t} \\ &= Ae^{\sigma t}(\cos\omega t + i\sin\omega t) + Be^{\sigma t}(\cos\omega t - i\sin\omega t) \\ &= (A+B)e^{\sigma t}\cos\omega t + i(A-B)e^{\sigma t}\sin\omega t \\ &= c_1 e^{\sigma t}\cos\omega t + c_2 e^{\sigma t}\sin\omega t. \end{aligned} \tag{1.6}$$

The last equality is possible because of the expectation that the solution of a physical system is real, so that A and B will be such that the combinations $(A+B)$ and $i(A-B)$ are indeed real numbers. For convenience, $(A+B)$ and $i(A-B)$ are denoted by the real coefficients c_1 and c_2, respectively. For this reason, we normally use the real-valued solutions

$$e^{\sigma t}\cos\omega t, \quad e^{\sigma t}\sin\omega t$$

as solutions that make up the general solution of the ODE.

EXAMPLE 1.3

Find the general solution of

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0.$$

The characteristic equation is $\gamma^2 + \gamma + 1 = 0$, which has two roots:

$$\gamma_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \gamma_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The general solution is

$$y(t) = c_1 e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t.$$

CASE 3: REPEATED REAL ROOTS

If the roots of the characteristic equation are not distinct, i.e., some of the roots are repeated, then some additional solutions to the ODE must be found to make up the general solution. Fortunately, it can be shown that they have simple forms. If the real root γ is repeated s times, then the corresponding solutions are not only

$$e^{\gamma t}, \tag{1.7}$$

as known before, but also

$$te^{\gamma t}, \quad t^2e^{\gamma t}, \ldots, \quad t^{s-1}e^{\gamma t}. \tag{1.8}$$

It is easy to verify that the above solutions do indeed satisfy the homogeneous ODE.

EXAMPLE 1.4
Find the general solution of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

The characteristic equation is $\gamma^2 + 2\gamma + 1 = (\gamma + 1)^2 = 0$, which has two repeated roots, $\gamma_1 = -1$ and $\gamma_2 = -1$. The general solution is

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

CASE 4: REPEATED COMPLEX ROOTS

If the complex root $\sigma + i\omega$ is repeated s times, then we have $2s$ solutions because the complex roots always appear as conjugate pairs (for characteristic equations with real coefficients). These roots can be shown to be

$$\begin{array}{llll} e^{\sigma t}\cos\omega t, & te^{\sigma t}\cos\omega t, & t^2 e^{\sigma t}\cos\omega t, \ldots, & t^{s-1}e^{\sigma t}\cos\omega t, \\ e^{\sigma t}\sin\omega t, & te^{\sigma t}\sin\omega t, & t^2 e^{\sigma t}\sin\omega t, \ldots, & t^{s-1}e^{\sigma t}\sin\omega t. \end{array} \tag{1.9}$$

Again, the above solutions can be easily shown to satisfy the homogeneous ODE.

EXAMPLE 1.5
Find the general solution of

$$\frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + y = 0.$$

The characteristic equation is $\gamma^4 + 2\gamma^2 + 1 = (\gamma + i)^2(\gamma - i)^2 = 0$, which has two repeated roots:

$$\gamma_1 = i, \quad \gamma_2 = i, \quad \gamma_3 = -i, \quad \gamma_4 = -i.$$

The general solution is

$$y(t) = c_1 \cos t + c_2 t \cos t + c_3 \sin t + c_4 t \sin t.$$

1.2.2 Multiple Roots of $\gamma^s - \alpha = 0$

In solving for the characteristic roots, it is sometimes necessary to solve for the multiple roots of a number α, which can be real or complex. In general, α may be written as

$$\alpha = Re^{i\theta}, \tag{1.10}$$

where R is the amplitude and θ is the phase angle. For a real number α, θ is zero or integer multipliers of 2π. Therefore, we need to solve for the s roots of the equation

$$\gamma^s = Re^{i\theta}. \tag{1.11}$$

One root is obviously,

$$\gamma_1 = R^{1/s} e^{i\theta/s}. \tag{1.12}$$

To find the remaining $s - 1$ roots, first realize that the angle of a complex number is determined up to only a multiple of 2π. This means that the number α can be represented by $\alpha = Re^{i(\theta+2n\pi)}$, where n can be zero or any positive or negative number. Thus, to be more general, the equation to be solved becomes

$$\gamma^s = Re^{i(\theta+2n\pi)}, \tag{1.13}$$

whose roots are

$$\gamma = R^{1/s} e^{i(\theta+2n\pi)/s}. \tag{1.14}$$

Setting $n = 0$ gives us the root γ_1, as before. Setting $n = 1, 2, \ldots, s - 1$ will produce the remaining $s - 1$ roots for a total of s roots. It can easily be verified that setting n equal to any other integer value, either positive or negative, will reproduce one of these s roots.

EXAMPLE 1.6

Find the four roots of $\gamma^4 + 1 = 0$. Application of Eq. (1.13) yields

$$\gamma^4 = -1 = e^{i(\pi+2n\pi)}.$$

Hence,

$$\gamma = e^{i(\pi/4+n\pi/2)}.$$

We obtain the four roots by setting $n = 0, 1, 2, 3$, which give $(1 + i)/\sqrt{2}$, $(1 - i)/\sqrt{2}$, $(-1 + i)/\sqrt{2}$ and $(-1 - i)/\sqrt{2}$. The four roots are located on the complex plane shown in Fig. 1.3.

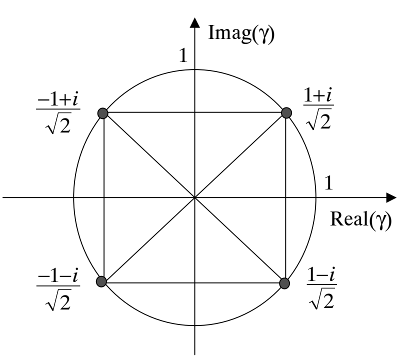

Figure 1.3. Four complex roots.

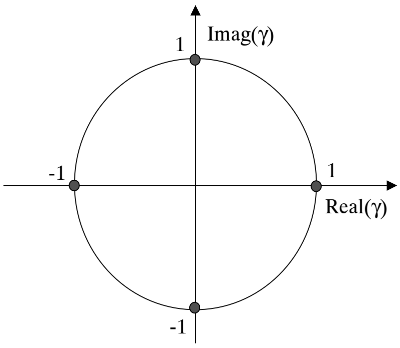

Figure 1.4. Two real and two pure imaginary roots.

EXAMPLE 1.7

Find the four roots of $\gamma^4 - 1 = 0$. Since $\gamma^4 = 1 = e^{i(2n\pi)}$ we have

$$\gamma = e^{i(n\pi/2)}.$$

Setting $n = 0, 1, 2, 3$, yields the four roots:

$$1,\ i,\ -1, -i.$$

The four roots are located on the complex plane shown in Fig. 1.4. Note that in both Examples 1.6 and 1.7 the roots are equally spaced in the complex plane. This is a general property of the complex roots of a number.

1.2.3 Determination of Coefficients

The general form of the solution to a homogeneous ODE is known from the roots of the characteristic equation. However, the general solution contains the unknown coefficients $c_1, c_2, \ldots, c_n$, which need to be determined. To be able to solve for the unknown coefficients, new information is needed. The new information comes in the form of the initial conditions that must be specified. The initial conditions are the specified values of

$$y(0),\quad \left.\frac{dy}{dt}\right|_{t=0},\quad \left.\frac{d^2y}{dt^2}\right|_{t=0},\ldots,\quad \left.\frac{d^{n-1}y}{dt^{n-1}}\right|_{t=0}. \tag{1.15}$$

Satisfying n initial conditions gives us n linear algebraic equations with n unknowns $c_1, c_2, \ldots, c_n$. Solving this set of algebraic equations will give us the values of $c_1, c_2, \ldots, c_n$. In a physical system, the initial conditions can be the values of its initial position and initial velocity. It is clear that the response of a system can be uniquely determined only after its initial conditions are specified.

EXAMPLE 1.8

Find the solution of the following ODE:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

subject to the initial conditions $y(0) = 0$ and $y'(0) = 1$, where $y'(0) = \frac{dy}{dt}|_{t=0} = 1$. The general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

To satisfy the initial conditions, the coefficients must satisfy

$$c_1 + c_2 = 0,$$
$$2c_1 + 3c_2 = -1.$$

Solving these equations gives $c_1 = 1$ and $c_2 = -1$. Thus, the solution that satisfies the initial conditions is

$$y(t) = e^{-2t} - e^{-3t}.$$

1.3 Nonhomogeneous ODE with Constant Coefficients

When the right-hand-side term is not zero, the ODE is said to be nonhomogeneous. Thus, a general nonhomogeneous ODE with constant coefficients has the form

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = f(t). \tag{1.16}$$

In a physical system, this may correspond to the case in which the dynamic system is subjected to some input or forcing function $f(t)$.

1.3.1 General Solution

The general solution of a nonhomogeneous ODE is the sum of the solution of the homogeneous part of the ODE and a particular solution of the nonhomogeneous ODE. It is simple to show mathematically why this is the case. Let the solution of the homogeneous part of the ODE be denoted by $y_h(t)$ and the particular solution by $y_p(t)$,

$$a_0 \frac{d^n y_h}{dt^n} + a_1 \frac{d^{n-1} y_h}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy_h}{dt} + a_n y_h = 0,$$

$$a_0 \frac{d^n y_p}{dt^n} + a_1 \frac{d^{n-1} y_p}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy_p}{dt} + a_n y_p = f(t). \tag{1.17}$$

Adding the two equations and recognizing the property

$$\frac{d^i (y_h + y_p)}{dt^i} = \frac{d^i y_h}{dt^i} + \frac{d^i y_p}{dt^i} \tag{1.18}$$

for any i, we have

$$a_0 \frac{d^n (y_h + y_p)}{dt^n} + a_1 \frac{d^{n-1} (y_h + y_p)}{dt^{n-1}} + \cdots + a_{n-1} \frac{d(y_h + y_p)}{dt} + a_n (y_h + y_p) = f(t), \tag{1.19}$$

Table 1.1: Forcing Functions and Particular Solutions

Forcing Function $f(t)$	Particular Solution $y_p(t)$ to Try
Constant	a
t	$at + b$
t^2	$at^2 + bt + c$
$\sin \omega t$	$a \sin \omega t + b \cos \omega t$
$e^{\sigma t}$	$ae^{\sigma t}$
$t^2 e^{\sigma t} \cos \omega t$	$at^2 e^{\sigma t} \cos \omega t + bt^2 e^{\sigma t} \sin \omega t$

which implies that

$$y(t) = y_h(t) + y_p(t) \tag{1.20}$$

is the general solution to the nonhomogeneous ODE. From previous sections, we know how to find the general form of the homogeneous solution. It is very important to realize that the unknown coefficients $c_1, c_2, \ldots, c_n$ in the homogeneous solution must not be determined at this stage. These coefficients can be determined only after a particular solution has been found and the general solution is constructed. This is because any initial conditions of the system are for $y_h(t) + y_p(t)$, not $y_h(t)$ alone.

1.3.2 Particular Solution

For a simple forcing function $f(t)$, it is sometimes possible to guess the form of the particular solution with a certain number of undetermined coefficients that will make this candidate solution satisfy the nonhomogeneous ODE. This method is known as the method of undetermined coefficients. Table 1.1 gives some simple cases that are commonly encountered in practice (for control applications).

EXAMPLE 1.9
Consider the nonhomogeneous ODE

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^t.$$

We try $y_p(t) = ae^t$ as a candidate particular solution. Substituting ae^t back to the differential equation will yield $a = 1/4$. Thus $y_p(t) = e^t/4$ is a particular solution.

In the following, we provide a justification for the above process. Because we know how to handle a homogeneous ODE, we try to find a way to turn our nonhomogeneous ODE into a homogeneous one by looking for a differential operator L such that

$$L\left\{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y\right\} = L\{e^t\} = 0.$$

Because it is required that e^t be a solution to the homogeneous problem $L\{e^t\} = 0$, $\gamma = 1$ must be a characteristic root. A corresponding characteristic equation is then $\gamma - 1 = 0$, which implies that the desired differential operator is $L = d(.)/dt - 1$. Having found L, we now apply it to the original nonhomogeneous ODE to turn it into a homogeneous one:

$$\left(\frac{d}{dt} - 1\right)\left(\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y\right) = 0.$$

The solution to this homogeneous ODE, which we know how to solve, will contain a particular solution to the original nonhomogeneous problem. The characteristic equation for this homogeneous ODE is

$$(\gamma - 1)(\gamma^2 + 2\gamma + 1) = (\gamma - 1)(\gamma + 1)^2 = 0.$$

Thus, the general solution has the form

$$y(t) = ae^t + c_1e^{-t} + c_2te^{-t}.$$

The constant a can be determined by substituting the above general solution back into the original nonhomogeneous ODE. Note that the part $c_1e^{-t} + c_2te^{-t}$ is simply the solution to the homogeneous part of the ODE and will be eliminated automatically. Completing this procedure will yield $a = 1/4$. Hence,

$$y_p(t) = \frac{1}{4}e^t$$

is a particular solution and $c_1e^{-t} + c_2te^{-t}$ is the solution to the homogeneous part of the ODE:

$$y_h(t) = c_1e^{-t} + c_2te^{-t}.$$

The general solution is the sum of the homogeneous solution and a particular solution, as claimed. The constants c_1, c_2 can be determined if the general solution $y(t) = y_h(t) + y_p(t)$ is made to satisfy the initial conditions

$$y(0) \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t=0},$$

which must be specified.

1.4 Coupled Ordinary Differential Equations

The earlier sections in this chapter address the ODE of only a single variable. This is the case when single-input–single-output (SISO) systems are considered. The dynamics of a multiple-input–multiple-output (MIMO) system can be described by a set of coupled ODEs. The set of equations

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + f_1(t),\\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + f_2(t)\end{aligned} \tag{1.21}$$